THE SET OF SECOND ITERATES IS NOWHERE DENSE IN C

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Abstract. Let C denote the set of continuous functions mapping [0, 1] into itself endowed with sup norm. It is proved that the set \{f \circ f : f \in C\} is nowhere dense in C.

Let C denote the set of continuous functions mapping [0, 1] into itself endowed with sup norm. Humke and Laczkovich [1, 2] investigated the structure of the set \(W^2 = \{f \circ f : f \in C\}\). They proved that \(W^2\) is an analytic non-Borel subset of C. They also proved that \(W^2\) is not everywhere dense in C. The author of this paper proved that \(W^2\) is a set of first category [5] and of zero Wiener measure [4]. Our aim here is to prove the

Theorem. \(W^2\) is nowhere dense in C.

For \(f \in C\) and \(n \in \mathbb{N}\) we define \(f^n(x)\) by induction:

\[ f^n(x) = f(f^{n-1}(x)) \quad \text{with} \quad f^1(x) = f(x). \]

For each \(n \in \mathbb{N}\) \(P^n_f\) denotes the set of periodic points of \(f\) with period \(n\) and \(\text{Fix}(f^n) = \{x : f^n(x) = x\}\); then \(P^n_f = \text{Fix}(f^n) \setminus \bigcup_{k<n} \text{Fix}(f^k)\). The closed interval with endpoints \(a\) and \(b\) will be denoted by \(\langle a, b \rangle\) even if \(b < a\). We put

\[ B(f, \varepsilon) = \left\{ h \in C : \sup_{x \in [0, 1]} |f(x) - h(x)| < \varepsilon \right\}. \]

Proof of theorem. Put

\[ K = \{f \in C : \text{for every } \varepsilon > 0, \text{there exists } \delta > 0, \; g \in B(f, \varepsilon) \text{ such that } B(g, \delta) \cap W^2 = \emptyset\}. \]

Thus \(K\) is the set of all functions \(f\) such that \(W^2\) is not dense in any neighbourhood of \(f\). It is enough to show that \(K = C\). For every \(f \in C\) we define \(D_f = \{x : f'(x) \text{ exists}\}\). Put

\[ \text{PL} = \{f \in C : f \text{ is piecewise linear}; \quad \text{Fix}(f) \subset D_f \cap (0, 1); \quad \text{for every } x \in D_f, |f'(x)| > 2\}. \]
It is easy to see that $\text{PL}$ is dense in $C$. We show that $\text{PL} \subseteq K$, from which it follows that $K = C$ since $K$ is closed in $C$. Let $f \in \text{PL}$ and $\varepsilon > 0$ be fixed. We show that there exists $\delta > 0$ and there is $g \in B(f, \varepsilon)$ such that $B(g, \delta) \cap W^2 = \emptyset$. Since $f \in \text{PL}$ card($\text{Fix}(f)$) is odd, assume that $2n + 1 = \text{card}(\text{Fix}(f))$. Furthermore let $q = p_1 \cdot p_2 \cdots \cdot p_{(n+1)}$ where $5 < p_1 < p_2 < \cdots < p_{(n+1)}$ are prime numbers. Then $\text{Fix}(f^q)$ is a finite set since $f \in \text{PL}$. Write $m = \text{card}(\text{Fix}(f^q) - \text{Fix}(f))$.

**Lemma 0.** There exist positive numbers $\delta_1$, $\delta_2$ such that $\varepsilon/2 > \delta_1 > \delta_2 > 0$ and

(i) if $x \in \text{Fix}(f)$ and $|t - x| < \delta_1$ then $f'(t)$ exists;

(ii) if $h \in B(f, \delta_2)$ then $\text{Fix}(h^q) \subseteq \bigcup_{x \in \text{Fix}(f)} (x - \delta_1, x + \delta_1)$;

(iii)$\quad h \in B(f, \delta_2), \quad 0 \leq l \leq k \leq q, \quad x, y \in \text{Fix}(f^q), \quad f^l(x) \neq f^k(y)$

