AMENABILITY OF LOCALLY COMPACT GROUPS
AND SUBSPACES OF $L^\infty(G)$

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(Communicated by J. Marshall Ash)

Abstract. If $G$ is a locally compact group, let $\mathcal{A}$ be the set of all the functions which left average to a constant, i.e. the function $f \in L^\infty(G)$ such that there is a constant in the $\| \cdot \|_\infty$-closed convex hull of $\{x f : x \in G\}$. We prove in this paper that $\mathcal{A}$ is a subspace of $L^\infty(G)$ if and only if $G$ is amenable as a discrete group. This answers a problem asked by Emerson, Rosenblatt and Yang, and Wong and Riazi. We also answer two other problems of Rosenblatt and Yang on whether the set $\mathcal{Z}$ of functions in $L^\infty(G)$ admitting a unique left invariant mean value is a subspace of $L^\infty(G)$ when $G$ is not amenable and whether there is a largest admissible subspace of $L^\infty(G)$ with a unique left invariant mean.

1. Introduction and notation

Let $G$ be a locally compact group with a fixed Haar measure $\lambda$ and let $L^p(G)$ be the associated real Lebesgue spaces $(1 \leq p \leq \infty)$. For each $f \in L^\infty(G)$ and $x \in G$, let $x f \in L^\infty(G)$ be defined by $x f(y) = f(xy)$ $(y \in G)$. A subspace $S$ of $L^\infty(G)$ is said to be admissible if it contains the constants and $x f$ for each $f \in S$ and $x \in G$. Let $LIM(S)$ be the set of all the left invariant means on $S$, i.e. all $m \in S^*$ with $m \geq 0$, $m(1) = 1$, and $m(x f) = m(f)$ $(x \in G$, $f \in S)$. If $LIM(L^\infty(G)) \neq \emptyset$, we say that $G$ is amenable. It is well known that, for any locally compact group $G$, $G$ is amenable if $G_d$ is amenable, where $G_d$ is the group $G$ with discrete topology. The converse is not always true. The purpose of this paper is to give a characterization of a locally compact group which is amenable as a discrete group and criteria for the amenability of a locally compact group and a discrete group in terms of the subspaces of $L^\infty(G)$. We also answer three of the problems raised by Rosenblatt and Yang in [7].

For $f \in L^\infty(G)$ and a constant $c$, we say that $f$ left averages to $c$ if $c \in \| \cdot \|_\infty$-closed convex of $\{x f : x \in G\}$. Let $\mathcal{A}$ denote the set of all functions...
which left average to some constant and \( \mathcal{A}_0 \) denote the set of functions which left average to 0. If \( f \in L^\infty(G) \), let \( S_f \) be the smallest admissible subspace generated by \( f \). \( f \in L^\infty(G) \) is said to have a **unique left invariant mean value** if \( LIM(S_f) \neq \phi \) and there is a constant \( c \) such that for each \( m \in LIM(S_f) \), \( m(f) = c \). See [7] for more details. The set of all functions with a unique left invariant mean value is denoted by \( \mathcal{U} \) and let \( H \) be the linear span of \( \{ f - \bar{f} : f \in L^\infty(G), \ x \in G \} \).

In §2, we use a theorem of Chou in [1] to prove that \( \mathcal{A}_0 \) is a subspace if and only if \( G_d \) is amenable. We also show that \( \mathcal{A}_0 \) is a subspace if and only if \( \mathcal{A} \) is a subspace. This answers a problem asked by Emerson in [3, p. 187], Wong and Riazi in [10, p. 494] and Rosenblatt and Yang in [7].

In §3, we prove that \( \mathcal{U} \) is not a subspace for any nonamenable group. This answers a problem asked by Rosenblatt and Yang in [7]. Consequently, we show that \( G \) is amenable if and only if \( \mathcal{U} \) is a subspace for a discrete group. We also settle a problem asked by Rosenblatt and Yang in [7] on whether there is a largest admissible subspace with a unique left invariant mean. We show that there is such a subspace if and only if \( G \) is amenable.

### 2. Locally compact groups which are amenable as discrete groups

Let \( \mathcal{D} \) be the maximal ideal space of \( L^\infty(G) \) with the Gelfand topology. Then \( \mathcal{D} \) is a compact Hausdorff space. The Gelfand transform \( \theta \) is an isometry of \( L^\infty(G) \) onto \( C(\mathcal{D}) \), the algebra of real-valued continuous functions on \( \mathcal{D} \) with the supremum norm. Note that if \( \theta \in \mathcal{D}, \ x \theta \) is defined by \( x\theta(f) = \theta(xf) \) for \( f \in L^\infty(G) \) and \( x \in G \).

