POWER ROOTS OF LINEARIZED POLYNOMIALS

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Abstract. In the present paper, we have discussed the number of power roots of linearized polynomials. For some cases, the exact formulas are given.

A polynomial of the form

\[ L(x) = \sum_{i=0}^{n} a_i x^{p^i} \]

with coefficients \( a_i \) in a finite field \( \mathbb{GF}(p) \) is called a \( p \)-polynomial, it is customary to speak of linearized polynomials. In this paper, we discuss the roots of \( L(x^r) \) (also \( y^r = L(x) \)) in \( \mathbb{GF}(p^m) \). The case of \( L(x) = x + x^p + \cdots + x^{p^{m-1}} \) is considered in [5, 8].

First we introduce a few definitions.

1. For linearized polynomials \( L_1(x) \), \( L_2(x) \), we define the symbolic multiplication \( \otimes \) by

\[ L_1(x) \otimes L_2(x) = L_1(L_2(x)). \]

2. The polynomials \( f(x) = \sum_{i=0}^{n} a_i x^i \), \( f^*(x) = \sum_{i=0}^{n} a_i x^{p^i} \) over \( \mathbb{GF}(p) \) are called \( p \)-associates of each other.

If \( f_1(x), f_2(x) \in \mathbb{GF}(p)[x] \), we can easily check

\[ (f_1(x) f_2(x))^* = f_1^*(x) \otimes f_2^*(x). \]

Therefore the set of \( p \)-polynomials over \( \mathbb{GF}(p) \) forms an integral domain under the symbolic multiplication and ordinary addition (for details see [4]), and the symbolic multiplication and ordinary multiplication are related by (\( \ast \)).

For a polynomial \( f(x) = \sum a_i x^i \in \mathbb{GF}(p)[x] \), let \( d(x) = (f(x), x^m - 1) \), then \( f^*(x) \) and \( d^*(x) \) have the same set of roots in \( \mathbb{GF}(p^m) \). So later we always suppose \( f(x) | x^m - 1 \). If \( \mathbb{GF}(p^m) \) is considered a vector space over \( \mathbb{GF}(p) \), \( f^*(x) \) induces a linear operator on \( \mathbb{GF}(p^m) \). \( f^*(\mathbb{GF}(p^m)) = \{ f^*(c) | c \in \mathbb{GF}(p^m) \} \) is an additive subgroup of \( \mathbb{GF}(p^m) \).

Let \( G \) be a finite Abelian group. By a character of \( G \), we mean a group homomorphism \( G \rightarrow \mathbb{C}^* \), where \( \mathbb{C}^* \) is the multiplicative group of the complex...
number field. The characters form an Abelian group \( G^\times \), called the dual of \( G \) (for the basic properties of characters, see [2]). The dual of the additive group of \( GF(p^m) \) is
\[
GF(p^m)^\sim = \{ \chi_u | u \in GF(p^m) \},
\]
where \( \chi_u(c) = e^{2\pi i T(uc)} \), \( c \in GF(p^m) \), \( T(x) = x + xp + \cdots + x^{p^m-1} \) is the absolute trace of \( GF(p^m) \) to \( GF(p) \). We denote the dual of the multiplicative group \( GF(p^m)^\times \) of \( GF(p^m) \) by \( GF(p^m)^{\sim \times} \).

For every \( \chi_u \), its restriction on \( f^*(GF(p^m)) \) induces a character \( \chi_u^* \) of \( f^*(GF(p^m)) \). We have

**Lemma 1.** The map \( \phi : \chi_u \to \chi_u^* \) is a surjective homomorphism of \( GF(p^m)^\sim \) to
\[
f^*(GF(p^m))^\sim \cdot \ker(\phi) = \{ \chi_u | f^*(u) = 0, u \in GF(p^m) \},
\]
where \( f_0(x) = x^n f(x^{-1}) \), the reciprocal polynomial of \( f(x) \).

