PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 1, May 1991

# STABILITY IN INTERPOLATION OF FAMILIES OF BANACH SPACES

WEI CAO AND YORAM SAGHER

(Communicated by J. Marshall Ash)

ABSTRACT. Let $D$ be a simply connected domain in the complex plane whose boundary $\Gamma$ is a rectifiable simple closed curve. Let $\{A(\gamma)/\gamma \in \Gamma\}$ and $\{B(\gamma)/\gamma \in \Gamma\}$ be interpolation families of Banach spaces. Let $T$ be a linear operator mapping $A(\gamma)$ continuously into $B(\gamma)$. For $z \in D$ let $T_z$ be the restriction of $T$ to the interpolation space $A_z$. Then $\{z \in D/\operatorname{cod}(T_z) = d < \infty$ and $\dim \operatorname{Ker}(T_z) = 0\}$ and $\{z \in D/\dim \operatorname{Ker}(T_z) = d < \infty$ and $T_z$ is onto $B_z\}$ are open sets.

## 1. INTRODUCTION

We begin with a brief review of the theory of interpolation for analytic families of Banach spaces. Throughout the paper, $D$ will denote a simply connected domain in the complex plane whose boundary $\Gamma$ is a rectifiable simple closed curve.

**Definition 1.1.** An interpolation family of Banach spaces on $\Gamma$ is a collection $\{A(\gamma)|\gamma \in \Gamma\}$, of Banach spaces which satisfies:

(1) Every $A(\gamma)$ is continuously embedded in a common Banach space $\mathscr{U}$ which is called a containing space for $\{A(\gamma)|\gamma \in \Gamma\}$.

(2) For every $a \in \cap_{\gamma\in\Gamma} A(\gamma)$, $\|a\|_{A(\gamma)}$ is a measurable function on $\Gamma$ with respect to $dP_z$, where $z \in D$ and $dP_z$ is a harmonic measure on $\Gamma$ with respect to $z$.

(3) Let $\mathscr{A} = \{a \in \cap_{\gamma\in\Gamma} A(\gamma)| \int_\Gamma \log^+ \|a\|_{A_\gamma} dP_z(\gamma) < \infty\}$, where $\log^+ x = \max(\log x, 0)$. $\mathscr{A}$ is a linear space which is called the log-intersection space for $\{A(\gamma)|\gamma \in \Gamma\}$. We assume that $\mathscr{A}$ is dense in each $A(\gamma)$ and that there exists a measurable function $\psi$ on $\Gamma$ which satisfies $\int_\Gamma \log^+ \psi(\gamma) dP_z < \infty$ such that $\|a\|_u \le \psi(\gamma)\|a\|_{A(\gamma)}$ for all $a \in \mathscr{A}$ and $\gamma \in \Gamma$.

We shall denote the spaces $A(\gamma)$ by $A_\gamma$ and their norms by $\|\cdot\|_\gamma$.

Received by the editors October 4, 1989 and, in revised form, February 2, 1990.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 47A53.

©1991 American Mathematical Society
0002-9939/91 $1.00 + $.25 per page

We denote by $N^+(D)$ the class of all functions $f(z)$ analytic on $D$ such that $f(\phi(w))$ belongs to the class $N^+ = N^+(\Delta)$, where $\Delta = \{w| \ |w| < 1\}$ and $\phi$ is a conformal map from $D$ to $\Delta$, see [7]. Thus $N^+(D)$ is closed under pointwise addition and multiplication and each $f(z) \in N^+(D)$ possesses a.e. non-tangential limits on $\Gamma$. If these non-tangential limits are essentially bounded on $\Gamma$, then $f \in H^\infty(D)$, the space of bounded analytic functions on $D$. A function $f \in N^+(D)$ is termed an outer function in $N^+(D)$, if $f(\phi(w))$ is an outer function in $N^+(\Delta)$.

Let $\mathscr{G}(A(\cdot), \Gamma) = \mathscr{G}(\mathscr{A}) = \mathscr{G}$ be the space of all functions of the form $g(z) = \Sigma_{j=1}^n \varphi_j(z) a_j$, where $a_j \in \mathscr{A}$ and $\varphi_j \in N^+(D)$, and such that

$$\|g(\cdot)\|_{\mathscr{G}} = \operatorname*{ess\,sup}_{\gamma \in \Gamma} \|g(\gamma)\|_\gamma < \infty .$$

Note that, in general, $(\mathscr{G}, \|\cdot\|_{\mathscr{G}})$ is not complete.