For \( \theta \in \mathcal{D} \), let \( \rho_\theta : L^\infty(G) \to \ell^\infty(G) \) be defined by \( (\rho_\theta f)(x) = \theta(xf), x \in G \). Here \( \ell^\infty(G) = L^\infty(G_d) \). Note that \( \rho_\theta \) is a unitary operator, \( \rho_\theta \) is a subspace of \( \ell^\infty(G) \) (see Chou [1]).

In [1], Chou proved the following:

**Theorem** (Chou). For a \( \sigma \)-compact locally compact group \( G \), \( G_d \) is amenable if and only if for each \( \theta \in \mathcal{D} \), the subspace \( \{ \rho_\theta(f) : f \in L^\infty(G) \} \) of \( \ell^\infty(G) \) has a left invariant mean.

Our proof of Theorem 2.3 depends on this theorem of Chou. We first establish some lemmas.

**Lemma 2.1.** For any locally compact group \( G \), the following statements are equivalent:

1. \( \mathcal{A} \) is a subspace;
2. \( \mathcal{A}_0 \) is a subspace;
3. \( \mathcal{A}_0 = H \).

**Proof.** (i) \( \Rightarrow \) (ii). If \( \mathcal{A} \) is a subspace, let \( f_1, f_2 \in \mathcal{A}_0 \subset \mathcal{A} \). There is a number \( c \) such that \( f_1 + f_1 \) left averages to \( c \). By Theorem 1.5 of [7], \( G \) is amenable. Let \( m \in LIM(L^\infty(G)) \), then \( m(f_1) = m(f_2) = 0 \) since \( f_1, f_2 \in \mathcal{A}_0 \). Therefore \( c = m(f_1 + f_2) = 0 \), i.e. \( f_1 + f_2 \in \mathcal{A}_0 \).
(ii) ⇒ (iii). For any \( x \in G \) and \( f \in L^\infty(G) \), since
\[
\lim_{n} \frac{1}{n} \sum_{k=1}^{n} x^k(f - xf) = 0
\]
f - xf \in \mathcal{A}_0. \] Hence if \( \mathcal{A}_0 \) is a subspace, then \( H \subseteq \mathcal{A}_0 \). It is easy to see that \( \mathcal{A}_0 \) is closed. So \( H \subseteq \mathcal{A}_0 \). Let \( f \notin H \), then there is \( m \in \text{LIM } (L^\infty(G)) \) with \( m(f) \neq 0 \). So \( f \notin \mathcal{A}_0 \). (See [5, (B37) and p. 236], i.e. \( \mathcal{A}_0 \subseteq H \).

(iii) ⇒ (ii) is clear. To prove (ii) ⇒ (i), let \( f_i \) left average to \( c_i \). Then \( f_i - c_i \) left averages to 0 \((i = 1, 2)\). If \( \mathcal{A}_0 \) is a subspace, then \( (f_1 - c_1) + (f_2 - c_2) \in \mathcal{A}_0 \), i.e. \( f_1 + f_2 \) left averages to \( c_1 + c_2 \).

Theorem 2.2. For any locally compact group \( G \) and \( \theta \in \mathcal{D} \), \( S_\theta = \{ \rho_\theta(f) : f \in L^\infty(G) \text{ is a simple function} \} \) is a subspace of \( L^\infty(G) \) with the following properties:

(i) \( |F| \in S_\theta \) for any \( F \in S_\theta \)

(ii) If \( m \in S^*_\theta \) is left invariant, then there are nonnegative left invariant \( m^+, m^- \in S^*_\theta \) such that \( m = m^+ - m^- \) where \( m^+ = \max(m, 0) \), \( m^- = -\min(m, 0) \) (see [5] (B34)).

Proof. (i) If \( f = \sum_{i=1}^{n} a_i 1_{E_i} \) is a measurable simple function, then \( |\rho_\theta(f)| = \rho_\theta(|f|) \) i.e. \( |\rho_\theta(f)| \in S_\theta \). Indeed, for any \( x \in G \), since \( \{ x^{-1} E_i : i = 1, 2, \ldots, n \} \) are pairwise disjoint, there is at most one \( i \) with \( \theta(x^{-1} E_i) \neq 0 \). Hence
\[
|\rho_\theta(f)(x)| = \theta \left( \sum_{i=1}^{n} |a_i| 1_{E_i} \right) = \sum_{i=1}^{n} |a_i| \theta(x^{-1} E_i) = \theta(\sum_{i=1}^{n} |a_i| 1_{E_i}) = \theta(|f|(x)).
\]

(ii) Since \( m \in S^*_\theta \) is bounded, there are nonnegative \( m^+, m^- \in S^*_\theta \) such that \( m = m^+ - m^- \), \( m^+ = \max(m, 0) \) and \( m^- = -\min(m, 0) \) by (i) and (B.37) of [5].