Then $\text{Fix}(f^q) = \{a_1, b_1, a_2, b_2, \ldots, a_n, b_n, a_{(n+1)}\}$ where $a_i < b_i < a_{(i+1)}$ for $i = 1, \ldots, n$ and put $U_i = (a_i - \delta_1, a_i + \delta_1)$, $V_j = (b_j - \delta_1, b_j + \delta_1)$ for

**Proof of Lemma 0.** Choose $0 < \eta_1$ so that it satisfies the following conditions:

$\eta_1 < \varepsilon/2$; $\eta_1 < \min\{|x - y| : x, y \in \text{Fix}(f^q), x \neq y\}$, and if $x \in \text{Fix}(f)$ with $|t - x| < \eta_1$ then $t \in D_f$. This is possible because $f \in \text{PL}$.

We define $0 < \eta_2$ so that $\eta_2 < \eta_1/10$ and

$$|x - y| < \eta_2 \Rightarrow \max_{0 \leq l \leq q} |f^l(x) - f^l(y)| < \eta_1/10$$

for every $x, y \in [0, 1]$. Put $\delta_1 = \eta_2$ and define $0 < \eta_3 < \delta_1$ so that $\eta_3 < \min\{|f^q(y) - y| : y \in [0, 1] \text{ and } \text{dist}(y, \text{Fix}(f^q)) \geq \delta_1\}$. We know that $\eta_3 > 0$, thus we can choose $0 < \eta_4$ so that $\eta_4 < \eta_3$ and

$$h \in B(f, \eta_4) \Rightarrow \min\{|h^q(y) - y| : \text{dist}(y, \text{Fix}(f^q)) \geq \delta_1\} > \frac{\eta_3}{2}. \quad (*)$$

Choose $0 < \eta_5$ so that $\eta_5 < \eta_4$ and if $h \in B(f, \eta_5)$ then $\max_{0 \leq l \leq q} \|f^l - h^l\| < \eta_4$. We put $\delta_2 = \eta_5$.

It is obvious that (i) is fulfilled and that (*) implies (ii). To see that (iii) holds let $h$, $l$, $k$, $x$, $y$ satisfy the conditions listed in (iii). Let $z_1 \in (x - \delta_1, x + \delta_1)$, $z_2 \in (y - \delta_1, y + \delta_1)$ be arbitrary. We have to show that $h^l(z_1) \neq h^k(z_2)$. We have $|f^l(x) - f^l(z_1)| < \eta_1/10$ and $|f^k(y) - f^k(z_2)| < \eta_1/10$ from the definition of $\eta_2$. Since $f^l(x), f^k(y) \in \text{Fix}(f^q)$ and $f^l(x) \neq f^k(y)$, $|f^l(x) - f^k(y)| > \eta_1$ follows from the definition of $\eta_1$. Thus $|f^l(z_1) - f^k(z_2)| > 8\eta_1/10$. We know that $|h^l(z_1) - f^l(z_1)| < \eta_2 < \eta_1/10$ and $|h^k(z_2) - f^k(z_2)| < \eta_2 < \eta_1/10$ from the definition of $\eta_2$. Thus $8\eta_1/10 < |f^l(z_1) - f^k(z_2)| < |h^l(z_1) - h^k(z_2)| + |h^l(z_1) - f^l(z_1)| + |h^k(z_2) - f^k(z_2)| < \eta_1/10 + |h^l(z_1) - h^k(z_2)| + \eta_1/10$ whenever $6\eta_1/10 < |h^l(z_1) - h^k(z_2)|$, which completes the proof.

Let $\text{Fix}(f) = \{a_1', b_1', a_2', b_2', \ldots, a_n', b_n', a_{(n+1)}'\}$ where $a_i' < b_i' < a'_{(i+1)}$ for $i = 1, \ldots, n$ and put $U_i = (a_i' - \delta_1, a_i' + \delta_1)$, $V_j = (b_j' - \delta_1, b_j' + \delta_1)$ for
every $i = 1, \ldots, n + 1$, $j = 1, \ldots, n$. Furthermore let $\text{Fix}(f^q) \setminus \text{Fix}(f) = \{c_1, \ldots, c_m\}$ and put $W_k = (c_k - \delta_1, c_k + \delta_1)$ for every $1 \leq k \leq m$.