**Definition 1.2.** For $z \in D$ and $a \in \mathscr{A}$, define

$$\|a\|_{A\{z\}} = \inf\{\|g\|_{\mathscr{G}} | g \in \mathscr{G}, \ g(z) = a\} .$$

The interpolation space $\{A_\gamma\}\{z\} = A\{z\}$ is defined to be the completion in $\mathscr{U}$ of $(\mathscr{A}, \|\cdot\|_{A\{z\}})$.

In most applications the family of dual spaces $\{A_\gamma^*\}$ is itself an interpolation family and $(A\{z\})^* = \{A_\gamma^*\}\{z\}$. In this paper we shall assume that the above duality result holds.

We consider in this paper linear operators $T$ mapping $\mathscr{A}$ into $\cap_{\gamma \in \Gamma} B(\gamma)$, with $\|Ta\|_{B(\gamma)} \le M(\gamma)\|a\|_{A(\gamma)}$ for all $a \in \mathscr{A}$ and $\gamma \in \Gamma$, where $\{A(\gamma)|\gamma \in \Gamma\}$ and $\{B(\gamma)|\gamma \in \Gamma\}$ are interpolation families on $\Gamma$ and

$$\int_\Gamma |\log M(\gamma)| \, dP_z(\gamma) < \infty .$$

We will denote the restriction of $T$ to $A\{z\}$ by $T_z$. If $\ker T_z = \{0\}$, denote by $T_z^{-1}$ the inverse of $T_z$, i.e., $T_z^{-1}$ is defined on $T_z A\{z\}$. If $T_z A\{z\}$ has finite codimension, by the open mapping theorem, $T_z A\{z\}$ is closed and $T_z^{-1}$ is a bounded linear operator.

The question of the stability of Fredholm property when one changes the parameters which determine the interpolation space has been considered by several authors [1, 4, 9, 12] in the context of interpolation of two spaces and by Vignati and Vignati [11] in the case of families of Banach spaces. We consider families of Banach spaces and prove that, if $\ker T_s = \{0\}$ and $\operatorname{codim} T_s = d < \infty$, then there exists a ball $B$ centered at $s$ such that $z \in B$ implies that $\ker T_z = \{0\}$ and $\operatorname{codim} T_z = d$. This generalizes the result in [11], where the theorem is proved in the case $d = 0$.

**Definition 1.3.** Suppose $U(A)$ is a subset of $\mathscr{G}(\mathscr{A})$. Let

$$U_s = \{a \in A\{s\} | \exists f \in U(A) \text{ such that } f(s) = a\} .$$

Define

$$\|a\|_{U_s(A)} = \inf\{\|f\|_{\mathscr{G}} | f(s) = a\,,\ f \in U(A)\}\,.$$

Clearly $\|a\|_{A\{s\}} \le \|a\|_{U_s(A)}$ for all $a \in U_s$.

Let $U_{\{s\}}$ be a completion in $A\{s\}$ of $\{U_s\,, \|\cdot\|_{U_s}\}$. Clearly $U_{\{s\}} \subset A\{s\}$. If also $\|a\|_{U_s(A)} < k\|a\|_{A\{s\}}$, for some $k$ and all $a \in U_s$ with $a \ne 0$, we call $U(A)$ an $s$-Calderón subset (with constant $k$). If $U(A)$ is an $s$-Calderón subset and a linear subspace of $\mathscr{G}(\mathscr{A})$, we will say that it is an $s$-Calderón subspace.

Clearly $U(A)$ is an $s$-Calderón subspace for some $k$ if and only if $U_{\{s\}}$ is a closed subspace of $A\{s\}$.

An important tool in handling finite dimensional subspaces is Auerbach's lemma:

**Lemma 1.4.** *Let $M$ be a Banach space with dimension $d < \infty$. Then there exist $\{e_1\,, \dots\,, e_d\} \subset M$ and $\{f_1\,, \dots\,, f_d\} \subset M^*$ such that*

(1) $\|e_i\|_M = 1$ *and* $\|f_i\|_{M^*} = 1\,,\ i = 1\,, \dots\,, d$.
(2) $f_i(e_j) = \delta_{ij}$.

For a proof see [8].

Given a finite set of vectors $\{e_1\,, \dots\,, e_d\}$ in a Banach space $A$, we define

$$b(e_1\,, \dots\,, e_d) = \min\left\{ \left\| \sum_{i=1}^{d} c_i e_i \right\|_A \,\middle|\, \max|c_i| = 1 \right\}.$$

Clearly $\{e_1\,, \dots\,, e_d\}$ are independent if and only if $b(e_1\,, \dots\,, e_d) > 0$. Given $\sum_{i=1}^{d} c_i e_i$, let $m = \max\{|c_i|\}$. Then $\|\sum_{i=1}^{d} c_i e_i\|_A = m\|\sum_{i=1}^{d} c_i e_i/m\|_A \ge mb(e_1\,, \dots\,, e_d)$, so that $|c_i| \le \|\sum_{i=1}^{d} c_i e_i\|_A / b(e_1\,, \dots\,, e_d)$.