Let \( F_0 \in S_\theta \) and \( F_0 \geq 0 \), since
\[
m^+(F_0) = \sup\{ m(F) : 0 \leq F \leq F_0, \ F \in S_\theta \}
\]
if \( x \in G \) and \( 0 \leq F \leq F_0 \), then \( 0 \leq x F \leq x F_0 \) and \( m(x F) = m(F) \). Hence \( m^+(F_0) = m^+(x F_0) \). Since for any \( F \in S_\theta \), there are \( F^+, F^- \in S_\theta \) such that \( F^+ \geq 0 \), \( F^- \geq 0 \) and \( F = F^+-F^- \), \( m^+ \) is left invariant. Similarly, we can prove that \( m^- \) is left invariant.

We are now ready to answer a question raised by Rosenblatt and Yang in [7] (Remark 3 after Theorem 1.5).
Theorem 2.3. For any locally compact group \(G\), \(\mathcal{A}\) is a subspace if and only if \(G_d\) is amenable.

Proof. If \(G_d\) is amenable, then \(\mathcal{A}\) is a subspace (see [7, Theorem 1.1]). Let \(\mathcal{A}\) be a subspace. By Lemma 2.1, \(\mathcal{A}_0 = H\). Since any countable subgroup of \(G\) is contained in a \(\sigma\)-compact, open and closed subgroup of \(G\) (see [6, Proposition 22.24]), it suffices to show that, for any \(\sigma\)-compact, open and closed subgroup \(G_0\) of \(G\), \((G_0)_d\) is amenable by \((D)\) and \((F)\) of [2, p. 516]. By Chou's theorem above, it suffices to show that the subspace \(L^\infty(G_0)\) has a left invariant mean for each \(\theta_0 \in \mathcal{D}_0\), where \(\mathcal{D}_0\) is the maximal ideal space of \(L^\infty(G_0)\).

Let \(\{x_\alpha G_0 : \alpha \in \Lambda\}\) be the set of all the left cosets of \(G_0\) in \(G\). For each \(f \in L^\infty(G_0)\), let \(f\) be defined by \(f(x_\alpha x) = f(x)\) for each \(x \in G_0, \alpha \in \Lambda\). Since \(G = \bigcup_{\alpha \in \Lambda} x_\alpha G_0\), \(f\) is a function on \(G\).

Claim. \(f \in L^\infty(G)\). Let \(c \in \mathbb{C}\) and \(K\) be any compact subset of \(G\). Since \(G_0\) is open, there are only finite many \(\alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda\) such that \(K \cap x_\alpha G_0 \neq \emptyset\) \((i = 1, 2, \ldots, n)\). Hence

\[
\{x \in G : f > c\} \cap K = \bigcup_{i=1}^n K \cap x_{\alpha_i} G_0 \cap \{x : f(x) > c\}
\]

is \(\lambda\)-measurable since \(f\) is measurable on \(G_0\). Therefore \(f \in L^\infty(G)\) (see [5, (11.31) and (15.8)]).

Let \(\theta_0 \in \mathcal{D}_0\) be given and \(\theta \in \mathcal{D}\) be defined by \(\theta(f) = \theta_0(f | G_0)\) for any \(f \in L^\infty(G)\). We first show that there is a left invariant mean \(m_\theta\) on \(S_\theta\) where \(S_\theta = \{\rho_\theta(f) : f \in L^\infty(G)\}\) is a simple function \}. Let

\[
H_\theta = \text{span}\{x F - F : x \in G, F \in S_\theta\} \quad \text{and} \quad h_\theta \in H_\theta.
\]

Then there is a \(h \in H\) such that \(h_\theta = \rho_\theta(h)\).

Claim. \(\|1 - h_\theta\|_\infty \geq 1\). Suppose \(\varepsilon > 0\) with \(\|1 - h_\theta\|_\infty < 1 - \varepsilon\). Since \(h \in H \subseteq \mathcal{A}_0\), there are \(\lambda_i > 0\) and \(x_i \in G\) \((i = 1, 2, \ldots, n)\) such that \(\sum_{i=1}^n \lambda_i = 1\) and \(\|\sum_{i=1}^n \lambda_i x_i h\|_\infty < \varepsilon/2\). Then

\[
\left\| \sum_{i=1}^n \lambda_i x_i \rho_\theta(h) \right\|_\infty = \left\| \rho_\theta \left( \sum_{i=1}^n \lambda_i x_i h \right) \right\|_\infty \leq \|\rho_\theta\| \left\| \sum_{i=1}^n \lambda_i x_i h \right\|_\infty \leq \varepsilon/2.
\]

It follows that

\[
\|1 - h_\theta\|_\infty \geq \left\| \sum_{i=1}^n \lambda_i x_i (1 - h_\theta) \right\|_\infty = \left\| 1 - \sum_{i=1}^n \lambda_i x_i \rho_\theta(h) \right\|_\infty \geq 1 - \varepsilon/2
\]

which is impossible. Hence \(\|1 - h_\theta\|_\infty \geq 1\) for any \(h_\theta \in H_\theta\). Now we can find \(M \in S^*_\theta\) such that \(M(1) = 1\) and \(M(h_\theta) = 0\) for any \(h_\theta \in H_\theta\), i.e. \(M\) is left
invariant. By Theorem 2.2, $M^+$ is a nonnegative left invariant functional on $S_\theta$. Put $m_\theta = M^+/M^+(1)$, then $m_\theta$ is a left invariant mean on $S_\theta$.