**Proof.** It is obvious that \( \phi \) is a homomorphism, which is surjective, since every character in \( f^*(GF(p^m)) \) can be extended to a character in \( GF(p^m)^\sim \).

Let \( \chi_u \in \ker(\phi) \). Then \( \chi_u(u) = 1 \) for \( u \in f^*(GF(p^m)) \), i.e. \( \chi_u(f^*(c')) = 1 \).

Hence \( T(u f^*(c')) = 0 \), for \( c' \in GF(p^m) \). Therefore \( \sum_{i=0}^{m-1} (u f^*(x))^i \equiv 0 \mod x^{p^m} - x \), i.e.
\[
\sum_{i=0}^{m-1} u^j f^*(x^i) \equiv \sum_{i=0}^{m-1} u^j \sum_{j=0}^{n} a_j x^{i+j,j}
\equiv \sum_{i=0}^{m-1} \sum_{j=0}^{n} u^j a_j x^{i+j,j}
\equiv x \sum_{j+i \equiv 0 \mod m} u^j a_j + x^p \sum_{j+i \equiv 1 \mod m} u^j a_j + \cdots + x^{p^m-1} \times \sum_{j+i \equiv m-1 \mod m} u^j a_j
\equiv 0 \mod x^{p^m} - x.
\]

The equation above holds if and only if
\[
(1) \quad \sum_{j+i \equiv t \mod m} u^j a_j = 0, \quad t = 0, 1, \ldots, m - 1.
\]

From this, we see that (1) holds if and only if
\[
\sum_{j+i \equiv 0 \mod m} u^j a_j = \left( \sum_{j=0}^{n} u^{p^m-1} a_j \right)^{p^m-n},
\]
i.e. \( \sum_{j=0}^{n} u^{p^m-1} a_j = 0 \). So \( u \) is a root of \( f_0^*(x) \). \( \Box \)

Now let \( R(f^*) \) denote the set of the roots of \( f^* \); then \( R(f^*) \) is an additive subgroup of \( GF(p^m) \).
Lemma 2. Let \( g(x) = (x^m - 1)/f(x) \).

(i) \( R(g^*) = f^*(\text{GF}(p^m)) \).

(ii) Additive group of \( \text{GF}(p^m) = R(f^*) + R(g^*) \) if \((f(x), g(x)) = 1\) (here the sum is direct sum).

Remark. If \((m, p) = 1\), \(x^{m-1}\) does not have multiple factors and \((f(x), g(x)) = 1\).

Proof. (i) Let \( f^*(c) \in f^*(\text{GF}(p^m)) \), then

\[
g^*(f^*(c)) = (fg)^*(c) = (x^m - 1)^*(c) = 0.
\]

So \( f^*(c) \in R(g^*) \), \( R(g^*) \) contains \( f^*(\text{GF}(p^m)) \). Also \( p^{m-n} = |R(g^*)| = |f^*(\text{GF}(p^m))| \), and hence \( R(g^*) = f^*(\text{GF}(p^m)) \).

(ii) Let \( c \in R(f^*) \cap R(g^*) \). Since \((f(x), g(x)) = 1\), there exist \( f_1(x), g_1(x) \) such that \( f(x)f_1(x) + g(x)g_1(x) = 1 \) and \( f^*(x) \otimes f_1^*(x) + g^*(x) \otimes g_1^*(x) = x \).

So \( c = 0 \), \( R(f^*) \cap R(g^*) = \{0\} \). But \( p^m = |R(f^*)||R(g^*)| \), and therefore (ii) holds. \( \square \)

From now on, we always suppose \((f(x), g(x)) = 1\).