**Lemma 1.5.** *Let $M$ be a Banach space with dimension $d < \infty$. Then $\{e_1\,, \dots\,, e_d\}$ can be found so that $\|e_i\|_M = 1$, and $b(e_1\,, \dots\,, e_d) = 1$.*

For a proof, see [4].

**Lemma 1.6.** *Let $E$ be a closed subspace of $A\{s\}$ with finite codimension $d$. Then there exists $\{v_i\,, i = 1, \dots\,, d\} \subset \mathscr{G}(\mathscr{A})$, such that $A\{s\} = E \oplus \operatorname{span}\{v_i(s)\}$.*

*Proof.* Since $E$ has finite codimension $d$, $A\{s\} = E \oplus M$ with $\dim M = d$. Assume that $\{e_i\,, 1 \le i \le d\}$ forms a basis for $M$ with $\|e_i\|_{A\{s\}} = 1$ and $b(e_1\,, \dots\,, e_d) = 1$. Let

$$\rho = \rho(e_1\,, \dots\,, e_d\,; E) = \inf\left\{ \left\| \sum_{i=1}^{d} c_i e_i - x \right\|_{A\{s\}} \,\middle|\, \max|c_i| = 1\,,\ x \in E \right\}.$$

Clearly $\rho \le 1$. If $\rho > 0$ then $E \cap M = \{0\}$ and $\{e_i\}$ are independent. Since $E$ is a closed subspace and $M$ has finite dimension, then $E \cap M = \{0\}$ implies $\rho > 0$.

Since $\{v(s), v \in \mathscr{G}(\mathscr{A})\}$ is dense in $A\{s\}$, we can find $\{v_i, i = 1, \dots, d\} \subset \mathscr{G}(\mathscr{A})$ with $\|v_i(s) - e_i\|_{A\{s\}} < \rho/2d$. Now suppose $\max\{|c_i|\} = 1$ and let $x \in E$; we have

$$\left\|x - \sum_{i=1}^{d} c_i v_i(s)\right\|_{A\{s\}} \geq \left\|x - \sum_{i=1}^{d} c_i e_i\right\|_{A\{s\}} - \left\|\sum_{i=1}^{d} c_i(v_i(s) - e_i)\right\|_{A\{s\}} > \rho/2.$$

Taking infimum we get $\rho(v_1(s), \dots, v_d(s); E) \geq \rho/2 > 0$. Hence $\{v_i(s)\}$ are linearly independent and $A\{s\} = E \oplus \operatorname{span}\{v_i(s)\}$. □

The function $\rho$ introduced in the last proof is, of course, a generalization of the function $b(e_1, \dots, e_d)$. Another application for the function $\rho$ is the following lemma.

**Lemma 1.7.** *Let $E$ be a closed subspace of a Banach space $B$, and let $M$ be a finite dimensional subspace of $B$. Then $E + M$ is closed.*

*Proof.* We may assume $E \cap M = \{0\}$. Let $\{e_1, \dots, e_d\}$ be a basis for $M$. Then $\rho = \rho(e_1, \dots, e_d; E) > 0$. Let $x_n \in E + M$, and $x_n \to x_0 \in B$. $x_n = y_n + \sum_{i=1}^{d} c_i^n e_i$. Let $M_n = \max\{|c_i^n| \mid i = 1, \dots, d\}$.

Given $\varepsilon > 0$, let $N$ be so that, for all $n \geq N$, $\|x_n\| \leq (1 + \varepsilon)\|x_0\|$. Then $\rho \leq \|\sum_{i=1}^{d} c_i^n e_i + y_n\|/M_n \leq (1 + \varepsilon)\|x_0\|/M_n$, and $M_n \leq (1 + \varepsilon)\|x_0\|/\rho$. Thus, $\limsup M_n \leq \|x_0\|/\rho$ and $\{\sum_{i=1}^{d} c_i^n e_i\}$ is compact. Denote $\sum_{i=1}^{d} c_i^n e_i = z_n$. Let $z_{n_k} \to z \in M$; then $y_{n_k} = x_{n_k} - z_{n_k} \to x_0 - z$ and, since $E$ is closed, $x_0 - z \in E$. □

We will also need the following well-known result due to Szegö [10].