Let $S_{\theta_0} = \{\rho_{\theta_0}(f) : f \in L^\infty(G_0) \text{ is a simple function}\}$. By using $m_\theta$, we can define a left invariant mean on $S_{\theta_0}$ as follows. For each $\rho_{\theta_0}(f) \in S_{\theta_0}$, define

$$m_{\theta_0}(\rho_{\theta_0}(f)) = m_\theta(\rho_{\theta_0}(f)).$$

Then $m_{\theta_0}$ is well defined. Indeed, for $f_1, f_2 \in L^\infty(G_0)$ and $f_i$ is a simple function $(i = 1, 2)$. Let $\rho_{\theta_0}(f_1) = \rho_{\theta_0}(f_2)$ i.e. $\rho_{\theta_0}(f) = 0$ where $f = f_1 - f_2$. Let $f = \sum_{i=1}^n a_i 1_{E_i}$. Note that for each $x \in G$, there is at most one $i$ such that $\rho_{\theta_0}(1_{E_i})(x) \neq 0$. Hence we have $\rho_{\theta_0}(1_{E_i})(x) = 0$ for $i = 1, 2, \ldots, n$ and $x \in G_0$. Let $1 \leq i \leq n$ be given,

$$m_{\theta_0}(\rho_{\theta_0}(f_1)) - m_{\theta_0}(\rho_{\theta_0}(f_2)) = m_\theta(\rho_{\theta}(f_1)) - m_\theta(\rho_{\theta}(f_2)) = m_\theta(\rho_{\theta}(f)) = 0$$

so $m_{\theta_0}(\rho_{\theta_0}(f_1)) = m_{\theta_0}(\rho_{\theta_0}(f_2))$. It is clear that $m_{\theta_0}$ is linear. To see that $m_{\theta_0}$ is left invariant, it suffices to show that for any $x \in G_0$ and measurable set $E$ of $G_0$,

$$(*) \quad x\rho_{\theta}(1_{E}) = \rho_{\theta}(x1_{E}).$$

Indeed, $(*)$ implies that

$$m_{\theta_0}(x\rho_{\theta_0}(1_{E})) = m_{\theta_0}(\rho_{\theta_0}(x1_{E})) = m_{\theta_0}(\rho_{\theta}(x1_{E})) = m_{\theta}(x\rho_{\theta}(1_{E})) = m_{\theta}(\rho_{\theta}(1_{E})) = m_{\theta_0}(\rho_{\theta_0}(1_{E}))$$

for any $x \in G_0$ and measurable subset $E$ of $G_0$. Hence $m_{\theta_0}(x\rho_{\theta_0}(f)) = m_{\theta_0}(\rho_{\theta_0}(f))$ for any $x \in G_0$ and simple function $f \in L^\infty(G_0)$. To prove $(*)$, note that

$$\rho_{\theta}(x1_{E}) = \rho_{\theta}(1_{\cup_{a \in A}Ax^{-1}E}) = \theta_0(1_{\cup_{a \in A}Ax^{-1}E} | G_0) = \theta_0(1_{x^{-1}E}),$$

and

$$x\rho_{\theta}(1_{E}) = \rho_{\theta}(x1_{\cup_{a \in A}Ax^{-1}E}) = \theta_0(1_{\cup_{a \in A}Ax^{-1}E} | G_0) = \theta_0(1_{x^{-1}E}).$$

Also $m_{\theta_0}$ is nonnegative. Indeed, let $f = \sum_{i=1}^n a_i 1_{E_i} \in L^\infty(G_0)$ be a simple function and $\rho_{\theta_0}(f) \geq 0$. Suppose that for each $i_0$ there exist $x \in G_0$ such
that $\theta_0(x_1E_i) \neq 0$ (otherwise take $a_{i_0} = 0$). Since $x_0^{-1}E_i \cap x_0^{-1}E_j = \emptyset$ ($i \neq j$), $\theta_0(x_1E_i) = 0$ if $j \neq i_0$. Hence

$$\rho_{\theta_0}(f)(x) = \sum_{i=1}^{n} \rho_{\theta_0}(a_{ix_1E_i}) = a_{i_0} \geq 0.$$ 

