AN EMBEDDING SPACE TRIPLE OF THE UNIT INTERVAL INTO A GRAPH AND ITS BUNDLE STRUCTURE

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Abstract. Let $l_2$ denote a Hilbert space, and let

\[ l^Q_2 = \{ (x_i) \in l_2 \mid \sup |i \cdot x_i| < \infty \} \]

\[ l^I_2 = \{ (x_i) \in l_2 \mid x_i = 0 \text{ except for finitely many } i \} . \]

We show that the triple $(H(X), H^{LIP}(X), H^{PL}(X))$ of spaces of homeomorphisms, of Lipschitz homeomorphisms, and of PL homeomorphisms of a finite graph $X$ onto itself is an $(l_2, l^Q_2, l^I_2)$-manifold triple, and that the triple $(E(I, X), E^{LIP}(I, X), E^{PL}(I, X))$ of spaces of embeddings, of Lipschitz embeddings, and of PL embeddings of $I = [0,1]$ into a graph $X$ is an $(l_2, l^Q_2, l^I_2)$-manifold triple.

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An $(l_2, l^Q_2, l^I_2)$-manifold triple is defined as a triple $(M, N, W)$ of an $l_2$-manifold, an $l^Q_2$-manifold, and an $l^I_2$-manifold which admits an open cover $\mathcal{U}$ of $M$ and open embeddings $\varphi_U: U \rightarrow l_2$, $U \in \mathcal{U}$, such that $\varphi_U(U \cap N) = \varphi_U(U) \cap l^Q_2$ and $\varphi_U(U \cap W) = \varphi_U(U) \cap l^I_2$ [SW$_2$]. If $X$ is a compact Euclidean polyhedron with dim $X > 0$ and $Y$ is an open set in $\mathbb{R}^n$, the triple $(C(X, Y), LIP(X, Y), PL(X, Y))$ of spaces of (continuous maps), of Lipschitz maps, and of PL maps of $X$ into $Y$ is such a manifold triple [Sa]. In this note, we find other examples of such manifold triples of function spaces where every function space has the compact-open topology; that is, we have the

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following:

**Theorem 1.** For a finite graph (polyhedron of dim = 1) \( X \subset \mathbb{R}^n \), the triple \((H(X), H^{\text{LIP}}(X), H^{\text{PL}}(X))\) of the spaces of homeomorphisms, of Lipschitz homeomorphisms and of PL homeomorphisms of \( X \) onto itself is an \((l_2, l_2^Q, l_2^I)\)-manifold triple.

**Theorem 2.** For a graph \( X \subset \mathbb{R}^n \), the triple \((E(I, X), E^{\text{LIP}}(I, X), E^{\text{PL}}(I, X))\) of spaces of embeddings, of Lipschitz embeddings, and of PL embeddings of \( I = [0, 1] \) into \( X \) is an \((l_2, l_2^Q, l_2^I)\)-manifold triple.

By Theorem 1, Conjecture 2.6 in [SW,] is true in the 1-dim case. Let 
\[ H^\partial(I) = \{ h \in H(I) | h|\partial I = \text{id} \}, \]
where \( \partial I = \{0, 1\} \). Similarly, \( H^{\text{LIP}}(I) \) and \( H^{\text{PL}}(I) \) are defined. First we prove the following:

**Theorem 3.** \((H^\partial(I), H^{\text{LIP}}(I), H^{\text{PL}}(I))\) is homeomorphic \((\cong)\) to \((l_2, l_2^Q, l_2^I)\).

**Proof.** For simplicity, let \( H = H^\partial(I) \), \( H' = H^{\text{LIP}}(I) \), \( H'' = H^{\text{PL}}(I) \). For each \( m \in \mathbb{N} \), let 
\[ L_m = \{ h \in H' | \text{bilip } h \leq 1 + m \}, \]
where \( \text{bilip } h \) is the minimum of \( k \geq 1 \) such that
\[ k^{-1} \cdot |x - y| \leq |h(x) - h(y)| \leq k \cdot |x - y| \quad \text{for each } x, y \in I. \]

