REPRESENTATIONS OF $SO(k, \mathbb{C})$ ON HARMONIC POLYNOMIALS ON A NULL CONE

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Abstract. The linear action of the group $SO(k, \mathbb{C})$ on the vector space $\mathbb{C}^{n \times k}$ extends to an action on the algebra of polynomials on $\mathbb{C}^{n \times k}$. The polynomials that are fixed under this action are called $SO(k, \mathbb{C})$-invariant. The $SO(k, \mathbb{C})$-harmonic polynomials are common solutions of the $SO(k, \mathbb{C})$-invariant differential operators. The ideal of all $SO(k, \mathbb{C})$-invariants without constant terms, the null cone of this ideal, and the orbits of $SO(k, \mathbb{C})$ on this null cone are studied in great detail. All irreducible holomorphic representations of $SO(k, \mathbb{C})$ are concretely realized on the space of $SO(k, \mathbb{C})$-harmonic polynomials.

1. Introduction

Let $G$ be a linear algebraic reductive subgroup of the group $GL(E)$ of all invertible linear transformations on a finite dimensional complex vector space $E$. If $S(E^*)$ is the symmetric algebra of all polynomial functions on $E$ then the action of $G$ on $E$ induces an action of $G$ on $S(E^*)$, denoted by $g \cdot p$, for $g \in G$ and $p \in S(E^*)$. We say that $p \in S(E^*)$ is $G$-invariant if $g \cdot p = p$ for all $g \in G$. The $G$-invariant polynomial functions form a subalgebra $J(E^*)$ of $S(E^*)$. Given $X \in E$, let $\partial_X$ denote the differential operator defined by

$$[\partial_X f](Y) = \left\{ \frac{d}{dt} f(Y + tX) \right\}_{t=0}, \quad t \in \mathbb{R},$$

for all smooth functions $f$ on $E$. The map $X \mapsto \partial_X$ induces an isomorphism of the algebra $S(E^*)$ onto the algebra of all differential operators on $E$ with constant coefficients. The image of an element $p \in S(E^*)$ under this isomorphism is denoted by $p(D)$. If $J_+(E^*)$ is the subset of all elements in $J(E^*)$ without constant term, then an element $f \in S(E^*)$ is said to be $G$-harmonic if $p(D)f = 0$ for all $p \in J_+(E^*)$. The subspace of all $G$-harmonic polynomial functions in $S(E^*)$ is denoted by $H(E^*)$. The study of $H(E^*)$ and the decomposition $S(E^*) = J(E^*)H(E^*)$ (the "separation of variable" theorem) was...
initiated by H. Maass in [M1] and [M2] and was extensively developed by S.
Helgason in [H1] and by B. Kostant in [K]. Several authors have investigated
the representation theory for specific types of Lie groups $G$ on $H(E^*)$. A non-
exhaustive list of publications on this subject includes [L], [G], [S], [T1], [T2],
[K-O], [K-V], [G-K], and [G-P-R].

It was shown in [T1] that, up to isomorphism, all irreducible holomorphic
representations of a Lie group $G$ of type $B_l$ or $D_l$ can be concretely realized
as $G$-submodules of $H(E^*)$ except for the case of the “mirror-conjugate repre-
sentations” of $D_l$ which was left unsettled (see [Z, Chapter XVI, §114] for the
definition of the “mirror-conjugate representations”). In this paper we will settle
this special case in conjunction with the description, in both cases $B_l$ and $D_l$,
of the ideal $J_+(E^*)S(E^*)$ and a detailed description of the orbit structure of
the $G$ action in the null cone $P$ of the common zeros of polynomial functions
in $J_+(E^*)S(E^*)$.