For each $i$, note that $\rho_{\theta}(1_{E_i}) = \rho_{\theta}(1_{\cup_{x \in A_i} x_{E_i}}) \geq 0$, so $m_{\theta}(\rho_{\theta}(1_{E_i})) \geq 0$ ($i = 1, 2, \ldots, n$). Therefore

$$m_{\theta_0}(\rho_{\theta_0}(f)) = \sum_{i=1}^{n} a_{i} m_{\theta_0}(\rho_{\theta_0}(1_{E_i})) = \sum_{i=1}^{n} a_{i} m_{\theta}(\rho_{\theta}(1_{E_i})) \geq 0.$$ 

It is clear that $m_{\theta_0}(1) = 1$. Hence $m_{\theta_0}$ is a left invariant mean on $S_{\theta_0}$.

Since $S_{\theta_0}$ is dense in \{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}, we can extend $m_{\theta_0}$ to \{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\} such that $m_{\theta_0}$ is a bounded functional. It is easy to see that $m_{\theta_0}$ is nonnegative and left invariant, i.e. $m_{\theta_0}$ is a left invariant mean on \{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\}. \quad \Box$

Let $X$ be a left invariant subspace of $L^\infty(G)$ and $D$ denote the set of all finite convex combinations of Dirac measures. As in [10], we denote the set \{\rho_{\theta_0}(f) : f \in L^\infty(G_0)\} by $\mathcal{N}_1(X)$ (see [10, pp. 480, 491]). It is clear that $\mathcal{N}_1(X) = X_{1\theta_0}$. See [5, 20.9].

**Corollary 2.4.** $\mathcal{N}_1(L^\infty(G))$ is closed under addition if and only if $G_d$ is amenable.

**Proof.** Since $\mathcal{N}_1(L^\infty(G)) = \mathcal{A}_0$, the Corollary is true by Lemma 2.1 and Theorem 2.3. \quad \Box

**Remark 1.** Wong and Riazi in [10, p. 493] proved that for any locally compact group $G$, $G$ is amenable if and only if $\mathcal{N}_1(UCB(G))$ is closed under addition where $UCB(G)$ is the set of all the uniformly continuous functions on $G$. Since there are amenable locally compact groups which are not amenable as discrete groups, we have answered the problem raised by Wong and Riazi in [10, p. 494 (Remark 3)].

**Remark 2.** Let $P(G) = \{\varphi \in L^1(G) : \varphi \geq 0 \text{ and } \|\varphi\|_1 = 1\}$. Emerson proved in [3] that $G$ is amenable if and only if \{\rho \in L^\infty(G) : \inf\{\|\rho * f\|_\infty : \rho \in P\} = 0\} is closed under addition. This corollary provides an answer to his problem in replacing $P(G)$ by $D$ [3, p. 187].

**Corollary 2.5.** (Granirer [4] and Rudin [9]). If $G$ is not discrete and $G_d$ is amenable, then

$$LIM(L^\infty(G)) \neq TLIM(L^\infty(G))$$

where $TLIM(L^\infty(G))$ is the set of all topological left invariant means on $L^\infty(G)$.

**Proof.** By Proposition 1.2 and the last proposition of [4], we can find an open dense subset $V$ in $G$ such that $m(1_V) < 1$ for all $m \in TLIM(L^\infty(G))$. By
Theorem 2.3, \( \mathcal{H} = H \). We can see that \( \|1_V - h\|_\infty \geq 1 \) for any \( h \in H \).
Indeed, if there is \( h \in H \) such that \( \|1_V - h\|_\infty < 1 - \epsilon \) for some \( \epsilon > 0 \), then we can find \( x_i \in G \) and \( \lambda_i > 0 \) \( (i = 1, 2, \ldots, n) \) such that \( \sum_{i=1}^{n} \lambda_i = 1 \) and \( \|\sum_{i=1}^{n} \lambda_i x_i h\|_\infty \leq \epsilon/2 \). It follows that
\[
\|1_V - h\|_\infty \geq \|\sum_{i=1}^{n} \lambda_i x_i (1_V - h)\|_\infty \geq \|\sum_{i=1}^{n} \lambda_i x_i 1_V\|_\infty - \|\sum_{i=1}^{n} \lambda_i x_i h\|_\infty \geq 1 - \epsilon/2
\]
which is impossible. Hence there is \( m \in LIM(L^\infty(G)) \) such that \( m(1_V) = 1 \), i.e. \( m \notin TLIM(L^\infty(G)) \).