**Lemma 1.8.** *Let $f(\gamma)$ be a positive $dP_z$ measurable function on $\Gamma$ such that*

$$\int_\Gamma |\log f(\gamma)| dP_z(\gamma) < \infty$$

*for some (and thus every) $z \in D$. Then there exists a non-vanishing outer function $G(z)$ in $N^+(D)$ whose a.e. non-tangential limits $G(\gamma) = \lim_{z \to \gamma} G(z)$ satisfy $|G(\gamma)| f(\gamma) = 1$ for a.e. $\gamma \in \Gamma$.*

## 2. Stability

An invariant form of Schwarz's lemma states that if $f(z)$ is analytic and $|f(z)| < 1$ on $\Delta$, then for any $z, \zeta \in \Delta$,

$$\left|\frac{f(z) - f(\zeta)}{1 - \overline{f(z)} f(\zeta)}\right| \leq \left|\frac{z - \zeta}{1 - \overline{z}\zeta}\right|.$$

This implies that if $z_0, \zeta_0, z \in \Delta$ and if we denote

$$m_{z_0}(z) = \frac{z - z_0}{1 - \overline{z_0} z} e^{i\alpha},$$

then

$$\left|\frac{m_{z_0}(z)-m_{z_0}(\zeta)}{1-\overline{m_{z_0}(z)}m_{z_0}(\zeta)}\right| = \left|\frac{z-\zeta}{1-\overline{z}\zeta}\right|.$$

This, in turn, implies that if $z, \zeta \in D$ and $d$ is a conformal map of $D$ onto $\Delta$, then

$$q(z, \zeta) = \left|\frac{d(z)-d(\zeta)}{1-\overline{d(z)}d(\zeta)}\right|$$

does not depend on $d$.

The invariant form of Schwarz's lemma also implies the following lemma.

**Lemma 2.1.** *Let $\phi$ be analytic on $D$, $|\phi(z)| \le M$. If $s, z \in D$, then*

$$|\phi(z)| \ge M\frac{|\phi(s)| - Mq(z, s)}{M - |\phi(s)|q(z, s)}.$$

For a proof see [9]. Of course, since $q(z, s) < 1$, the inequality above implies $|\phi(z)| \ge |\phi(s)| - q(z, s)M$.

**Lemma 2.2.** *If $f \in \mathscr{G}(\mathscr{A})$ and $z, s \in D$, then*

$$\|f(z)\|_{A\{z\}} \ge \|f(s)\|_{A\{s\}} - q(z, s)\|f\|_{\mathscr{G}}. \tag{2.1}$$

*Proof.* Let us fix $s \in D$. Since $f(s) \in A\{s\}$, it is possible to find $\psi \in A^*\{s\}$ such that $\|\psi\|_{A^*\{s\}} = 1$ and $\psi(f(s)) = \|f(s)\|_{A\{s\}}$.

Given $\varepsilon > 0$, we can find $G \in \mathscr{G}(A^*)$ such that $\|G(s) - \psi\|_{A^*\{s\}} < \varepsilon$ and $\|G\|_{\mathscr{G}(\mathscr{A}^*)} < 1+\varepsilon$. Thus $\phi(z) = G(z)(f(z))$ is analytic in $D$, $\phi \in H^\infty(D)$, and

$$|\phi(z)| \le (1+\varepsilon)\|f(z)\|_{A\{z\}} \le (1+\varepsilon)\|f\|_{\mathscr{G}}.$$

By Lemma 2.1, we have

$$\begin{aligned}\|f(z)\|_{A\{z\}} &\ge \frac{|\phi(z)|}{1+\varepsilon} \ge \frac{|\phi(s)|}{1+\varepsilon} - q(z, s)\|f\|_{\mathscr{G}} \\ &\ge \frac{1-\varepsilon}{1+\varepsilon}\|f(s)\|_{A\{s\}} - q(z, s)\|f\|_{\mathscr{G}}.\end{aligned}$$

Letting $\varepsilon \to 0$, we get the theorem. □

**Theorem 2.3.** *Let $\{v_1, \dots, v_d\} \subset \mathscr{G}(\mathscr{A})$, and assume that $\{v_i(s)\}$ are independent. Denote $b_z = b(v_1(z), \dots, v_d(z))$. Then there exists $\delta > 0$ so that, for all $|z-s| < \delta$, $b_z > b_s/2$. Consequently, if we denote by $U$ the span of $\{v_i\}$ in $\mathscr{G}(\mathscr{A})$ then, for these $z$, $\dim(U_{\{z\}}) = d$.*