If \( m = m' \), then \( L_m \) is a Z-set in \( L_{m'} \). In fact, we have a homotopy \( \varphi : L_{m'} \times I \to L_m \) defined as follows:

\[
\varphi_t(h)(s) = \left( 1 - \frac{t}{2} \right) \cdot h \left( \frac{s}{1 - t/2} \right) \quad \text{for } 0 \leq s \leq 1 - \frac{t}{2},
\]

\[ \varphi_t(h) \left( 1 - \frac{t}{4} \right) = 1 - \frac{(1 + m') \cdot t}{4}, \]
and

\[ \varphi_t(h) \text{ is linear on } \left[ 1 - \frac{t}{2}, 1 - \frac{t}{4} \right] \text{ and on } \left[ 1 - \frac{t}{4}, 1 \right], \]

which satisfies \( \varphi_0 = \text{id} \) and \( \text{Im}(\varphi_t) \cap L_m = \emptyset \) if \( t > 0 \). Since \( H' = \bigcup_{m \in \mathbb{N}} L_m \) and each \( L_m \) is a compact convex set in the Banach space \( C(I, \mathbb{R}) \), which contains an infinite-dimensional, \( \sigma \)-fd-compact, convex set \( H^{\text{PL}} \cap L_m \) as a dense subset, \( (L_m, H^{\text{PL}} \cap L_m) \cong (Q, \sigma) \) by [Do, Theorem 2(i)]. Thus the tower \( \{L_m\}_{m \in \mathbb{N}} \) satisfies the condition \((**)\) in [SW2]. Let \( \psi_n : H \to H^{\text{PL}}, n \in \mathbb{N}, \) be maps such that \( \psi_n(h)(\frac{i}{n}) = h(\frac{i}{n}) \) and \( \psi_n(h) \) is linear on each \( \left[ \frac{i-1}{n}, \frac{i}{n} \right] \). Then \( \psi_n \) converges to \( \text{id} \) as \( n \to \infty \). For each \( h \in H \), \( \text{bilip } \psi_n(h) = \max\{a, b^{-1}\} \), where
\[ a = \max\{n \cdot (\psi_n(h)(\frac{i}{n}) - \psi_n(h)(\frac{i-1}{n})) | i = 1, \ldots, n\}, \]
\[ b = \min\{n \cdot (\psi_n(h)(\frac{i}{n}) - \psi_n(h)(\frac{i-1}{n})) | i = 1, \ldots, n\}. \]

Hence \( \text{bilip } \psi_n(h) \leq \text{bilip } h \). It follows that \( H' \) is map dense in \( H \) and \( \{L_m\}_{m \in \mathbb{N}} \) satisfies the condition \((**')\) in [SW2]. Since \( H \cong l_2 \) by [An] [cf.
Ke], \((H, H', H'') \cong (l_2, l_2^Q, l_2^I)\) by [SW, Lemma 1.5 and Theorems 2.1 and 2.2].

By the arguments in [An], Theorem 1 follows from Theorem 3. Since [An] is unpublished, we give a sketch of the proof for the reader: Let \(A, B,\) and \(C\) be the sets of isolated points, of end points, and of branch points of \(X\), respectively. Let \(D\) be the set of components of \(X\) that are simple closed curve. Let \(E\) and \(F\) be the sets of maximal free open arcs in \(X \setminus \bigcup D\) such that the closure of each member of \(E\) is an arc and the closure of each member of \(F\) a simple closed curve. Let \(W\) be the finite set of all permutations of the union \(A \cup B \cup C \cup D \cup E \cup F\) onto itself which carries each \(A, B, C, D, E,\) and \(F\) onto itself and preserves incidence in \(X\), and let \(n(W), n(D), n(E),\) and \(n(F)\) denote the numbers of elements in \(W, D, E,\) and \(F\), respectively. Let \(T\) be a finite space of \(n(W) \cdot 2^{n(D)+n(F)}\) points, and let \(V\) be the product space of \(n(D)\) circles except where \(n(D) = 0\), in which case it is a single point. Then it is not hard to see that

\[ (H(X), H^{LIP}(X), H^{PL}(X)) \cong (H \times T \times V, H' \times T \times V, H'' \times T \times V), \]

which implies Theorem 1 by Theorem 3. \(\square\)