3. AMENABLE LOCALLY COMPACT GROUPS

For a locally compact group \( G \), it is well known that if \( G_d \) is amenable, then \( \mathcal{U} = \| \cdot \|_\infty \)-closed linear span \( \{ x_f - f : x \in G, f \in L^\infty(G) \} \cup C \) (see [7]). Rosenblatt and Yang in [7] proved that if \( G \) is discrete and contains \( F_2 \), the free group on two generators, then \( \mathcal{U} \) is not a subspace. They asked if \( \mathcal{U} \) is a subspace when \( G \) is not amenable. Our Theorem 3.4 answers this problem negatively.

The existence and uniqueness of the left invariant mean on \( L^\infty(G) \) have been discussed in many papers. It is natural to ask whether there exists a largest admissible subspace \( S_M \) of \( L^\infty(G) \) with a unique left invariant mean (see [7, p. 5, Problem (d)]). It is proved in [7] that such space does not exist for any discrete group containing \( F_2 \). Our Theorem 3.6 answers this problem completely.

We need the following lemma, probably known, for which we were unable to find a reference.

A set \( E \) of \( G \) is called a permanently positive subset (P.P.) if \( \cap_{i=1}^{n} x_i E \) is not locally null for any \( x_1, x_2, \ldots, x_n \in G \).

**Lemma 3.1.** Let \( E \) be a P.P. set in \( G \). Then there exists \( m \in LIM(S_{1_E}) \) with \( m(1_E) = 1 \).

**Proof.** Put \( m(1) = m(x_1 E) = 1 \) for any \( x \in G \) and linearly extend \( m \) to \( S_{1_E} \). Then \( m \) is well defined. Indeed, let \( h = \alpha_0 + \sum_{i=1}^{n} \alpha_i x_i 1_E = 0 \) \( (x_i \in G \), \( i = 1, 2, \ldots, n) \). Since \( \cap_{i=1}^{n} x_i^{-1} E \) is not locally null and \( h = \sum_{i=0}^{n} \alpha_i \) on \( \cap_{i=1}^{n} x_i^{-1} E \), \( \sum_{i=0}^{n} \alpha_i = 0 \). Hence \( m(h) = \sum_{i=0}^{n} \alpha_i = 0 \). Similarly, if \( h \geq 0 \) and \( h \in S_{1_E} \), then \( m(h) \geq 0 \). Since \( m \) is nonnegative and \( m(1) = 1 \), \( m \) is bounded. Therefore \( m \in LIM(S_{1_E}) \).

**Lemma 3.2.** For any infinite locally compact group \( G \), there is a subset \( A \) in \( G \) such that both \( A \) and \( A^c \) are P.P. sets, where \( A^c = G \sim A \).

**Proof.** If \( G \) is discrete or \( G \) is \( \sigma \)-compact and nondiscrete, then there is a subset \( A \) in \( G \) such that both \( A \) and \( A^c \) are P.P. sets in \( G \) (see [7, p. 5] and [8, Proposition 3.4]). Let \( G \) be non-\( \sigma \)-compact and nondiscrete. We can find an open and closed \( \sigma \)-compact subgroup \( G_0 \) of \( G \) (see [6, Proposition 22.24]). Let
be a subset of \( G_0 \) such that both \( A_0 \) and \( A_0^c = G_0 \sim A_0 \) are P.P. sets in \( G_0 \).

Suppose \( \{x_\alpha G_0 : \alpha \in \Lambda \} \) is all the left cosets of \( G_0 \) in \( G \). Let \( A = \bigcup_{\alpha \in \Lambda} x_\alpha A_0 \).

Since for any compact subset \( K \) of \( G \), there are \( \alpha_1, \alpha_2, \ldots, \alpha_n \in \Lambda \) such that \( K \subseteq \bigcup_{i=1}^n x_{\alpha_i} G_0 \), \( K \cap A = \bigcup_{i=1}^n x_{\alpha_i} A_0 \cap K \) is measurable. Hence \( A \) is measurable (see [5, (11.31) and (15.8)]). For any \( g_i \in G \) \( (i = 1, 2, \ldots, n) \), since \( G = \bigcup_{\alpha \in \Lambda} G_0 x_{\alpha}^{-1} \), there is \( \alpha_i \in \Lambda \) such that \( g_i = y_i x_{\alpha_i}^{-1} \) for some \( y_i \in G_0 \) \( (i = 1, 2, \ldots, n) \). Hence

\[
\bigcap_{i=1}^n g_i A_0 \supseteq \bigcap_{i=1}^n y_i^{-1} x_{\alpha_i}^{-1} x_{\alpha_i} A_0 = \bigcap_{i=1}^n y_i A_0.
\]

Since \( \bigcap_{i=1}^n y_i A_0^c \) is a P.P. set in \( G_0 \), \( \bigcap_{i=1}^n g_i A \) is a P.P. set in \( G \). Similarly, \( A^c = \bigcup_{\alpha \in \Lambda} x_{\alpha} A_0^c \) is a P.P. set in \( G \), where \( A_0^c = G_0 \sim A_0 \). \( \square \)

