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THE GOTTLIEB GROUP OF FINITE LINEAR QUOTIENTS OF ODD DIMENSIONAL SPHERES

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ABSTRACT. Let G be a finite, freely acting group of homeomorphisms of the odd-dimensional sphere S^{2n-1}. John Oprea has proven that the Gottlieb group of S^{2n-1}/G equals $Z(G)$, the centre of G. The purpose of this short paper is to give a considerably shorter, more geometric proof of Oprea's theorem in the important case where G is a linear group.

In [G1], [G2] Gottlieb introduced subgroups $G_n(X) \subseteq \pi_n(X)$ of the homotopy groups of a connected space X, which have come to be known as the Gottlieb groups of X. The group $G_1(X)$ is usually referred to as the Gottlieb group of X and has been extensively studied (cf., e.g., [Ga, L, P]). In [G1], Gottlieb provided the following characterization of $G_1(X)$. Let $\widetilde{X}$ be the universal cover of X and identify $\Pi \simeq \pi_1(X)$ with the group of covering transformations in the usual way. Then, we have

Proposition 1. *The Gottlieb group, $G_1(X)$, is the subgroup of Π consisting of all z which are equivariantly homotopic to the identity.*

Remark. To show that z is in $G_1(X)$ we must find a homotopy $L_t\colon \widetilde{X} \to \widetilde{X}$, $0 \le t \le 1$, such that

$$L_0 = id, \qquad L_1 = z \tag{1}$$

and

$$L_t g = g L_t, \quad \text{for all } g \in \Pi \text{ and for } 0 \le t \le 1. \tag{2}$$

From this characterization it is clear that the Gottlieb group is a characteristic subgroup, lying in the centre of Π. In this paper we prove the following theorem.

Theorem. *Let the finite group G act freely and linearly on the odd-dimensional sphere S^{2n-1}. Then, the Gottlieb group of the quotient space S^{2n-1}/G is isomorphic to $Z(G)$, the centre of G, under the canonical isomorphism of G with $\pi_1(S^{2n-1}/G)$.*

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In [O], J. Oprea has proven a more general theorem in which it is only assumed that G is a finite, freely acting group of homeomorphisms of S^{2n-1}. The methods in [O] use rather complicated algebro-topological arguments because of the general nature of the action. The sole purpose of this paper is to show that if we restrict our attention to linear actions, a much simpler, geometric proof is possible. Before proceeding to the proof, we recall the following fact about linear representations of a finite group on a real vector space. For all the results on linear representations that we use, we refer to Isaacs's monograph [I1].

Proposition 2. *Let the finite group G act linearly and irreducibly, over $\mathbf{R}$, on the real vector space V. Let $\mathrm{Cent}_G(V)$ denote the commuting algebra of the G-action on V, i.e., the algebra of endomorphisms of V which commute with G. Then, $\mathrm{Cent}_G(V)$ is a division algebra isomorphic to one of $\mathbf{R}$, $\mathbf{C}$, or $\mathbf{H}$ (quaternions). Furthermore, $\mathrm{Cent}_G(V) \simeq \mathbf{R}$ if and only if the representation of G on V is absolutely irreducible, i.e., the complex representation of G on $\mathbf{C} \otimes V$ is irreducible.*

Proof of theorem. Suppose that the free action of G on the odd-dimensional sphere S^{2n-1} is induced by an orthogonal linear representation on the $2n$-dimensional real vector space V. In the notation of Proposition 1, we take $X = S^{2n-1}/G$, $\widetilde{X} = S^{2n-1}$, and $\Pi = G$. To prove the theorem we must produce, for each $z \in Z(G)$, a homotopy L_t satisfying (1) and (2) above. If G consists of the identity map and the antipodal map, then a homotopy from the identity map to the antipodal map may be constructed by letting L_t be a path from I to $-I$ in the unitary matrix group $U(n)$, which is path-connected. We may now assume that G is not this group.

Let $g \in G$ have order n, the eigenvalues of g are all primitive nth roots of unity, otherwise some nonidentity power of g fixes a point. From this condition on the eigenvalues we get

(3) The only possible involution in G is the antipodal map: $v \to -v$.

From the freeness of the G-action we also get

(4) If $W \subseteq V$ is a nonzero G-invariant subspace, then the restricted representation of G on W is faithful.

Write $V = V_1 \oplus \cdots \oplus V_s$, an orthogonal direct sum of irreducible G-invariant subspaces over $\mathbf{R}$. Suppose that the commuting algebra of one or more of the V_i's is $\mathbf{R}$. Then, by (3), (4), and Proposition 2, G is a group with only one involution and an absolutely irreducible faithful real representation. First, J. Malzan [M] and then M. Isaacs [I2] (using simpler methods) have shown that such a group has order 2. In this case, G would consist of the identity map and the antipodal map, but we have already handled this possibility.

Let $z \in Z(G)$. For $1 \le i \le s$, let L_t^i be a path in $\mathrm{Cent}_G(V_i)$ starting at the identity and ending at $z|_{V_i}$. Since we may assume that all of the centralizers $\mathrm{Cent}_G(V_i)$ are division algebras of real dimension at least two, we may

construct the paths so that for each t and i the map L_t^i is an invertible linear transformation of V_i. Define the homotopy L_t by

$$L_t(v) = M_t(v)/|M_t(v)|, \quad \text{for } v \in S^{2n-1},$$

where

$$M_t(v) = L_t^1(v_1) + \cdots + L_t^s(v_s),$$

$|\ |$ denotes the norm in V, and $v = v_1 + \cdots + v_s$ is the decomposition of $v \in V$ induced by the decomposition of V into G-invariant subspaces. By construction, each M_t is invertible, so L_t is well-defined. Since the V_i's are G-invariant, then the equivariance condition (2) holds with L_t replaced by M_t. Since the norm is G-invariant and for each $g \in G$, $gM_t(v) = M_t(gv)$, then $|M_t(v)| = |gM_t(v)| = |M_t(gv)|$. It now follows that (1) and (2) hold for all L_t. All is now proven.

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