*Proof.* Without loss of generality, we can assume that $\|v_i\|_{\mathscr{G}} = 1$. As we consider $v_i(z)$ in $A\{z\}$, $b_z$ will be controlled by inequality (2.1), which will prove the theorem:

$$\left\|\sum_{i=1}^d c_i v_i(z)\right\|_{A\{z\}} \ge \left\|\sum_{i=1}^d c_i v_i(s)\right\|_{A\{s\}} - q(z, s)\left\|\sum_{i=1}^d c_i v_i\right\|_{\mathscr{G}}.$$

Since $\{v_i(s)\}$ are independent, we have $b_s > 0$. Suppose $\max\{|c_i|\} = 1$; then

$$b_s \le \left\| \sum_{i=1}^{d} c_i v_i(s) \right\|_{A\{s\}} \quad \text{and} \quad \left\| \sum_{i=1}^{d} c_i v_i \right\|_{\mathscr{G}} \le \sum_{i=1}^{d} |c_i| \cdot \|v_i\|_{\mathscr{G}} \le d.$$

Hence,

$$\left\| \sum_{i=1}^{d} c_i v_i(z) \right\|_{A\{z\}} \ge b_s - dq(z, s).$$

We can choose $\delta$ small enough such that $|z - s| < \delta$ implies

$$b_s - dq(z, s) > b_s/2.$$

It follows that $b_z > b_s/2$. Therefore, if $|z - s| < \delta$, $\{v_i(z)\}$ are independent and $\dim(U_{\{z\}}) = d$. □

**Theorem 2.4.** *Let $\{v_1, \dots, v_d\} \subset \mathscr{G}(\mathscr{A})$, and assume that $\{v_i(s)\}$ are independent. Denote by $U$ the span of $\{v_i\}$ in $\mathscr{G}(\mathscr{A})$. Then $\exists \delta > 0$, such that $|z - s| < \delta$ implies $U$ is a $z$-Calderón space with a uniform constant $k$.*

*Proof.* Without loss of generality, we can assume that $\|v_i(s)\|_{A\{s\}} = 1$. Let $m = \max\{\|v_i\|_{\mathscr{G}}\}$ and $b_z = b(v_1(z), \dots, v_d(z))$. Then, as we have seen before, if $b_z > 0$, we have

$$|c_i| \le \frac{\| \sum_{i=1}^{d} c_i v_i(z) \|_{A\{z\}}}{b_z}$$

so that

$$\left\| \sum_{i=1}^{d} c_i v_i \right\|_{\mathscr{G}} \le \frac{md}{b_z} \left\| \sum_{i=1}^{d} c_i v_i(z) \right\|_{A\{z\}}.$$

Since $b_s > 0$, by Theorem 2.3, $\exists \delta > 0$ such that $|z - s| < \delta$ implies that $b_z > b_s/2$ and $U$ is a $z$-Calderón subspace with constant $2md/b_s$. □

**Theorem 2.5.** *Let $U(A)$, $V(A)$ be two $z$-Calderón subspaces with constant $k$, for all $z$ in a ball $S$ which is contained in $D$. Then the function*

$$\chi_z = \inf\{d(f(z), U_{\{z\}}) | f \in V(A) \text{ and } \|f(z)\|_{A\{z\}} = 1\}$$

*is continuous in $S$.*

*Proof.* Given $0 < \varepsilon < 1$ and $s \in S$, there are $v \in V(A)$, $u \in U(A)$ so that

$$1 = \|v(s)\|_{A\{s\}} \le \|v\|_{\mathscr{G}} \le k\|v(s)\|_{A\{s\}} = k$$
$$\|u(s) - v(s)\|_{A\{s\}} < \chi_s + \varepsilon/2$$

with

$$\|u\|_{\mathscr{G}} \le k\|u(s)\|_{A\{s\}}.$$

Therefore

$$\|u\|_{\mathscr{G}} \le k(\|u(s) - v(s)\|_{A\{s\}} + \|v(s)\|_{A\{s\}}) < k(\chi_s + 2).$$

Clearly we have

$$\chi_z \leq \frac{\|u(z) - v(z)\|_{A\{z\}}}{\|v(z)\|_{A\{z\}}}.$$

By Lemma 2.2, we have

$$\|v(z)\|_{A\{z\}} \geq \|v(s)\|_{A\{s\}} - q(z, s)\|v\|_{\mathscr{G}} = 1 - q(z, s)\|v\|_{\mathscr{G}}.$$