**Lemma 3.3.** If \( \mathcal{U} \) is a subspace, then \( \mathcal{U} \supseteq H_s \) where

\[ H_s = \text{span}\{x - f : x \in G, f \in L^\infty(G) \text{ is a simple function}\}. \]

**Proof.** It suffices to show that for each measurable subset \( E \) of \( G \) and \( x \in G \), \( x 1_E - 1_E \in \mathcal{U} \). Assume that \( G \) is infinite. By Lemma 3.2, we can find a subset \( A \) in \( G \) such that both \( A \) and \( A^c \) are P.P. sets. Put \( E_A = (E \cap A) \cup A^c \), then \( E_A \) is a P.P. set. By Lemma 3.1, \( \text{LIM}(S_{E_A}) \neq \phi \) and \( \text{LIM}(S_{E^c}) \neq \phi \).

Put \( \xi_1 = x 1_{E_A} - 1_{E_A} \) and \( \xi_2 = x 1_{A^c} - 1_{A^c} \), then \( S_{\xi_1} \subseteq S_{E_A} \) and \( S_{\xi_2} \subseteq S_{E^c} \).

Hence \( \text{LIM}(S_{\xi_i}) \neq \phi \) \( (i = 1, 2) \). Also \( \xi_1, \xi_2 \in \mathcal{U}_0 \), hence \( m(\xi_i) = 0 \) for all \( m \in \text{LIM}(S_{\xi_i}) \) \( (i = 1, 2) \), i.e. \( \xi_1, \xi_2 \in \mathcal{U} \). Since \( \mathcal{U} \) is a subspace and

\[ \xi_1 - \xi_2 = (x 1_{E_A} - 1_{E_A}) - (x 1_{A^c} - 1_{A^c}) = x 1_{E \cap A} - 1_{E \cap A^c}. \]

\[ x 1_{E \cap A} - 1_{E \cap A} \in \mathcal{U}. \]

Similarly, \( x 1_{E \cap A^c} - 1_{E \cap A^c} \in \mathcal{U} \). Note that

\[ (x 1_{E \cap A} - 1_{E \cap A}) + (x 1_{E \cap A^c} - 1_{E \cap A^c}) = x 1_E - 1_E. \]

Hence \( x 1_E - 1_E \in \mathcal{U} \). \( \square \)

The following answers question (b) in the Remark following Corollary 1.4 of [7].

**Theorem 3.4.** If \( \mathcal{U} \) is a subspace, then \( G \) is amenable.

**Proof.** By Lemma 3.3, \( \mathcal{U} \supseteq H_s \). If \( G \) is not amenable, \( \overline{H_s} = \overline{H} = L^\infty(G) \).

For each \( f_0 \in L^\infty(G) \), there exist \( f_n \in H_s \) \( (n = 1, 2, \ldots) \) such that \( f_n \rightarrow f_0 \) in \( \| \cdot \|_\infty \). Since \( f_n \in H_s \subseteq \mathcal{U} \), \( \text{LIM}(S_{f_n}) \neq \phi \). So

\[ 1 = m(f_n (1 - h_{f_n}) \leq \|1 - h_{f_n}\|_\infty \quad (n = 1, 2, \ldots) \]

for \( m_{f_n} \in \text{LIM}(S_{f_n}) \) and \( h_{f_n} \in H_{f_n} \), where

\[ H_{f_n} = \text{span}\{x f_n - f_n : x \in G\} \quad (n = 0, 1, 2, \ldots). \]

If \( h_{f_0} = \sum_{i=1}^m \alpha_i (x, f_0 - f_0) \in H_{f_0} \) for \( \alpha_i \in \mathbb{C} \) and \( x_i \in G \) \( (i = 1, 2, \ldots, m) \), put \( h_{f_n} = \sum_{i=1}^m \alpha_i (x, f_n - f_n) \), then \( h_{f_n} \in H_{f_n} \) \( (n = 1, 2, \ldots) \) and \( h_{f_n} \rightarrow h_{f_0} \) in
Hence \(|1 - h_f|_\infty \to |1 - h_{f_0}|_\infty \) i.e. \(|1 - h_{f_0}|_\infty \geq 1 \) for any \(h_{f_0} \in H_{f_0}\). But \(G\) is not amenable, by [3, Theorem 2.12], there exists \(f_0 \in UCB(G)\), the set of uniformly continuous function on \(G\), and \(t_i, s_i \in G\) (\(i = 1, 2, \ldots, N\)) such that

\[
\sum_{i=1}^{N} t_i f_0 - s_i f_0 = \sum_{i=1}^{N} (t_i f_0 - f_0) + (f_0 - s_i f_0) \geq 1
\]

i.e. there exists \(h_{f_0} \in H_{f_0}\) such that \(h_{f_0} \geq 1\). Take \(h_{f_0}^* = (2||A||_1)^{-1} h_{f_0}\) then \(h_{f_0}^* \in H_{f_0}\) and

\[
(2||h_{f_0}||_\infty)^{-1} \leq h_{f_0}^* \leq 1/2
\]

so \(|1 - h_{f_0}^*|_\infty < 1\) which is impossible. 