2. Description of the ideal $J_+(E^*)S(E^*)$ and its null cone

In this article $E$ denotes $C^{n \times k}$ and $G$ is $SO(k, C)$. Then $G$ acts linearly
on $E$ by right multiplication and leaves the nondegenerate symmetric bilinear
form $(X, Y) \to \text{tr}(XY^t)$, $X, Y \in E$, invariant. It follows that the function
$X^*$ defined by $X^*(Y) = \text{tr}(XY^t)$ is an element of $E^*$, the dual of $E$. It was
shown in [T1] that the algebra $S(E^*)$ can be equipped with the inner product

$$\langle p_1, p_2 \rangle = p_1(D)p_2(\overline{X})|_{X = 0}, \quad p_1, p_2 \in S(E^*),$$

which is invariant under the restriction of the action of $G$ to $G_0 = SO(k)$.

A slight modification of the techniques in [H2, Chapter III], leads to the
following results concerning $S(E^*)$ and $H(E^*)$.

The algebra $S(E^*)$ is decomposed into an orthogonal direct sum with respect
to the inner product given above as $S(E^*) = J_+(E^*)S(E^*) \oplus H(E^*)$. If $H_1(E^*)$
denotes the subspace of $H(E^*)$ spanned by the polynomial functions of the
form $(X^*)^m$, $X \in P$, $m = 0, 1, 2, \ldots$ and if $H_2(E^*)$ denotes the subspace
of $H(E^*)$ of all polynomial functions which vanish on $P$ then we have the
orthogonal direct sum decomposition $H(E^*) = H_1(E^*) \oplus H_2(E^*)$. Moreover,
the linear subspace $J_+(E^*)S(E^*) \oplus H_2(E^*)$ is the ideal in $S(E^*)$ of all polynomial
functions which vanish on $P$, i.e., the ideal $\sqrt{J_+(E^*)S(E^*)}$.

We will now study this ideal $J_+(E^*)S(E^*)$. Recall ([W], [D-P, Theorem
5.6ii]) that it is generated by the $n(n + 1)/2$ polynomials

$$p_{ij}(X) = \sum_{s=1}^{k} X_{is} X_{js}, \quad 1 \leq i \leq j \leq n,$$

together with the $(k \times k)$-minors of the matrix $X$ (which are 0 when $k > n$). We also derive geometric properties of the null cone $P$ of $E$ defined by
$J_+(E^*)S(E^*)$. 

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The following theorem sums up our results, which extend earlier results of [T1] (case $k > 2n$) and [H] (case $k$ odd, $k < 2n$).

**Theorem 2.1**

(i) For $k > 2n$, the ideal $J_+(E^*)S(E^*)$ is prime. The scheme $P$ is a complete intersection, with one open dense orbit.

(ii) For $k = 2n$, the ideal $J_+(E^*)S(E^*)$ is the intersection of two prime ideals, hence it is radical. The scheme $P$ is a complete intersection, with two open orbits.

(iii) For $k < 2n$, the ideal $J_+(E^*)S(E^*)$ is not radical, except for $k \leq 2$. The orbits are nowhere dense, except for $k = 1$. For $k$ odd $>1$, $P$ is irreducible and nowhere reduced. For $k$ even, $P$ has two irreducible components and is generically reduced (but not reduced except for $k = 2$).

**Proof.** As a set, the scheme $P$ is $\{X \in \mathbb{C}^{n \times k} | XX^t = 0 \text{ and } \text{Rank}(X) < k\}$. Therefore, a matrix $X$ is in $P$ if and only if the image $\Lambda(X)$ of the morphism $X^t : \mathbb{C}^n \to \mathbb{C}^k$ is totally isotropic for the quadratic form $\sum_{1 \leq i \leq k} Y_i^2$ (such a space is automatically of dimension $\leq k/2$). The space $\Sigma_{r,k}$ of all $r$-dimensional totally isotropic spaces for a non-degenerate quadratic form has a long history. We borrow the following facts from [G-H, pp. 735–739]:

(a) $\Sigma_{r,k}$ is empty for $r > k/2$.

(b) $\Sigma_{r,k}$ has dimension $r(k - (3r + 1)/2)$ if $r \leq k/2$. It is irreducible for $r < k/2$ but has two irreducible components for $k = 2r$.