Lemma 3. Let \( f(x), g(x) \) be in Lemma 2, \((f(x), g(x)) = 1\). Then

\[
f^*(\text{GF}(p^m)) = \{x_u : u \in R(g_0^*(x))\}.
\]

Proof. If \( \chi_u := \chi_{x_u} \), then \( u - c \in R(f_0^*) \) by Lemma 1, \( u - c = 0 \). So the set on the right has \(|R(g_0^*)|\) elements. Also \( |f^*(\text{GF}(p^m))| = |f^*(\text{GF}(p^m))| = |R(g^*)| = |R(g_0^*)| \), so Lemma 3 holds. \( \square \)

Lemma 4. Let \( c \in \text{GF}(p^m) \). Then

\[
\sum_u \chi_u(c) = \begin{cases} p^{m-n} & \text{if } f^*(c) = 0, \\ 0 & \text{if } f^*(c) \neq 0, \end{cases}
\]

where the sum is taken over \( R(g_0^*) \).

Proof. By Lemma 2, \( c = c_1 + c_2, c_1 \in R(g^*), c_2 \in R(f^*) \), then \( \chi_u(c_2) = 1 \) by Lemmas 1 and 3. Hence

\[
\sum_u \chi_u(c) = \sum_u \chi_u(c_1) = \begin{cases} p^{m-n} & \text{if } c_2 = 0, \\ 0 & \text{if } c_2 \neq 0. \end{cases}
\]

This is the reformulation of Lemma 4. \( \square \)

Now let \( \psi \) be a multiplicative and \( \chi \) an additive character of \( \text{GF}(p^m) \). Then the Gaussian sum is defined by

\[
G(\psi, \chi) = \sum_{c \in \text{GF}(p^m)} \psi(c)\chi(c).
\]

There are many important results on Gaussian sums. Here we give one from the Stickelberg theorem as a lemma.
Lemma 5 [8]. Let \( \psi \) be an \( r \)th order multiplicative character of \( \text{GF}(p^{2ab}) \), \( r | p^n + 1 \). Then

\[
G(\psi, \chi_j) = \begin{cases} 
(-1)^{b-1}p^{ab} & \text{if } r \text{ odd or } (p^a + 1)/r \text{ even}, \\
(-1)^{b-1+b_j}p^{ab} & \text{if } r \text{ even and } (p^a + 1)/r \text{ odd},
\end{cases}
\]

\( j = 1, 2, \ldots, r - 1 \).

For the subset \( S \) of \( \text{GF}(p^m) \), let \( N_r(S) = \{|c \in \text{GF}(p^m)| c^r = s, s \in S\} \). If \( S = \{s\} \), we denote the number by \( N_r(s) \).

Lemma 6 [2]. Let \( \psi \) be an \( r \)th order multiplicative character of \( \text{GF}(p^m) \), \( c \in \text{GF}(p^m) \). Then

\[
N_r(c) = \sum_{j=0}^{r-1} \psi^j(c). \tag*{\square}
\]

Now we can give the following result which interprets the relationship between \( N_r(R(f^*)) \) and \( N_r(c \sim R(g_0^*)) \) for some \( c \sim \in \text{GF}(p^m) \).

Theorem 1. Let \( f(x) = \sum a_i x^i \in \text{GF}(p)(x) \), \( r | p^n + 1 \), \( g_0(x) \) be the reciprocal polynomial of \( g(x) = x^m - 1/f(x) \), \( (f(x), g(x)) = 1 \). Then

\[
N_r(R(f^*)) = p^n + (-1)^{b} p^{ab} + (-1)^{b-1}p^{ab} N_r(c \sim R(g_0^*)),
\]

where \( c \sim \in \text{GF}(p^m) \) and such that: (1) \( c \sim = 1 \) if \( r \) odd or \( p^a + 1/r \) even or \( b \) even, and (2) \( \psi(c \sim) = -1 \) otherwise.