**Open problem.** Does amenability of \(G\) imply that \(\mathcal{U}\) is a subspace? (This is the case when \(G\) is discrete (see [7, Theorem 1.1]).

**Corollary 3.5.** If \(G\) is a discrete group, the following statements are equivalent

(a) \(G\) is amenable,
(b) \(\mathcal{U}\) is a subspace,
(c) \(\mathcal{A}\) is a subspace,
(d) \(\mathcal{A}_0 = \overline{H}\),
(e) \(\mathcal{U}_0 = \overline{H}\) where \(\mathcal{U}_0 = \{f \in \mathcal{U} : m(f) = 0 \text{ for all } m \in \text{LIM}(S_f)\}\).

**Proof.** By Theorem 2.3 and Lemma 2.1, (a) \(\iff\) (c) \(\iff\) (d). By Theorem 3.4 and Theorem 1.1 of [7], (a) \(\iff\) (b).

It is clear that if \(\mathcal{U}_0\) is a subspace, then \(\mathcal{U}\) is a subspace. Hence (e) \(\Rightarrow\) (b). To see that (d) \(\Rightarrow\) (e). Let \(f \notin \overline{H}\), there exists \(m \in \text{LIM}(L^\infty(G))\) such that \(m(f) \neq 0\), i.e. \(f \notin \mathcal{U}_0\) and \(\mathcal{U}_0 \subseteq \overline{H}\). Since \(\text{LIM}(L^\infty(G)) \neq \emptyset\) by Theorem 2.3, \(\overline{H} = \mathcal{A}_0 \subseteq \mathcal{U}_0\). Therefore \(\mathcal{U}_0 = \overline{H}\). \(\Box\)

Now we come to discuss the existence of the largest admissible subspace of \(L^\infty(G)\) with a unique left invariant mean.

**Theorem 3.6.** There is a largest admissible subspace \(S_M\) in \(L^\infty(G)\) with a unique left invariant mean if and only if \(G\) is amenable. In this case, \(S_M = H + \mathcal{C}\).

**Proof.** Suppose that such \(S_M\) exists. Note that if \(\xi \in \mathcal{A}_0\) and \(\text{LIM}(S_\xi) \neq \emptyset\), then \(\text{LIM}(S_\xi)\) is a singleton. Using this fact and the same proof of Lemma 3.3, we have \(S_M \supseteq H_s\) (see Lemma 3.3 for \(H_s\)). Let \(m \in \text{LIM}(S_M)\), we can extend \(m\) to \(\overline{S}_M\) such that \(m \in \text{LIM}(\overline{S}_M)\) which is also a singleton. Hence \(\overline{S}_M\) is closed and \(S_M \supseteq \overline{H}_S = \overline{H}\), i.e. \(S_M \supseteq \overline{H} + \mathcal{C}\). For any \(x \in G\) and \(f \in L^\infty(G)\), since \(xf - f \in \mathcal{A}_0\), \(m(xf - f) = 0\), i.e. \(m(h) = 0\) for all \(h \in H\). Therefore

\[
1 = m(1 - h) \leq ||1 - h||_\infty
\]

for any \(h \in H\). Hence \(G\) is amenable.

Conversely, let \(G\) be amenable. Take \(S_M = \overline{H} + \mathcal{C}\). Let an admissible subspace \(S\) with a unique left invariant mean \(m\) be given. For each \(f \in S\), \(f - m(f) \in \overline{H}\), i.e. \(S \subseteq \overline{H} + \mathcal{C}\). Indeed, if \(f - m(f) \notin \overline{H}\), there exists \(M \in \text{LIM}(L^\infty(G))\) such that \(M(f - m(f)) \neq 0\). Note that \(M|S \in \text{LIM}(S)\) and
\[ M(f) = M|S(f) \neq m(f) \] which is impossible. For any \( m \in LIM(S_M) \), since \( xf - f \in \mathcal{A}_0 \quad (x \in G, \ f \in L^\infty(G)) \), \( m(h) = 0 \) for any \( h \in H \). Consequently, \( S_M \) must have a unique left invariant mean. \( \square \)

ACKNOWLEDGMENTS

I would like to thank my supervisor Dr. Anthony T. Lau. This paper will form a part of my thesis under his supervision. I am deeply indebted to Professor Lau for his valuable suggestions and encouragement. Special thanks are also due to my colleague, Dr. M. Skantharajah for showing me the references [3, 10].

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