Now, by Witt's theorem [A, Chapter III], two elements $X_1$ and $X_2$ of $P$ are in the same $G$-orbit if and only if $\Lambda(X_1)$ and $\Lambda(X_2)$ are of the same rank $r$ and are in the same component of $\Sigma_{r,k}$. We get the following description of the subspaces $P_r$ of $P$ consisting of matrices of rank $r$:

(2.1) $P_r$ is empty for $r > n$ or $r > k/2$. For $r \leq n$ and $r \leq k/2$, it has dimension

$$\dim\{\text{surjections } \mathbb{C}^n \to \mathbb{C}^r\} + \dim \Sigma_{r,k} = r(n + k - (3r + 1)/2).$$

(2.2) For $r < k/2$ and $r \leq n$, $P_r$ is covered by the following (non-disjoint) $G$-orbits: where $A$ is any $(n - r) \times r$ matrix.

$$r \begin{cases} \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & i \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \begin{bmatrix} i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & i \end{bmatrix} = \begin{bmatrix} 1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 1 \end{bmatrix} \end{cases} \cdot G.$$
(2.3) For \( k = 2r \) and \( r \leq n \), \( P_r \) is covered by the following two families of (non-disjoint) \( G \)-orbits:

\[
\begin{bmatrix}
1 & \cdots & i \\
\vdots & \ddots & \vdots \\
A & & iA
\end{bmatrix} \cdot G
\]

and

\[
\begin{bmatrix}
1 & \cdots & i \\
\vdots & \ddots & \vdots \\
A & & A'
\end{bmatrix} \cdot G
\]

where, again, \( A \) is any \((n-r) \times r\) matrix and \( A' \) is the matrix obtained by switching signs in the last column of \( iA \).

It follows easily from (2.2) that \( P_r \) is contained in the closure of \( P_{r+1} \) whenever the latter is non-empty. This yields the following geometric description of \( P \):

(2.4) For \( k \geq 2n \), the maximal \( r \) for which \( P_r \) is non-empty is \( n \). The stratum \( P_n \) is dense in \( P \), which therefore has dimension \( nk - n(n + 1)/2 \) or codimension \( n(n + 1)/2 \) in \( E \) (by (2.1)). It follows moreover, from (2.2) and (2.3), that

(a) For \( k > 2n \), \( P_n \) is just one orbit.

\[
\begin{bmatrix}
1 & \cdots & i \\
\vdots & \ddots & \vdots \\
1 & & i
\end{bmatrix} \cdot G
\]

(b) For \( k = 2n \), \( P_n \) is the union of 2 orbits:

\[
P_n^+ = \begin{bmatrix}
1 & \cdots & i \\
\vdots & \ddots & \vdots \\
1 & & i
\end{bmatrix} \cdot G
\]

and

\[
P_n^- = \begin{bmatrix}
1 & \cdots & i \\
\vdots & \ddots & \vdots \\
1 & & -i
\end{bmatrix} \cdot G.
\]
(2.5) For $k < 2n$, write $k = 2k' + \epsilon$ with $\epsilon = 0$ or $1$, i.e., $k'$ is the rank of the group $SO(k, C)$. The maximal $r$ for which $P_r$ is non-empty is $k'$. The stratum $P_{k'}$ is dense in $P$, which therefore has dimension $k'(n + k - (3k' + 1)/2)$ or codimension $k'n - k'(k' - 1)/2 + (n - k')\epsilon$ in $E$ (by (2.1)). Moreover, by (2.2) and (2.3), $P$ has 1 irreducible component when $k$ is odd, 2 when $k$ is even. As suspected in [T1, Remark 2.10 (1)], $P_{k'}$ contains, in general, infinitely many orbits. More precisely, each such orbit is determined by the $(n - k')$-dimensional linear subspace $\text{Ker} X'$ of $C^n$, hence $P_{k'}/G$ is isomorphic to the $k'(n - k')$-dimensional Grassmannian $G(k', n - k')$ and is infinite for $k > 1$.

We now turn our attention to the scheme structure of $P$, given by the ideal $I = J_+(E^*)S(E^*)$.