Hence we can choose $\eta > 0$ small enough such that $|z - s| < \eta$ implies both $\|v(z)\|_z^{-1} < 1 + \varepsilon$ and $z \in S$. Again, invoking Lemma 2.2, we have

$$\begin{aligned}\|u(z) - v(z)\|_{A\{z\}} &\leq \|u(s) - v(s)\|_{A\{s\}} + q(z, s)\|u - v\|_{\mathscr{G}} \\ &\leq \|u(s) - v(s)\|_{A\{s\}} + q(z, s)(\|u\|_{\mathscr{G}} + \|v\|_{\mathscr{G}}) \\ &< (\chi_s + \varepsilon/2) + q(z, s)(\chi_s + 3)k.\end{aligned}$$

Therefore, if necessary, choosing a smaller $\eta > 0$, we have that, for $|z - s| < \eta$,

$$\chi_z \leq \frac{\|u(z) - v(z)\|_{A\{z\}}}{\|v(z)\|_{A\{z\}}} \leq (1 + \varepsilon)(\chi_s + \varepsilon).$$

Similarly $\chi_s \leq (1 + \varepsilon)(\chi_z + \varepsilon)$ in $|z - s| < \eta_1$, and hence $\chi_z$ is continuous in $S$. □

Note that when $V(A)$ is finite dimensional, as it is in the proof below, $\chi_z$ is equivalent to the function $\rho$ defined in the proof of Lemma 1.6.

**Theorem 2.6.** *Let $U$ be a $z$-Calderón subspace with constant $k$, for all $z$ in a ball $S$ contained in $D$. Assume that, for an $s \in S$, $U_{\{s\}}$ has codimension $d < \infty$. Then there exists $\delta > 0$, so that, for all $z$ with $|z - s| < \delta$, we have* $\operatorname{codim} U_{\{z\}} = d$.

*Proof.* By Lemma 1.6, there exist $\{v_i | i = 1, \dots, d\} \subset \mathscr{G}(\mathscr{A})$ so that $\{v_i(s)\}$ forms a basis for $M_s$ and $A\{s\} = U_{\{s\}} \oplus M_s$. We can assume $\|v_i\|_{\mathscr{G}} = 1$. Denote by $M_z$ the space spanned by $\{v_i(z)\}$. Since $A\{s\} = U_{\{s\}} \oplus M_s$ and $U_{\{s\}}$, $M_s$ are closed, there exists a projection $P$ of $A\{s\}$ onto $U_{\{s\}}$ with $\ker P = M_s$, and $\|P\| < \infty$.

Since $U$ is a $z$-Calderón subspace, $U_{\{z\}}$ is a closed subspace of $A\{z\}$. By Lemma 1.7, since $M_z$ is finite-dimensional, $U_{\{z\}} + M_z$ is closed. Hence, if $A\{z\} \neq U_{\{z\}} + M_z$ then there exists $e \in \mathscr{G}(\mathscr{A})$, with $\|e(z)\|_{A\{z\}} = 1$ and $1 \leq \|e\|_{\mathscr{G}(\mathscr{A})} < 2$, such that $\operatorname{dist}(e(z), U_{\{z\}} + M_z) > 2/3$.

Consider $e(s) \in A\{s\}$. We have $e(s) = y_s + z_s$, where $y_s = Pe(s) \in U_{\{s\}}$, $z_s = (I - P)e(s) = \sum_{i=1}^d c_i v_i(s) \in M_s$. We have

$$\begin{aligned}\|y_s\|_{A\{s\}} &\leq \|P\|\,\|e(s)\|_{A\{s\}} < 2\|P\| \\ \|z_s\|_{A\{s\}} &\leq (\|P\| + 1)\|e(s)\|_{A\{s\}} < 2(\|P\| + 1).\end{aligned}$$

Since $U$ is $s$-Calderón, there exists $f \in U(A)$ such that $\|f(s) - y_s\|_{A\{s\}} \leq \min\{1/3, \|y_s\|_{A\{s\}}\}$ and

$$\|f\|_{\mathscr{G}} \leq k\|f(s)\|_{A\{s\}} \leq 2k\|y_s\|_{A\{s\}} < 4k\|P\|. \tag{2.2}$$

On the other hand we have, denoting $b = b(v_1(s), \dots, v_d(s))$,

$$|c_i| \le \frac{\|\sum_{i=1}^d c_i v_i(s)\|_{A\{s\}}}{b} = \frac{\|z_s\|_{A\{s\}}}{b} < 2\frac{\|P\|+1}{b}.$$