We prove Theorem 2 in a more general setting. To this end, we extend the piecewise linearity to maps from \(I\) to a metric space \(X = (X, d)\). A map \(f: [a, b] \to X\) is said to be linear if

\[ \frac{d(f(t), f(a))}{d(f(t), f(b))} = \frac{|t-a|}{|t-b|} \quad \text{for each } a < t < b, \]

and \(f: [a, b] \to X\) is piecewise linear (PL) if there is a sequence \(a = s_0 < s_1 < \cdots < s_n = b\) such that each \(f|_{[s_{i-1}, s_i]}\) is linear. The space of PL embeddings of \(I\) into a metric space \(X\) is also denoted by \(E^{PL}(I, X)\). In case \(X\) is a connected polyhedron in \(\mathbb{R}^n\), we adopt the arc-length metric \(d\) defined by using Euclidean metric. Then the piecewise linearity of \(f: I \to X\) defined above coincides with the usual sense. In this case, \(E^{LIP}(I, X)\) is not changed, since \(d\) is locally Lipschitz-equivalent to the Euclidean metric by [LV, Theorem 2.34]. Note that this metric \(d\) is convex; that is, for each \(x, y \in X\) there is some \(z \in X\) such that \(d(x, z) = d(y, z) = d(x, y)/2\). Let \(a(X)\) denote the hyperspace of arcs in \(X\) with the Vietoris topology (Hausdorff metric) and \(\text{Im}: E(I, X) \to a(X)\) the natural map defined by \(\text{Im}(h) = h(I)\). Theorem 2 is a corollary to the following:

**Theorem 4.** Let \(X\) be a locally compact 1-dim ANR with a metric \(d\) which is convex on a neighborhood of each point. Then \((E(I, X), E^{LIP}(I, X), E^{PL}(X, Y))\) is an \((l_2, l_2^Q, l_2^I)\)-manifold triple and the map \(\text{Im}: E(I, X) \to a(X)\) is a locally trivial bundle with fiber \(\mathbb{Z}_2 \times l_2\) and \(\text{Im}|E^{LIP}(I, X)\) and \(\text{Im}|E^{PL}(I, X)\) are subbundles with fibers \(\mathbb{Z}_2 \times l_2^Q\) and \(\mathbb{Z}_2 \times l_2^I\), respectively.
Before the proof, note that any connected locally compact 1-dim ANR has a convex metric. In fact, it has a Peano compactification with locally nonseparating remainder by [Cu] and any Peano continuum admits a convex metric by [Bi] or [Mo].

In Theorem 4, for each \( h \in E(I, X) \), there is a dendrite \((=\text{compact 1-dim AR})\) \( Y \) such that \( h(I) \subseteq \text{int} Y \), whence \( E(I, Y) \) is a neighborhood of \( h \) in \( E(I, X) \). The arc-length metric of \( Y \) defined by using the metric of \( X \) is convex and locally coincides with the metric of \( X \). Thus Theorem 4 reduces to the case in which \( X \) is a dendrite with a convex metric.

**Theorem 5.** For a dendrite \( X \) with a convex metric, \( (E(I, X), E^L(I, X), E^{PL}(I, X)) \) is an \((l_2, l^2_2, l^2_2)\)-manifold triple and the map \( \text{Im}: E(I, X) \to a(X) \) is a locally trivial bundle with fiber \( \mathbb{Z}_2 \times l_2 \) and \( \text{Im}\mid E^{L}(I, X) \) and \( \text{Im}\mid E^{PL}(I, X) \) are subbundles with fibers \( \mathbb{Z}_2 \times l^2_2 \) and \( \mathbb{Z}_2 \times l^2_2 \), respectively.