Case $k \geq 2n$. By (2.4), $P$ has codimension equal to the number of its defining equations. In other words, $P$ is the complete intersection of the $p_{ij}$'s. Since the Unmixedness Theorem holds in the polynomial ring $S(E^*)$ [M, p. 107 and Theorem 31, p. 108], to check that $P$ is reduced (i.e., that $I$ is radical), it is enough to find one point on each component of $P$ at which $P$ is reduced. But this follows from [T1, Lemma 2.9], where it is shown, using the Jacobian criterion, that $P$ is smooth on the dense stratum $P_n$.

Case $k < 2n$. The situation is very different. Recall that we wrote $k = 2k' + \epsilon$, where $\epsilon = 0$ or $1$. By (2.5), the codimension of $P$ is $(k'n - k'(k' - 1)/2 + (n - k')\epsilon)$, which is strictly less than the number of its defining equations, except for $k = 2n - 1 > 1$. Except for this case, where $P$ is still the complete intersection of the $p_{ij}$'s and $I$ is primary, $P$ may well have embedded primes (and it does, at least for $k$ even $> 2$).

The same argument used in the above mentioned Lemma 2.9 of [T1] shows that, on the dense stratum $P_{k'}$, the rank of the Jacobian matrix of the $p_{ij}$'s is $(k'n - k'(k' - 1)/2)$. If $k' < k - 1$, that is if $k > 2$, the derivatives of the $(k \times k)$-minors are $0$ on $P_{k'}$. It follows that, for $k$ odd $> 1$, $P$ is nowhere reduced, hence $I$ is not radical but that, for $k$ even, $P$ is generically smooth!

This is therefore not enough to conclude in the case $k$ even and we will use a direct computation, based on (2.1) only, to show that $I$ is not radical whenever $2 < k < 2n$.

Recall that any element of $P$ has rank $\leq k'$. Thus, if $X' = (X_{ij})_{1 \leq i, j \leq k' + 1}$, $(\det X')$ vanishes on $P$, hence is in the radical of $I$ (recall that $k' + 1 \leq n$). Suppose $(\det X') \in I$. Substitute $0$ for all the $X_{ij}$'s except for $X_{12}$, $X_{22}$, and $X_{ii}$ for $1 \leq i \leq k' + 1$. When $k > k' + 1$, that is when $k > 2$, all the $(k \times k)$-minors of $X$ vanish and the only non-zero $p_{ij}$'s left are $p_{11} = X_{11}^2 + X_{12}^2$, $p_{12} = X_{11}X_{21} + X_{12}X_{22}$, $p_{22} = X_{21}^2 + X_{22}^2$ and $p_{ii} = X_{ii}^2$ for $3 \leq i \leq k' + 1$. 

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Modulo $(X_{33}^2, \ldots, X_{k'+1,k'+1}^2)$, we get a congruence of the type
\[
(X_{11}X_{22} - X_{12}X_{21})X_{33} \cdots X_{k'+1,k'+1} = \sum_{1 \leq i < j \leq 2} q_{ij}p_{ij}.
\]

By comparing coefficients of $X_{33} \cdots X_{k'+1,k'+1}$, we get $X_{11}X_{22} - X_{12}X_{21} \in (p_{11}, p_{12}, p_{22})$, which is untrue. Therefore, $(\det X')$ is not in $I$ when $k \geq 3$, and $I$ is not radical. Direct calculations show that, for $k = 1$, $I = (X_{11}, \ldots, X_{n1})$ is prime and that, for $k = 2$, $I$ is radical. $\square$

It follows immediately from Theorem 2.1 and the remarks made earlier that, for the case $k \geq 2n$, the ideal $J_+(E^*)S(E^*)$ is radical (prime when $k > 2n$) and hence $H_2(E^*) = \{0\}$. Thus we have the following result which extends an earlier result in [T1, Theorem 2.5].