**The main steps of the proof**

To facilitate the presentation we will briefly outline the elements involved in our proof. There are two basic steps.

**Step 1.** In order to define a function $g \in B(f, \varepsilon)$ and find a suitable $\delta > 0$ so that $B(g, \delta) \cap W^2 = \emptyset$ we will first describe a technique for construction of a function $\Phi^p_{(i)}$, for a preassigned $p > 5$ and interval $I \subset [0, 1]$. Then we pick intervals $I_i \subset U_i$ for $1 \leq i \leq n + 1$. The function $g$ will differ from $f$ only on $\bigcup_{i=1}^{n+1} U_i$. On each $I_i$, it will be a special function $\Phi^p_{(i)}$ derived from the technique. On the remainder of each $U_i \setminus I_i$ $g$ will be defined so that

(i) it is continuous on $[0, 1]$;

(ii) it has no odd order periodic points in $\bigcup_{i=1}^{n+1}(U_i \setminus I_i)$.

Then the $\delta$ is chosen so that any function $v \in B(g, \delta)$ will have at least $m + 1$ periodic orbits of period $p_i$ in $U_i$, $1 \leq i \leq n + 1$.

**Step 2.** We will examine, via 4 lemmas, the possible locations of periodic orbits of period $p_i$ for any function $v \in B(g, \delta)$. It will be shown that the assumptions $v \in B(g, \delta)$ and $v = h^p$ for some $h \in C$ are self-contradictory.

**Step 1**

Let $|I|$ denote the length of the interval $I$. Put $\gamma = |I| \cdot [2(2p + 1)]^{-1}$ and define the pairwise disjoint closed intervals

\[ A_i = [d_i - \gamma, d_i + \gamma] \quad (i = 0, 1, 2, \ldots, p - 1) \]

where $d_0 = \gamma$, $d_i = d_{(i-1)} + 4\gamma$, $1 \leq i \leq p - 2$, and $d_{(p-1)} = d_{(p-2)} + 8\gamma$. Write $d' = d_{(p-2)} + 4\gamma$ and denote by $\Omega$ the increasing linear function mapping $I$ onto $[d' - \gamma, d' + \gamma]$. Next we define a function $\Phi^p$ on $[0, 1]$ which is to be linear on the intervals $A_i$ ($i = 0, 1, \ldots, p - 1$), $[d' - \gamma, d' + \gamma]$ and on the intervals contiguous to them: put $\Phi^p(d_i - \gamma) = d_{(i+1)} - 9\gamma/10$, $\Phi^p(d_i + \gamma) = d_{(i+1)} + 9\gamma/10$ ($i = 0, 1, \ldots, p - 2$) $\Phi^p(d_{(p-1)} - \gamma) = d_0 + 9\gamma/10$, $\Phi^p(d_{(p-1)} + \gamma) = d_0 - 9\gamma/10$.

We define $\gamma' = \gamma/(2p + 1)$ from $\gamma'/|\Omega([0, 1])| = \gamma$ and put

\[ \Phi^p(d' - \gamma) = d' - \gamma + 4.1\gamma', \hspace{0.5cm} \Phi^p(d' + \gamma) = d' - \gamma + 0.1\gamma'. \]

Thus we have defined $\Phi^p$ (see Figure 1, p. 1144). It is easy to see that $\Phi^p$ has the following properties:

(A) If $\|h - \Phi^p\| < \gamma \cdot (0.9)^p/30$ then there exists a periodic orbit of $h$ with period $p$ in $\bigcup_{i=0}^{p-1}(d_i - \gamma/3, d_i + \gamma/3)$.