It follows that

$$\left\|\sum_{i=1}^d c_i v_i\right\|_{\mathscr{G}} \le \sum_{i=1}^d |c_i|\,\|v_i\|_{\mathscr{G}} \le \frac{2d(\|P\|+1)}{b}. \tag{2.3}$$

Let $g = f + \sum_{i=1}^d c_i v_i$. Clearly $\|g(s) - e(s)\|_{A\{s\}} < 1/3$ and

$$\begin{aligned}\|g - e\|_{\mathscr{G}} &\le \|f\|_{\mathscr{G}} + \left\|\sum_{i=1}^d c_i v_i\right\|_{\mathscr{G}} + \|e\|_{\mathscr{G}} \\ &\le 4k\|P\| + \frac{2d(\|P\|+1)}{b} + 2.\end{aligned}$$

Choose $\delta$ such that $|z - s| < \delta$ implies

$$q(z, s)\left[4k\|P\| + \frac{2d(\|P\|+1)}{b} + 2\right] < 1/3.$$

Then, by Lemma 2.2,

$$\begin{aligned}\|e(z) - g(z)\|_{A\{z\}} &\le \|e(s) - g(s)\|_{A\{s\}} + q(z, s)\|e - g\|_{\mathscr{G}} \\ &\le 1/3 + q(z, s)\left[4k\|P\| + \frac{2d(\|P\|+1)}{b} + 2\right] < 2/3\end{aligned}$$

which contradicts $\operatorname{dist}(e(z), U_{\{z\}} + M_z) > 2/3$. Therefore $U_{\{z\}} + M_z = A\{z\}$.

Since $A\{s\} = U_{\{s\}} \oplus M_s$, $U_{\{s\}}$ is closed and $M_s$ has finite dimension, we have $\chi_s > 0$. By Theorem 2.5, there exists $\delta > 0$ such that $|z - s| < \delta$ implies $\chi_z > 0$, which implies $U_{\{z\}} \cap M_z = \{0\}$. Hence the codimension of $U_{\{z\}}$ is $d$. □

**Theorem 2.7.** *If* $\ker T_s = \{0\}$ *and* $\operatorname{codim}(T_s A\{s\}) = d < \infty$, *then there exists* $\delta > 0$, *so that, for all* $|z - s| < \delta$, $\ker T_z = \{0\}$ *and* $\operatorname{codim} T_z A\{z\} = d$.

*Proof*. The control over $\ker T_z$ is achieved by considering the following function:

$$r(T_z) = \inf\{\|Ta\|_{B\{z\}} \mid \|a\|_{A\{z\}} = 1\}.$$

Clearly if $r(T_z) > 0$, $\ker T_z = \{0\}$ and the range of $T_z$ is closed. Conversely, if the range of $T_z$ is closed and $\ker T_z = \{0\}$ then, by the open mapping theorem, $r(T_z) = \|T_z^{-1}\|^{-1} > 0$. Since $\operatorname{codim} T_s = d < \infty$, the range of $T_s$ is closed, and, since $\ker T_s = \{0\}$, we have $r(T_s) > 0$. Theorem 2.3 in [11] says that $r(T_z) > r(T_s)/2 > 0$ for all $z$ which satisfy $|z - s| < \delta$, some $\delta > 0$.

Since $\|T\|_{A(\gamma)\to B(\gamma)} \le M(\gamma)$ with $\int_\Gamma |\log M(\gamma)| dP_z(\gamma) < \infty$, by Lemma 1.8, there exists an outer function $G(\cdot)$ such that

$$|G(\gamma)M(\gamma)| = 1 \quad \text{for a.e. } \gamma \in \Gamma. \tag{2.4}$$

We claim that $G(\cdot)T\mathscr{G}(\mathscr{A})$ is a $z$-Calderón subspace of $\mathscr{G}(\mathscr{B})$ with constant $c$, for all $z$ such that $|z-s|<\delta$, where

$$c = 4\|T_s^{-1}\| \sup_{\eta} \left\{ \exp\left( \int \log M(\gamma)\, dP_\eta(\gamma) \right) \middle| \ |\eta - s| < \delta \right\}.$$

In fact, let $y \in G(\cdot)T\mathscr{G}(\mathscr{A})_z$. Then $y \in T_z\mathscr{A}$, and so there exists $x \in \mathscr{A}$ such that $Tx = y$ and $\|x\|_{A\{z\}} \le \|T_z^{-1}\|\,\|y\|_{B\{z\}}$. We choose $g \in \mathscr{G}(\mathscr{A})$ such that $g(z) = x$ and

$$\|g\|_{\mathscr{G}(\mathscr{A})} \le 2\|x\|_{A\{z\}} \le 2\|T_z^{-1}\|\,\|y\|_{B\{z\}}. \tag{2.5}$$