**Corollary 2.2** ("Separation of variables" theorem for $S(E^*)$, $E = C^{n \times k}$, $k \geq 2n$).

(i) The algebra of all polynomial functions on $C^{n \times k}$, $k \geq 2n$, can be decomposed as
\[
S(E^*) = J_+(E^*)S(E^*) \oplus H(E^*)(\text{orthogonal direct sum})
\]
\[
S(E^*) = J(E^*)H(E^*).
\]

(ii) The space $H(E^*)$ is generated by all powers of all polynomial functions $f$ satisfying
\[
f(X) = \sum_{i,j} A_{ij}X_{ij} \text{ with } AA^t = 0, \quad 1 \leq i \leq n, \quad 1 \leq j \leq k.
\]

(iii) The space $H_2(E^*)$ of all $G$-harmonic polynomial functions that vanish on the null cone $P$ of the common zeros of polynomial functions in $J_+(E^*)S(E^*)$ is zero.

For the case $k = 2n$, Corollary 2.2 will play a crucial role in the proof of Theorem 3.1 of the next section.

3. **The harmonic representations of $SO(k, C)$**

In [T1] it was shown that all irreducible holomorphic representations of $G = SO(k, C)$ can be explicitly realized as $G$-submodules of $H(E^*)$ with the exception of the "mirror-conjugate representations" of $G$. In this section we will show that these representations too can be realized on $H(E^*)$ and have a very interesting characterization on the orbits of $G$ in $P$, which was studied in Theorem 2.1. Since "the mirror-conjugate representations" can only occur when $k$ is even we shall assume henceforth that $k = 2n$. 
Let 

\[ \gamma = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \cdots & 1 \\ -i & \cdots & -i \\ \vdots & \ddots & \vdots \\ -i & \cdots & i \\ 1 & \cdots & -i \end{bmatrix} \]

Then

\[ \gamma^{-1} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & \cdots & i \\ \vdots & \ddots & \vdots \\ 1 & \cdots & -i \\ -i & \cdots & i \end{bmatrix}, \]

where the entries not exhibited are 0. Then

\[ (\gamma^{-1})(\gamma^{-1})' = \begin{bmatrix} 1 \\ \vdots \\ 1 \end{bmatrix} = \sigma. \]

If \( \tilde{G} = \gamma^{-1} G \gamma \), then it can be easily verified that

\[ \tilde{G} = \{ \tilde{g} \in \text{GL}(2n, \mathbb{C}) : \tilde{g} \sigma(\tilde{g})' = \sigma, \det \tilde{g} = 1 \}. \]

In general we shall denote, by \( \tilde{g} \), the image of \( g \) under the conjugation \( g \to \gamma^{-1} g \gamma \), and vice versa. It follows that \( \tilde{G} \) is the connected component of the identity in the group of linear transformations which preserve the symmetric bilinear form \( \text{tr}(x \sigma y') = x_1 y_{2n} + x_2 y_{2n-1} + \cdots + x_{2n} y_1 \), for all \( x = (x_1, \ldots, x_{2n}) \) and \( y = (y_1, \ldots, y_{2n}) \) in \( \mathbb{C}^{2n} \). It is well-known (cf. [Z, Chapter XVI, §114]) that \( \tilde{G} \) has the Gauss decomposition induced by \( \text{GL}(2n, \mathbb{C}) \)

\[ (3.1) \quad \tilde{G} = \tilde{Z}_- \tilde{D} \tilde{Z}_+, \]

where the components \( \tilde{Z}_- \), \( \tilde{D} \), \( \tilde{Z}_+ \) are the intersections of \( \tilde{G} \) with the subgroups \( \tilde{Z}_-(2n) \), \( \tilde{D}(2n) \), and \( \tilde{Z}_+(2n) \) of all lower triangular unipotent, of all diagonal, and of all upper triangular unipotent matrices of \( \text{GL}(2n, \mathbb{C}) \), respectively. It follows that \( \tilde{B} = \tilde{Z}_- \tilde{D} \) is a Borel subgroup of \( \tilde{G} \) which consists of all lower triangular matrices of the form