(B) If $\|h - \Phi^p\| < \gamma \cdot (0.9)^p/30$ and $\{x_i\}_{i=0}^{p-1}$ is a periodic orbit of $h$ with period $p$ in $\bigcup_{i=0}^{p-1} A_i$ then $\{x_i\}_{i=0}^{p-1} \subset \bigcup_{i=0}^{p-1}(d_i - \gamma/3, d_i + \gamma/3)$.
Figure 1

(C) If \( \{x_i\}_{i=0}^{p-1} \subset I \setminus \Omega(I) \) is a periodic orbit of \( \Phi^p \) and \( p \) is odd then the smallest interval which contains \( \{x_i\}_{i=0}^{p-1} \) also contains the point \( d_{p-2} \) (see Figure 1).

\textbf{Proof of (A) and (B).} Let \( d_0 - \gamma < x < d_0 - \gamma/3 \) and \( d_0 + \gamma/3 < y < d_0 + \gamma \). Since the absolute value of the slope of \( \Phi^p \) above \( A_i \) (0 \( \leq i \leq p-1 \)) equals to 0.9 it is implied that if \( \|h - \Phi^p\| < \gamma \cdot (0.9)^p/30 \) then \( h^p(x) > d_0 \) and \( h^p(y) < d_0 \). So the interval \( (x, y) \) contains a periodic point of \( \Phi^p \) with period \( p \) (this proves (A)), but \( x, y \) are not periodic points of \( \Phi^p \) with period \( p \) so we have proved (B) as well.

\textbf{Proof of (C).} The fact that \( \{x_i\}_{i=0}^{p-1} \) is a periodic orbit of \( \Phi^p \) implies that there exists \( 0 \leq 1 \leq p - 1 \) such that \( \Phi^p(x_i) < x_i \) whence \( x_i > d' + \gamma \). Without loss of generality we may assume that \( i = 0 \). If \( x_k > d_{p-2} \) for every \( 0 \leq k \leq p - 1 \) then \( x_1, x_3, \ldots, x_{p-2}, x_0 \in [d_{p-2} + \gamma, d' - \gamma] \) which contradicts the fact that \( x_0 > d' + \gamma \) (see Figure 1).

For an arbitrary interval \( I \) let \( \tau_I \) denote the similarity transformation mapping \([0, 1]\) onto \( I \). We define

\[ \Phi^p_{\tau_I}(x) = \tau_I(\Phi^p(\tau_I^{-1}(x))) \quad (x \in I). \]
Put
\[ \Phi^p_{\Omega^i(I)}(x) = \begin{cases} \Phi_{\Omega^i(I)}(x), & \text{if } x \in \Omega^i(I) \setminus \Omega^{i+1}(I) \text{ and } 0 \leq i < m; \\ \Phi_{\Omega^m(I)}(x), & \text{if } x \in \Omega^m(I). \end{cases} \]

We next define \( g \in C \) as follows (see Figure 2). We put \( g(x) = f(x) \) if \( x \notin \bigcup_{i=1}^{n+1} U_i \).

Let \( 1 \leq i \leq n + 1 \) be fixed. We choose an interval \( I_i \subset U_i \) with midpoint \( a_i \) such that \( |I_i| < \delta_2/(2|f'(a_i)|) \). Let \( g \) be linear decreasing on both components of \( U_i \setminus I_i \) and
\[ g(x) = \Phi^p_{\Omega^i(I)}(x), \quad (x \in I_i). \]
Thus we have \( g \in C \cap B(f, \delta_2) \). Put \( \delta = \min_{1 \leq i \leq n+1} |\Omega^m(I_i)|/100 \). Let \( 1 \leq i \leq n + 1 \) be fixed and let
\[ A_k^i = \tau_{\lambda^i}(A_k), \quad (k = 0, 1, 2, \ldots, p_i - 1). \]

**Proposition 1.** We can see that if \( \|h - g\| < \delta \) then:

(a) For every \( 1 \leq i \leq n + 1 \) \( h \) has at least one periodic orbit with period \( p_i \)
in \( \bigcup_{j=0}^{p_i-1} \Omega^j(A_k^i) \) \((j = 0, 1, \ldots, m)\).

(b) If \( l \) is a divisor of \( q \) then \( P_h^l \cap (U_i \setminus I_i) = \emptyset \) (see Figure 2).
Proof of Proposition 1. (a) It is easy to see (a) is implied by (A) above.