Proof. Let \( \psi \) be a \( r \)th order multiplicative character of \( \text{GF}(p^{2ab}) \). By Lemmas 4 and 6, we have

\[
N_r(R(f^*)) = p^{-(2ab-n)} \sum_{c \in \text{GF}(p^m)^*} \sum_{j=0}^{r-1} \sum_{u \in R(g_0^*)} \psi^j(c) \chi_u(c) + 1
\]

\[
= p^{-(2ab-n)} \sum_{j=0}^{r-1} \sum_{u \in R(g_0^*)} G(\psi^j, \chi_u) + 1
\]

\[
= p^{-(2ab-n)} \left\{ p^{2ab} - 1 + \sum_{j=1}^{r-1} G(\psi^j, \chi_0) + \sum_{u \in R(g_0^*) \setminus \{0\}} G(\psi^0, \chi_u) \right\} + 1,
\]

where

\[
\sum_{j=1}^{r-1} G(\psi^j, \chi_0) = 0, \quad \sum_{u \in R(g_0^*) \setminus \{0\}} G(\psi^0, \chi_u) = -p^{(2ab-n)} + 1,
\]
if \( r \) is odd or \( p^a + 1/r \) even or \( b \) even; then, by Lemma 5,

\[
\text{the last term} = \sum_{j=1}^{r-1} \sum_{u \in R(g_0^*) \setminus \{0\}} G(\psi^j, \chi_u)
\]

\[
= \sum_{j=1}^{r-1} \sum_{u \in R(g_0^*) \setminus \{0\}} \psi^j(u) G(\psi^j, \chi_1)
\]

\[
= \sum_{j=1}^{r-1} \sum_{u \in R(g_0^*) \setminus \{0\}} \psi^j(u) (-1)^{b-1} p^{ab}
\]

\[
= (-1)^{b-1} p^{ab} \left\{ \sum_{u \in R(g_0^*) \setminus \{0\}} \sum_{j=0}^{r-1} \psi^j(u) - p^{2ab-n} + 1 \right\}
\]

\[
= (-1)^{b-1} p^{ab} \{ N_r(R(g_0^*)) - p^{2ab-n} \}
\]

if \( r \) even, \( p^a + 1/r \) odd and \( b \) odd, and similarly

\[
\text{the last term} = (-1)^{b-1} p^{ab} \{ N_r(c \tilde{R}(g_0^*)) - p^{2ab-n} \};
\]

here \( c \tilde{c} \in \mathbb{GF}(p^m) \) such that \( \psi(c \tilde{c}) = -1 \). Collecting the results above, we prove the theorem. \( \square \)

Let \( f(x) \in \mathbb{GF}(p^m)[x] \), \( f(0) \neq 0 \). The least integer \( e \) such that \( f(x)|x^e - 1 \) is called the order of \( f(x) \). The least integer \( e \) such that \( f(x)|x^e + 1 \) is called the suborder of \( f(x) \). Now we give several applications of Theorem 1.

**Theorem 2.** Let \( f(x) \in \mathbb{GF}(p)[x] \), \( f(0) \neq 0 \), \( \deg(f(x)) = n \), the order of \( g(x) \) be \( e \), \( (f(x), g(x)) = 1 \), \( r|p^a + 1 \). If \( r|(p^{2ab} - 1)/(p^e - 1) \), then the number of the roots of \( f^*(x^r) \) in \( \mathbb{GF}(p^{2ab}) \)

\[
N_r(R(f^*)) = p^n + (-1)^{b-1} \delta(r, b) p^{n-1} (p^{2ab-n} - 1);
\]

where

\[
\delta(r, b) = \begin{cases} 
  r - 1 & \text{if } r \text{ odd or } p^a + 1/r \text{ even or } b \text{ even,} \\
  -1 & \text{if } r \text{ even, } p^a + 1/r \text{ odd and } b \text{ odd.}
\end{cases}
\]