\[ (3.2) \quad \tilde{b} = \begin{bmatrix} b_{11} & \cdots & b_{nn} \\ & \ddots & \vdots \\ & & b_{nn}^{-1} \end{bmatrix}. \]
Let \((m)^+\) denote an \(n\)-tuple of positive integers \((m_1, m_2, \ldots, m_n)\) satisfying the condition \(m_1 \geq m_2 \geq \cdots \geq m_n > 0\), and set \((m)^- = (m_1, \ldots, m_{n-1}, -m_n)\). Define the holomorphic characters \(\pi^{(m)^+}\) and \(\pi^{(m)^-}\) on \(\tilde{B}\) by
\[
\pi^{(m)^+}(\tilde{b}) = b_{11}^{m_1} \cdots b_{nn}^{m_n} \quad \text{and} \quad \pi^{(m)^-}(\tilde{b}) = b_{11}^{m_1} \cdots b_{nn}^{-m_n}
\]
for all \(\tilde{b} \in \tilde{B}\). Set
\[
\mathcal{V}^{(m)^+} = \{f : \tilde{G} \rightarrow \mathbb{C} : f \text{ holomorphic and } f(\tilde{b}\tilde{g}) = \pi^{(m)^+}(\tilde{b})f(\tilde{g}) \quad \forall(\tilde{b}, \tilde{g}) \in \tilde{B} \times \tilde{G}\}
\]
and
\[
\mathcal{V}^{(m)^-} = \{f : \tilde{G} \rightarrow \mathbb{C} : f \text{ holomorphic and } f(\tilde{b}\tilde{g}) = \pi^{(m)^-}(\tilde{b})f(\tilde{g}) \quad \forall(\tilde{b}, \tilde{g}) \in \tilde{B} \times \tilde{G}\}.
\]
Let \(\tilde{R}_\pi^{(m)^+}\) (respectively \(\tilde{R}_\pi^{(m)^-}\)) denote the representation of \(\tilde{G}\) obtained by right translation on \(\mathcal{V}^{(m)^+}\) (respectively \(\mathcal{V}^{(m)^-}\)). Then, by the Borel-Weil theorem, \(\tilde{R}_\pi^{(m)^+}\) (respectively \(\tilde{R}_\pi^{(m)^-}\)) is irreducible with signature \((m)^+\) (respectively \((m)^-\)). These representations are termed “mirror-conjugate representations” of \(G\) in [Z, Chapter XVI]. Moreover, if \(\tilde{g} \in \tilde{G}\) then, in the Gauss decomposition of \(\tilde{G}\), \(\tilde{g} = \tilde{b}[\tilde{g}]\tilde{z}[\tilde{g}]\) with \(\tilde{b}[\tilde{g}] \in \tilde{B}\) and \(\tilde{z}[\tilde{g}] \in \tilde{Z}_+\), and \((\tilde{b}[\tilde{g}])_{ii} = \Delta_i(\tilde{g})/\Delta_{i-1}(\tilde{g})\), where \(\Delta_i(\tilde{g})\) is the \(i\)th principal minor of \(\tilde{g}\), \(\Delta_0(\tilde{g}) = 1\), \(1 \leq i \leq n\), so that the highest weight vector of \(\mathcal{V}^{(m)^+}\) (respectively \(\mathcal{V}^{(m)^-}\)) is given by
\[
f^{(m)^+}(\tilde{g}) = \pi^{(m)^+}(\tilde{b}[\tilde{g}]) = \Delta_1^{-m_1}(\tilde{g})\Delta_2^{-m_2}(\tilde{g})\cdots\Delta_n^{-m_n}(\tilde{g})
\]
and
\[
f^{(m)^-}(\tilde{g}) = \pi^{(m)^-}(\tilde{b}[\tilde{g}]) = \Delta_1^{-m_1}(\tilde{g})\Delta_2^{-m_2}(\tilde{g})\cdots\Delta_n^{-m_n}(\tilde{g})
\]
for all \(\tilde{g} \in \tilde{G}\).