**Proof.** For simplicity, let \( E = E(I, X) \), \( E' = E^{L}(I, X) \) and \( E'' = E^{PL}(I, X) \) and let \( H, H' \) and \( H'' \) be as in the proof of Theorem 1.1. Let \( b(X) = X^2 \setminus \Delta X \), where \( \Delta X \) is the diagonal of \( X^2 \), and let \( \beta: E \to b(X) \) be the map defined by \( \beta(h) = (h(0), h(1)) \). We have the map \( \alpha: b(X) \to a(X) \) such that \( \alpha(x, y) \) is the unique arc in \( X \) connecting \( x \) and \( y \). Then \( \alpha \circ \beta = \text{Im}: E(I, X) \to a(X) \), and \( \alpha \) is a locally trivial bundle with fiber \( \mathbb{Z}_2 \). (Geometrically, \( b(X) \) can be considered as the space of oriented arcs in \( X \).) Hence it suffices to construct a homeomorphism

\[
\varphi: (E, E', E'') \to (b(X) \times H, b(X) \times H', b(X) \times H'')
\]

so that \( p \circ \varphi = \beta \), where \( p: b(X) \times H \to b(X) \) is the projection. Then \( (E, E', E'') \) is an \((l_2, l^2_2, l^2_2)\)-manifold triple by Theorem 3 and the result of [SW2]. From the uniquely arcwise connectedness of \( X \), there exists a map \( \lambda: X^2 \times I \to X \) such that

\[
d(x, \lambda(x, y, t)) = t \cdot d(x, y) \quad \text{and} \quad d(y, \lambda(x, y, t)) = (1 - t) \cdot d(x, y)
\]

for each \( x, y \in X \) and \( t \in I \). We define the map \( \tau: b(X) \to E \) by \( \tau(x, y)(t) = \lambda(x, y, t) \). As is easily observed, \( \tau(b(X)) \subset E'' \). From the uniquely arcwise connectedness of \( X \), \( \tau \circ \beta(h)(I) = h(I) \) for each \( h \in E \). Then the desired homeomorphism \( \varphi \) and its inverse are defined by

\[
\begin{align*}
\varphi(h) & = (\beta(h), (\tau \circ \beta(h))^{-1} \circ h) \quad \text{and} \quad \varphi^{-1}(x, y, g) = \tau(x, y) \circ g.
\end{align*}
\]

**Example.** If \( X \) contains a two-disk, both \( \text{Im}: E(I, X) \to a(X) \) and \( \text{Im}: E^{L}(I, X) \to a(X) \) are not locally trivial bundles. To show this, let \( g: I^2 \to X \) be an embedding and \( A_0 = g(I \times \{0\}) \). For each \( n \in \mathbb{N} \), let

\[
A_n = g \left( I \times \left\{ \frac{1}{2n}, \frac{1}{2n - 1} \right\} \cup \{1\} \times \left[ \frac{1}{2n}, \frac{1}{2n - 1} \right] \right) \subseteq a(X).
\]

Then \( A_n \) converges to \( A_0 \) in \( a(X) \). However, any \( h_n \in \text{Im}^{-1}(A_n) \) \( (n \in \mathbb{N}) \) does not converge to any \( h \in \text{Im}^{-1}(A_0) \), because \( \{h_n(0), h_n(1)\} \) converges to
AN EMBEDDING SPACE TRIPLE OF THE UNIT INTERVAL

\{g(0, 0)\}, but \{h(0), h(1)\} = \{g(0, 0), g(1, 0)\}. In case \(X\) is a polyhedron with \(\dim X > 1\), \(\text{Im}: E^\text{PL}(I, X) \rightarrow a^\text{Pol}(X)\) is not a locally trivial bundle, where \(a^\text{Pol}(X)\) is the subspace of \(a(X)\) consisting of polyhedral arcs.

**Problem.** Let \(X\) and \(Y\) be Euclidean polyhedra such that \(X\) is compact and \(E(X, Y) \neq \emptyset\). Is \((E(X, Y), E^\text{LIP}(X, Y), E^\text{PL}(X, Y))\) an \((l_2, l_2^Q, l_2^I)\)-manifold triple? Is each space of this triple an ANR? If \(X = I\) and \(\dim Y > 1\), is it then an ANR?

**Remark.** In Theorem 5, \(\text{Im}: E(I, X) \rightarrow a(X)\) is nontrivial in the case \(X \neq I\). In fact, \(\text{Im} = \alpha \circ \beta\) and \(\beta\) is trivial as shown in the proof, but \(\alpha\) is nontrivial since \(b(X) = X^2 \setminus \Delta X\) is connected in this case. It should be noted that \(a(X)\) is not contractible in general. For example, in the case \(X\) is the simple triod, \(a(X)\) has the homotopy type of \(S^1\).

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**References**