Then $g(\zeta)/G(z) \in \mathscr{G}(\mathscr{A})$ and $H(\zeta) = G(\zeta)T(g(\zeta)/G(z)) \in G(\cdot)T\mathscr{G}(\mathscr{A})$. Clearly $H(z) = y$, and, by (2.4), (2.5),

$$\|H\|_{\mathscr{G}(\mathscr{B})} \le \frac{2\|T_z^{-1}\|\,\|y\|_{B\{z\}}}{|G(z)|} \le c\|y\|_{B\{z\}}.$$

Therefore $G(\cdot)T\mathscr{G}(\mathscr{A})$ is a $z$-Calderón subspace with constant $c$ for all $z$ such that $|z-s|<\delta$.

It is also clear that

$$(G(\cdot)T\mathscr{G}(A))_{\{z\}} = T_z A\{z\}.$$

Thus, by Theorem 2.6,

$$\begin{aligned} \operatorname{codim} T_z A\{z\} &= \operatorname{codim}(G(\cdot)T\mathscr{G}(\mathscr{A}))_{\{z\}} \\ &= \operatorname{codim}(G(\cdot)T\mathscr{G}(\mathscr{A}))_{\{s\}} \\ &= \operatorname{codim} T_s A\{s\}; \end{aligned}$$

$\ker T_z = \{0\}$, and the theorem is proved. □

**Corollary 2.8.** *If* $\dim\ker T_s = d < \infty$ *and* $T_s$ *is surjective, then there exists* $\delta > 0$, *so that for all* $|z-s|<\delta$, $\dim\ker T_z = d < \infty$ *and* $T_s$ *is surjective.*

*Proof.* Consider $T^* : B^*(\gamma) \to A^*(\gamma)$. Clearly $\operatorname{codim} T_s^* = \dim\ker T_s = d < \infty$, $\dim\ker T_s^* = \operatorname{codim} T_s = 0$ and $\|T^*\|_{B^*(\gamma)\to A^*(\gamma)} = \|T\|_{A(\gamma)\to B(\gamma)}$. By assumption, $\{A_\gamma^*\}$, $\{B_\gamma^*\}$ are interpolation families and $(A\{z\})^* = \{A_\gamma^*\}\{z\}$, $(B\{z\})^* = \{B_\gamma^*\}\{z\}$.

By Theorem 2.7, there exists $\delta > 0$ so that $|z-s|<\delta$ implies $\ker T_z^* = \{0\}$ and $\operatorname{codim} T_z^* = d$. Therefore, for all these $z$ we have $\dim\ker T_z = d$ and $T_z$ is surjective. □

## References


1. E. Albrecht, *Spectral interpolation*, Oper. Theory: Adv. Appl. **14** (1984), 13–37.
2. J. Bergh and J. Löfström, *Interpolation spaces*, Springer-Verlag, New York and Berlin, 1976.
3. A. P. Calderón, *Intermediate spaces and interpolation, the complex method*, Studia Math. **24** (1964), 113–190.
4. W. Cao and Y. Sagher, *Stability of Fredholm properties in interpolation scales*, Ark. für Mat. (to appear).

5. R. Coifman, M. Cwikel, R. Rochberg, Y. Sagher, and G. Weiss, *The complex method for interpolation of operators acting on families of Banach spaces*, Lecture Notes in Math. vol. 779, Springer-Verlag, Berlin, Heidelberg, and New York, 1980, 123–153.

6. ——, *A theory of complex interpolation for families of Banach spaces*, Adv. in Math. **33** (1982), 203–229.

7. P. L. Duren, *Theory of $H^p$ spaces*, Academic Press, New York and London, 1970.

8. J. Lindenstraus and L. Tzafriri, *Classical Banach spaces*, Springer-Verlag, Berlin, Heidelberg and New York, 1977.

9. I. Ya. Sneiberg, *Spectral properties of linear operators in interpolation families of Banach spaces*, Mat. Issled. **9** (1974), 214–229.

10. G. Szegö, *Über die Randwerte einer analytischen Funktion*, Math. Ann. **84** (1921), 482–492.

11. A. T. Vignati and M. Vignati, *Spectral theory and complex interpolation*, J. Funct. Anal. **80** (1988), 383–397.

12. M. Zafran, *Spectral theory and interpolation of operators*, J. Funct. Anal. **36** (1980), 185–204.

Department of Mathematics, University of Illinois at Chicago, Chicago, Illinois 60680