(b) Using Lemma 0 we can see that if \( \{x_k\}_{k=0}^{l-1} \) is a periodic orbit of \( h \) and \( \{x_k\}_{k=0}^{l-1} \cap U_i \neq \emptyset \) then \( \{x_k\}_{k=0}^{l-1} \subset U_i \). The same remains valid for \( I_i \), because \( h(i_j) \subset I_i \), thus if (b) is false then \( \{x_k\}_{k=0}^{l-1} \subset U_i \setminus I_i \). However, since \( l \) is odd \( x_0 \) and \( x_l \) are in different components of \( U_i \setminus I_i \) and thus \( x_0 \neq x_l \).

We now proceed to our Step 2.

**Step 2**

Suppose that \( \psi^2 \in B(g, \delta) \). Let \( \eta > 0 \) be so small that \( \varphi^2 \in B(g, \delta) \) holds for every \( \varphi \in B(\psi, \eta) \). In [3, Theorem 1] it is proved that: there exists a residual subset \( A \) of \( C \) such that for every \( \varphi \in A \) and \( u, v \in \mathbb{N} \), every neighbourhood of any periodic point of \( \varphi \) with period \( v \) contains periodic points of \( \varphi \) with period \( uv \).

Choose \( \varphi \in B(\psi, \eta) \cap A \) and put \( h = \varphi^2 \). Let \( 1 \leq i \leq n + 1 \) and \( 0 \leq j \leq m \) arbitrary. From Proposition 1 we know that \( \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k) \) contains a periodic orbit \( \{x_k\}_{k=0}^{p_i-1} \) of \( h \) with period \( p_i \). Since \( h = \varphi^2 \) one can easily see that either there is a periodic orbit \( \{a_k\}_{k=0}^{2p_i-1} \) of \( \varphi \) with period \( 2p_i \) such that \( a_{2k} = x_k \) (\( k = 0, 1, \ldots, p_i - 1 \)) or \( \{x_k\}_{k=0}^{p_i-1} \) is a periodic orbit of \( \varphi \) itself. In the latter case, using the previous statement with \( u = p_i \) and \( v = 1 \), we get that there is a periodic orbit \( \{a_k\}_{k=0}^{2p_i-1} \) of \( \varphi \) such that \( \{a_k\}_{k=0}^{2p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k) \). Hence we have shown that

**Proposition 2.** For every \( 1 \leq i \leq n + 1 \) and \( 0 \leq j \leq m \) there exists a periodic orbit \( \{a_k\}_{k=0}^{2p_i-1} \) of \( \varphi \) so that

\[
\{a_{2k}\}_{k=0}^{p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k).
\]

Since \( \{a_{2k-1}\}_{k=0}^{p_i} \) is a periodic orbit of \( \varphi \) with period \( 2p_i \), \( \{a_{2k-1}\}_{k=0}^{p_i} \) is a periodic orbit of \( h \) with period \( p_i \). Thus Lemma 0 implies that

\[
\{a_{2k-1}\}_{k=0}^{p_i} \subset \bigcup_{z=1}^{m} W_z \cup \bigcup_{l=1}^{n} V_l \cup \bigcup_{k=1}^{n+1} U_k.
\]

We will show this is impossible via the four lemmas.

We use the notation of Proposition 2; clearly we may assume that \( a_{2k} \in \Omega^j(A_i^k) \) (\( 0 \leq k \leq p_i - 1 \)).

**Lemma 1.** \( \{a_{2k-1}\}_{k=1}^{p_i} \) is not contained in \( U_i \).

**Proof of Lemma 1.** We actually prove more; namely,

(i) \( \{a_{2k-1}\}_{k=1}^{p_i} \) is not contained in \( \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k) \),
(ii) \( \{a_{2k-1}\}_{k=1}^{p_i} \) is not contained in \( \Omega^j(I_i) \) and then
(iii) \( \{a_{2k-1}\}_{k=1}^{p_i} \) is not contained in \( U_i \).

