THE GOTTLIEB GROUP OF FINITE LINEAR QUOTIENTS
OF ODD DIMENSIONAL SPHERES

S. ALLEN BROUGHTON

(Communicated by Frederick R. Cohen)

Abstract. Let $G$ be a finite, freely acting group of homeomorphisms of the odd-dimensional sphere $S^{2n-1}$. John Oprea has proven that the Gottlieb group of $S^{2n-1}/G$ equals $Z(G)$, the centre of $G$. The purpose of this short paper is to give a considerably shorter, more geometric proof of Oprea's theorem in the important case where $G$ is a linear group.

In [G1], [G2] Gottlieb introduced subgroups $G_n(X) \subseteq \pi_n(X)$ of the homotopy groups of a connected space $X$, which have come to be known as the Gottlieb groups of $X$. The group $G_1(X)$ is usually referred to as the Gottlieb group of $X$ and has been extensively studied (cf., e.g., [Ga, L, P]). In [G1], Gottlieb provided the following characterization of $G_1(X)$. Let $\tilde{X}$ be the universal cover of $X$ and identify $\Pi \cong \pi_1(X)$ with the group of covering transformations in the usual way. Then, we have

Proposition 1. The Gottlieb group, $G_1(X)$, is the subgroup of $\Pi$ consisting of all $z$ which are equivariantly homotopic to the identity.

Remark. To show that $z$ is in $G_1(X)$ we must find a homotopy $L_t : \tilde{X} \to \tilde{X}$, $0 \leq t \leq 1$, such that

\begin{align*}
    L_0 &= \text{id}, \\
    L_1 &= z
\end{align*}

and

\begin{align*}
    L_t g &= g L_t, \quad \text{for all} \ g \in \Pi \ \text{and for} \ 0 \leq t \leq 1.
\end{align*}

From this characterization it is clear that the Gottlieb group is a characteristic subgroup, lying in the centre of $\Pi$. In this paper we prove the following theorem.

Theorem. Let the finite group $G$ act freely and linearly on the odd-dimensional sphere $S^{2n-1}$. Then, the Gottlieb group of the quotient space $S^{2n-1}/G$ is isomorphic to $Z(G)$, the centre of $G$, under the canonical isomorphism of $G$ with $\pi_1(S^{2n-1}/G)$.

Received by the editors January 8, 1990.

1980 Mathematics Subject Classification (1985 Revision). Primary 55Q52, 57S17, 57S25.

Key words and phrases. Gottlieb group, linear group actions.

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0002-9939/91 $1.00 + .25 per page
In [O], J. Oprea has proven a more general theorem in which it is only assumed that \( G \) is a finite, freely acting group of homeomorphisms of \( S^{2n-1} \). The methods in [O] use rather complicated algebro-topological arguments because of the general nature of the action. The sole purpose of this paper is to show that if we restrict our attention to linear actions, a much simpler, geometric proof is possible. Before proceeding to the proof, we recall the following fact about linear representations of a finite group on a real vector space. For all the results on linear representations that we use, we refer to Isaacs's monograph [I1].

**Proposition 2.** Let the finite group \( G \) act linearly and irreducibly, over \( \mathbb{R} \), on the real vector space \( V \). Let \( \text{Cent}_G(V) \) denote the commuting algebra of the \( G \)-action on \( V \), i.e., the algebra of endomorphisms of \( V \) which commute with \( G \). Then, \( \text{Cent}_G(V) \) is a division algebra isomorphic to one of \( \mathbb{R} \), \( \mathbb{C} \), or \( \mathbb{H} \) (quaternions). Furthermore, \( \text{Cent}_G(V) \cong \mathbb{R} \) if and only if the representation of \( G \) on \( V \) is absolutely irreducible, i.e., the complex representation of \( G \) on \( C \otimes V \) is irreducible.

**Proof of theorem.** Suppose that the free action of \( G \) on the odd-dimensional sphere \( S^{2n-1} \) is induced by an orthogonal linear representation on the \( 2n \)-dimensional real vector space \( V \). In the notation of Proposition 1, we take \( X = S^{2n-1}/G \), \( \bar{X} = S^{2n-1} \), and \( \Pi = G \). To prove the theorem we must produce, for each \( z \in Z(G) \), a homotopy \( L_t \) satisfying (1) and (2) above. If \( G \) consists of the identity map and the antipodal map, then a homotopy from the identity map to the antipodal map may be constructed by letting \( L_t \) be a path from \( I \) to \( -I \) in the unitary matrix group \( U(n) \), which is path-connected. We may now assume that \( G \) is not this group.

Let \( g \in G \) have order \( n \), the eigenvalues of \( g \) are all primitive \( n \)th roots of unity, otherwise some nonidentity power of \( g \) fixes a point. From this condition on the eigenvalues we get

\( (3) \) The only possible involution in \( G \) is the antipodal map: \( v \mapsto -v \).

From the freeness of the \( G \)-action we also get

\( (4) \) If \( W \subseteq V \) is a nonzero \( G \)-invariant subspace, then the restricted representation of \( G \) on \( W \) is faithful.

Write \( V = V_1 \oplus \cdots \oplus V_s \), an orthogonal direct sum of irreducible \( G \)-invariant subspaces over \( \mathbb{R} \). Suppose that the commuting algebra of one or more of the \( V_i \)'s is \( \mathbb{R} \). Then, by (3), (4), and Proposition 2, \( G \) is a group with only one involution and an absolutely irreducible faithful real representation. First, J. Malzan [M] and then M. Isaacs [12] (using simpler methods) have shown that such a group has order 2. In this case, \( G \) would consist of the identity map and the antipodal map, but we have already handled this possibility.

Let \( z \in Z(G) \). For \( 1 \leq i \leq s \), let \( L_t^i \) be a path in \( \text{Cent}_G(V_i) \) starting at the identity and ending at \( z|_{V_i} \). Since we may assume that all of the centralizers \( \text{Cent}_G(V_i) \) are division algebras of real dimension at least two, we may
construct the paths so that for each $t$ and $i$ the map $L_i^t$ is an invertible linear transformation of $V_i$. Define the homotopy $L_t$ by

$$L_t(v) = M_t(v) / |M_t(v)|, \quad \text{for } v \in S^{2n-1},$$

where

$$M_t(v) = L_i^1(v_1) + \cdots + L_i^s(v_s),$$

$||$ denotes the norm in $V$, and $v = v_1 + \cdots + v_s$ is the decomposition of $v \in V$ induced by the decomposition of $V$ into $G$-invariant subspaces. By construction, each $M_t$ is invertible, so $L_t$ is well-defined. Since the $V_i$'s are $G$-invariant, then the equivariance condition (2) holds with $L_t$ replaced by $M_t$. Since the norm is $G$-invariant and for each $g \in G$, $gM_t(v) = M_t(gv)$, then $|M_t(v)| = |gM_t(v)| = |M_t(gv)|$. It now follows that (1) and (2) hold for all $L_t$. All is now proven.

REFERENCES


Department of Mathematics, Cleveland State University, Cleveland, Ohio 44115

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