**Proof.** Since \( f(x)|x^e - 1 \), \( f^*(x)|x^{e^r} - x \), \( R(f^*) \subseteq \mathbb{GF}(p^e) \). Suppose \( \eta \) is a primitive root of \( \mathbb{GF}(p^{2ab}) \). Then \( \zeta = \eta^s \) is a primitive root of \( \mathbb{GF}(p^e) \); here \( s = (p^{2ab} - 1)/(p^e - 1) \), and hence \( \psi(\zeta) = 1 \). For \( c \in \mathbb{GF}(p^e) \), so for \( c \in R(f^*) \), \( \psi(c) = 1 \). Now by (2), we can directly get the desired result. \( \square \)

**Corollary 1.** Suppose \( (p, 2ab) = 1 \), \( r|p^a + 1 \).

(i) Let \( f(x) = (x^{2ab} - 1)/(x - 1) \). Then

\[
N_r(R(f^*)) = p^{2ab-1} + (-1)^{b-1} \delta(r, b) p^{ab-1} (p - 1);
\]

here \( \delta(r, b) \) is the one in Theorem 2.
(ii) Let the order of \( g(x) \) be \( q \), an odd prime number, \( \deg(f(x)) = n \). Then
\[
N_r(R(f^*)) = p^n + (-1)^{b-1}\delta(r, b)p^{n-ab}(p^{2ab-n} - 1);
\]
here \( \delta(r, b) \) as before.

Proof. We only need to prove \( r|(p^{2ab} - 1)/(p - 1) \). It holds because
\[
(p^{2ab} - 1)/(p - 1) = ((p^{2ab} - 1)/(p^{2a} - 1))((p^a - 1)/(p - 1))(p^a + 1)
\]
and
\[
((p^{2ab} - 1)/(p^{2a} - 1))((p^a - 1)/(p - 1))
\]
is an integer. So (i) holds.

For (ii), we must have \( q|ab \) because \( q \) is the order of \( g(x) \). If \( q|a \),
\[
(p^{2ab} - 1)/(p^a - 1) = ((p^{2ab} - 1)/(p^{2a} - 1))((p^a - 1)/(p^a - 1))(p^a + 1).
\]
If \( q \nmid a \), then \( (q, 2a) = 1 \) and \( (p^{2a} - 1, p^q - 1) = p - 1 \). Hence
\[
(p^{2ab} - 1)(p^a - 1)/(p^{2a} - 1)(p^q - 1)
\]
is an integer. Also
\[
(p^{2ab} - 1)/(p^q - 1) = ((p^{2ab} - 1)(p^a - 1)(p^{2a} - 1)(p^q - 1))(p^a + 1),
\]
so \( r|(p^{2ab} - 1)/(p^q - 1) \). □

Remark. For \( q = 2 \), \( f(x) = (x^{2a} - 1)/(x^2 - 1) \). If \( a \) or \( b \) even or \( (p + 1, r) = 1 \), we also have that (ii) holds.

With the application of (ii), e.g., suppose \( f(x) = (x^{2ab} - 1)/(x^2 + x + 1) \), \( 3|ab \), then \( N_r(R(f^*)) = p^{2ab-2} + (-1)^{b-1}\delta(r, b)p^{ab-2}(p^2 - 1) \).

Theorem 3. Let \( f(x) \in GF(p)[x] \), \( f(0) \neq 0 \), \( \deg(f(x)) = n \), the suborder of \( g(x) \) be \( e \), \( (f(x), g(x)) = 1 \), \( r|p^{a+1} \). If \( r|(p^{2ab} - 1)/(p^e - 1) \), then the number of the roots of \( f^*(x^t) \)
\[
N_r(R(f^*)) = p^n + (-1)^{b-1}\delta(r, b)p^{n-ab}(p^{2ab-n} - 1);
\]
here if \( r|(p^{2ab} - 1)/(p^e - 1) \), \( e(r, b) = \delta(r, b) \); if \( r \nmid (p^{2ab} - 1)/(p^e - 1) \),
\[
e(r, b) = \begin{cases} 
-1 & \text{if } r \text{ odd or } p^a + 1/r \text{ even or } b \text{ even}, \\
-1 & \text{if } r \text{ even, } p^a + 1/r \text{ odd and } b \text{ odd}.
\end{cases}
\]