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Assume first that 

\[ \{a_{2k-1}\}_{k=1}^{p_i} \subseteq \bigcup_{k=0}^{p_i-1} \Omega^k_i(A_i^k). \]

Since \( \varphi(a_{2p_i-1}) = a_0 \) and \( a_0 \in \Omega^j_0(A_0) \), \( a_{2p_i-2} \in \Omega^j(A_{p_i-1}) \) (see Figure 1) we have \( \{a_0, a_{2p_i-2}\} \subseteq \varphi(\langle a_{2p_i-1}, a_{2p_i-3} \rangle) \) thus \( \varphi(\langle a_{2p_i-1}, a_{2p_i-3} \rangle) \) contains \( \{a_{2k-1}\}_{k=1}^{p_i} \) except for (at most) two elements. Thus \( \varphi^2(\langle a_{2p_i-1}, a_{2p_i-3} \rangle) \supseteq \{a_2, a_4, \ldots, a_{2p_i-4}\} \) and this implies that

(a) \( a_{2p_i-1} \approx a_{2p_i-2} \) or

(b) \( a_{2p_i-3} \approx a_{2p_i-2} \) (see Figure 1),

where \( x \approx y \) iff there are indices \( i, j, k \) such that \( x, y \in \Omega^j(A_i^k) \). If (a) then \( a_3 \in \langle a_0, a_{2p_i-1} \rangle \subset \varphi(\langle a_{2p_i-1}, a_{2p_i-2} \rangle) \) which is impossible. If (b) then \( a_1 \in \langle a_{2p_i-2}, a_{2p_i-1} \rangle \subset \varphi(\langle a_{2p_i-3}, a_{2p_i-2} \rangle) \) thus \( a_2 \in \varphi^2(\langle a_{2p_i-3}, a_{2p_i-2} \rangle) \) which is impossible.

Next assume that \( \{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^j(I_i) \). Then from (1) we get \( \{a_{2k-1}\}_{k=1}^{p_i} \subset I_i^j = \Omega^j(I_i) \setminus \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k) \) (see Figure 1). But \( a_{2p_i-1} \in I_i^j \subset \{a_0, a_{2p_i-2}\} \subset \varphi(\langle a_{2p_i-1}, a_{2p_i-3} \rangle) \) and thus \( a_4 \in \varphi^2(\langle a_{2p_i-1}, a_{2p_i-2} \rangle) \subset h(I_i) \) contradicting \( a_0 \approx \Omega^j(d_0) \).

Finally suppose \( \{a_{2k-1}\}_{k=1}^{p_i} \subset U_i \setminus \Omega^j(I_i). \) Since \( p_i \) is a divisor of \( q \) it follows from Proposition 1 that it is enough to show that \( \{a_{2k-1}\}_{k=1}^{p_i} \) is not contained in \( I_i \). Suppose that

\[ \{a_{2k-1}\}_{k=1}^{p_i} \subset I_i. \]

Then there exists \( 0 \leq l \leq j \) such that \( \{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^j(I_i) \setminus \Omega^j(I_i), \) this follows from (ii). Since \( \{a_{2k-1}\}_{k=1}^{p_i} \) is a periodic orbit of \( h \) with period \( p_i \), there are odd numbers \( 1 \leq m_1, m_2 \leq 2p_i-1 \) such that \( h(a_{m_1}) = a_{m_1} \) and \( h(a_{m_2}) = a_{m_2} \).

From (C) it follows that

\[ a_{2p_i-4} \in [a_{m_1}, a_{m_2}] \subset \varphi(\langle a_0, a_{2p_i-2} \rangle). \]

Therefore \( a_{2p_i-5} \in \varphi^2(\langle a_0, a_{2p_i-2} \rangle) \subset h(\Omega^j(I_i)) \subset \Omega^j(I_i), \) which is a contradiction. This completes the proof of Lemma 1.

Lemma 2. If \( j_1 \neq j \), \( 0 \leq j_1 \leq m \) is fixed, and if \( \{y_k\}_{k=0}^{p_i-1} \subset \bigcup_{k=0}^{p_i-1} \Omega^j(A_i^k) \) is a periodic orbit of \( h \) and \( \{b_i\}_{i=0}^{p_i-1} \) is a periodic orbit of \( \varphi \) such that \( y_k = b_{2k} \in \Omega^j(A_i^k) \) \((k = 0, 1, \ldots, p_i-1)\), then there do not exist indices \( 0 \leq j_2, j_3 \leq 2p_i-1, 1 \leq j_4 \leq m \) such that \( a_{j_2}, b_{j_3} \in W_{j_4} \).

Proof of Lemma 2. Assume that there exists a periodic orbit \( \{b_i\}_{i=1}^{2p_i-1} \) of \( \varphi \) such that \( b_{2k} \in \Omega^j(A_i^k) \) \((k = 0, 1, \ldots, p_i-1)\) and \( a_{j_2}, b_{j_3} \in W_{j_4}. \) We can assume \( j < j_1 \). Then it follows from Lemma 0 that there are indices \( j_5, j_6 \) such that \( 0 \leq j_5 \leq 2p_i-1, 1 \leq j_6 \leq m \) and such that \( a_{j_1}, b_{j_5} \in W_{j_6}. \)
Thus, since \( \varphi(a_1) = a_2 \) and \( \varphi(b_j) \in \Omega^j(I_i) \), the point \( a_4 \) belongs to \( \varphi(a_1, b_j) \) and this implies \( a_5 \) is an element of \( \varphi^2(a_1, b_j) \) of \( h(W_{j_i}) \). But \( a_1 \in W_{j_i} \) which implies that \( a_3 \in h(W_{j_i}) \) and \( a_5 \in h^2(W_{j_i}) \). Thus \( h^2(W_{j_i}) \cap h(W_{j_i}) \neq \emptyset \) and this contradicts Lemma 0.

It should be noted that Lemma 2 implies that there exists \( 0 \leq l \leq m \) so that for every periodic orbit \( \{b_i\}_{k=0}^{2p_i-1} \) of \( \varphi \) if \( b_{2k} \in \Omega^j(A_k^j) \) \( (k = 0, 1, \ldots, p_i-1) \), then \( \{b_{2k-1}\}_{k=0}^{2p_i-1} \cap \bigcup_{z=0}^{m} W_z = \emptyset \).

**Lemma 3.** \( \{a_{2k-1}\}_{k=1}^{p_i-1} \) is not contained in \( U_i \) where \( 1 \leq i \leq n+1 \) and \( i \neq i' \).

**Proof of Lemma 3.** Assume that \( \{a_{2k-1}\}_{k=1}^{p_i-1} \subset U_i \). First we suppose that \( \{a_{2k-1}\}_{k=1}^{p_i-1} \) is not contained in \( \Omega^{m+1}(I_{i}) \). Then there exists \( 0 \leq w \leq m \) such that
\[
\{a_{2k-1}\}_{k=1}^{p_i-1} \subset \Omega^w(I_i) \backslash \Omega^{w+1}(I_i).
\]
Assume that \( \{b_l\}_{l=0}^{2p_i-1} \) is a periodic orbit of \( \varphi \) such that \( b_{2l} \in \Omega^w(A_i^j) \) for every \( 0 \leq l \leq p_i - 1 \). Then
\[
b_{2p_i-4} \in \varphi(\langle a_0, a_{2p_i-2} \rangle)
\]
(this follows from property (C) of \( \Phi^p(I) \)). Thus
\[
b_{2p_i-3} \in \varphi^2(\langle a_0, a_{2p_i-2} \rangle) \subset \Omega^j(I_i)
\]
and hence that
\[
\{b_{2l-1}\}_{l=1}^{p_i-1} \subset \Omega^j(I_i) \bigcup_{k=0}^{p_i-1} \Omega^j(A_k^j) = I_i
\]
because \( p_i \neq p_i' \) and \( p_i, p_i' \) are prime numbers. Further
\[
a_{2p_i-1} \in \langle b_0, b_{2p_i-2} \rangle \subset \varphi(\langle b_{2p_i-1}, b_{2p_i-3} \rangle)
\]
and thus \( a_0 \in \varphi^2(\langle b_{2p_i-1}, b_{2p_i-3} \rangle) \). This is a contradiction since \( a_0 \notin \varphi^2(I_i) \) (see Figure 1).