Proof. First we must have \( e|ab \). Let \( \eta \) is a primitive root of \( GF(p^{2ab}) \). The roots of \( x^{2(p^e - 1)} - 1 = 0 \) form a cyclic subgroup \( E \) of generator \( \zeta = \eta^t \) with \( 2(p^e - 1) \) elements; here \( t = (p^{2ab} - 1)/2(p^e - 1) \). \( \psi \) induces a character of \( E \). So if \( r|(p^{2ab} - 1)/(p^e - 1) \), then for \( c \in E \), \( \psi(c) = 1 \), similar as Theorem 2; if \( 2r \nmid (p^{2ab} - 1)/(p^e - 1) \), \( \psi \) induces a quadratic character of \( E \). \( \psi(c) = 1 \), \( c \in GF(p^e)^* \); \( \psi(c) = -1 \), \( c \in R((x^e + 1)^*) \), \( c \neq 0 \). Using this result to the proof of Theorem 1, we prove this case. □

Corollary 2. Suppose \( (p, 2ab) = 1 \), \( r|p^a + 1 \).
(i) Let \( f(x) = (x^{2ab} - 1)/(x + 1) \). Then
\[
N_r(R(f^*)) = p^{2ab-1} + (-1)^{b-1} \tau(r, b)p^{ab-1}(p - 1).
\]
Here \( \tau(r, b) = \delta(r, b) \) if \( a \) or \( b \) even; \( r - 1 \) otherwise.

(ii) Let the suborder of \( g(x) \) be \( q \), an odd prime number, \( \text{deg}(f(x)) = n \). Then
\[
N_r(R(f^*)) = p^n + (-1)^{b-1} \tau(r, b)p^{n-ab}(p^{2ab-n} - 1);
\]
here \( \tau(r, b) \) as (i).

Proof. Following the proof of Corollary 1, we have
\[
(p^{2ab} - 1)/(p - 1) = ((p^{2a} - 1)/(p - 1))(p^a - 1)/(p - 1)(p^a + 1)
\]
and
\[
((p^{2ab} - 1)/(p^{2a} - 1))(p^a - 1)/(p - 1))
\]
is an integer. If \( a \) or \( b \) is even, \( r|(p^{2ab} - 1)/2(p - 1) \) and
\[
N_r(R(f^*)) = p^{2ab-1} + (-1)^{b-1} \delta(r, b)p^{ab-1}(p - 1);
\]
if \( a \) and \( b \) are odd, \( r|(p^{2ab} - 1)/2(p^e - 1) \) if and only if \( p^a + 1/r \) even and
\[
N_r(R(f^*)) = p^{2ab-1} + (-1)^{b-1}(r - 1)p^{ab-1}(p - 1).
\]

Similarly, from the proof of Corollary 1 we see that if \( a \) or \( b \) even, then \( r|(p^{2ab} - 1)/2(p - 1) \) and
\[
N_r(R(f^*)) = p^n + (-1)^{b-1} \delta(r, b)p^{ab-n}(p^{2ab-n} - 1);
\]
if \( a \) and \( b \) odd, \( r|(p^{2ab} - 1)/2(p^e - 1) \) if and only if \( p^a + 1/r \) even and
\[
N_r(R(f^*)) = p^n + (-1)^{b-1}(r - 1)p^{ab-n}(p^{2ab-n} - 1). \quad \Box
\]

Remark. When \( q \) is a composite number, we can use (3) or (4) to obtain some partial results.

In fact, by Lemma 2, we also obtain the number of roots of \( y^r = g^*(x) \) in \( \text{GF}(p^{2ab}) \).

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References


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