Next suppose that \( \{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^{m+1}(I_{i}) \) (see Figure 3). Then for every \( 0 \leq z \leq m \) we choose a periodic orbit of \( \varphi \), \( \{b_{2l}\}_{l=0}^{2p_i-1} \), so that
\[
b_{2l} \in \Omega^z(A_i^j) \quad (l = 0, 1, \ldots, p_i - 1).
\]
It follows from Lemma 2 that there exists \( 0 \leq z_0 \leq m \) such that \( \{b_{2l-1}\}_{l=1}^{p_i} \) does not intersect any \( W_j \). Since \( \{b_{2l-1}\}_{l=1}^{p_i} \) is a periodic orbit of \( h \), it intersects some \( U_k \) or \( V_k \). Since \( h(\Omega^{j+1}(I_i)) \) does not contain \( a_{2l} \), \( (l = 0, \ldots, p_i - 1) \), we get from Lemma 0 and the assumption \( \{a_{2k-1}\}_{k=1}^{p_i} \subset \Omega^{m+1}(I_{i}) \), that
\[
\{b_{2l-1}\}_{l=1}^{p_i} \subset I_i \backslash \Omega^{j+1}(I_i).
\]
But since \( j \leq m \) an analogous case with changing
rules between \( \{a_k\}_{k=0}^{2^{p_i-1}} \) and \( \{b_l^0\}_{l=0}^{2^{p_i-1}} \) was investigated in the first part of this proof showing that this is impossible.

We shall write \( U_i \sim V_j \) if there is a periodic orbit \( \{a_k\}_{k=0}^{2^{p_i-1}} \) of \( \varphi \) such that \( \{a_{2l}^0\}_{l=0}^{p_i-1} \subset U_i \) and \( \{a_{2l-1}^1\}_{l=1}^{p_i-1} \subset V_j \). Lemmas 1, 2, and 3 imply that for every \( 1 \leq i \leq n + 1 \) there exists \( 1 \leq j \leq n \) such that \( U_i \sim V_j \).

**Lemma 4.** There exist indices \( i, j, k \) such that \( 1 \leq i < j < k \leq n + 1 \), with \( U_i \sim V_j \) and \( U_k \sim V_j \).

**Proof of Lemma 4.** Since the number of the \( U_j \)'s is greater than the number of the \( V_j \)'s, there exist \( i, j \) and \( k \) such that

\[
U_i \sim V_j, \quad U_k \sim V_j.
\]

If the desired condition: \( (i \leq j < k) \), does not hold for these \( i, j \) and \( k \), we can assume that \( i < k \leq j \). (The situation when \( j < i, k \) is similar). Then we claim there is no \( 1 \leq l \leq n + 1 \) such that \( V_l \sim U_i \). For if \( V_l \sim U_l \) then,
since \( \varphi(V_j) \cap U_i \neq \emptyset \) and \( \varphi(V_j) \cap U_k \neq \emptyset \), we would get \( \varphi(V_j) \supset V_i \). Thus \( \varphi^2(V_j) \cap U_i \neq \emptyset \) which contradicts Lemma 0. Therefore deleting both \( U_i \) and \( V_j \) the remaining \( U_j \)'s and \( V_j \)'s still have the property that for each \( j \) there is a \( k \) with \( U_j \sim U_k \).

We continue with this process of checking the condition and deleting until we have either the condition fulfilled or have deleted down to two \( U \)'s and only one \( V \). Then the desired condition will hold.

Let \( i, j, k \) be as in Lemma 4. Then, since \( i < j < k \), we have both \( \varphi(V_j) \cap U_i \neq \emptyset \) and \( \varphi(V_j) \cap U_k \neq \emptyset \) and \( \varphi(V_j) \supset V_j \). This implies \( \varphi^2(V_j) \cap U_i \neq \emptyset \) and \( \varphi \) in turn contradicts Lemma 0. This completes the proof of the theorem.

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REFERENCES


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