For the same \(n\)-tuple, \((m_1, m_2, \ldots, m_n) = (m)^+\), define a holomorphic character of \(B \equiv B(n)\), the lower triangular subgroup of \(\text{GL}(n, \mathbb{C})\), by
\[
\xi^{(m)}(b) = b_{11}^{m_1} \cdots b_{nn}^{m_n}, \quad b \in B.
\]
Let \(H(E^*, (m))\) denote the subspace of all \(G\)-harmonic polynomial functions \(p\) which also satisfy the covariant condition
\[
p(bx) = \xi^{(m)}(b)p(X), \quad \forall(b, X) \in B \times E.
\]
Let
\[
X_0^+ = \begin{bmatrix} 1 & \cdots & i \\ \cdot & \ddots & \cdot \\ 1 & \cdots & i \end{bmatrix}
\]
and
\[
X_0^- = \begin{bmatrix}
1 & \cdots & i \\
\cdot & \cdot & \cdot \\
1 & \cdots & i \\
\end{bmatrix}.
\]

Then it follows from Equation (2.7) of Theorem 2.1 that \( P_n \) is the union of two orbits, \( P_n^+ = X_0^+ \cdot G \) and \( P_n^- = X_0^- \cdot G \). Let \( H^+(E^*, (m)) \) (respectively \( H^-(E^*(m)) \)) denote the subspace of all functions in \( H(E^*, (m)) \) which vanish on the orbit \( P_n^- \) (respectively \( P_n^+ \)). Then we have

**Theorem 3.1.** (i) If \( H(E^*, (m)) \) is the subspace of \( H(E^*) \) consisting of all \( G \)-harmonic polynomial functions \( p \) which also satisfy the covariant condition \( p(bX) = \xi((m)(b)p(X) \) for all \( (b, X) \in B \times E \), then \( H(E^*, (m)) \) is decomposed into an orthogonal direct sum as
\[
H(E^*, (m)) = H^+(E^*, (m)) \oplus H^-(E^*, (m)).
\]

(ii) If \( R^m \) denotes the representation of \( G \) obtained by right translation on \( H(E^*, (m)) \), then the restriction \( R^{(m)+} \) (respectively \( R^{(m)-} \)) of \( R^m \) to \( H^+(E^*, (m)) \) (respectively \( H^-(E^*, (m)) \)) is irreducible with signature \( (m)^+ = (m_1, \ldots, m_n) \) (respectively \( (m)^- = (m_1, \ldots, m_{n-1}, -m_n) \)).

**Proof.** We define a representation \( R^m \) on \( V^+(m) \oplus V^-(m) \) by
\[
[R^m(g_0)(f^+ + f^-)](\tilde{g}) = f^+(\tilde{g} g_0) + f^-(\tilde{g} g_0),
\]
for all \( f^+ \in V^+(m) \), \( f^- \in V^-(m) \), \( g_0 \in G \), \( \tilde{g} \in \tilde{G} \). Using the “Weyl’s unitarian trick” (cf. [V, §4.11]), we may equip \( V^+(m) \oplus V^-(m) \) with an inner product which is invariant under the compact real form \( G_0 = SO(k) \) of \( G \), and, using Schur’s orthogonality relations, we can show that since \( V^+(m) \) and \( V^-(m) \) are inequivalent \( G_0 \)-simple modules they form an orthogonal direct sum relative to the \( G_0 \)-invariant inner product. Set
\[
S_0 = 2 \begin{bmatrix}
1 & \cdots & 1 \\
\cdot & \cdot & \cdot \\
1 & \cdots & 1 \\
\end{bmatrix}, \quad \Pi = \begin{bmatrix}
1 & \cdots & 0 \\
\cdot & \cdot & \cdot \\
0 & \cdots & 0 \\
\end{bmatrix}, \quad \begin{bmatrix}
1 & \cdots & 1 \\
\cdot & \cdot & \cdot \\
1 & \cdots & 1 \\
\end{bmatrix}
\]
and define a linear map \( \Lambda : H(E^*, (m)) \to V^+(m) \oplus V^-(m) \) by
\[
\Lambda p = \Lambda^+ p + \Lambda^- p, \quad \text{for all } p \in H(E^*, (m)),
\]
where

\[ \Lambda^+ p(\tilde{g}) = p(\Pi \tilde{g} \gamma^{-1}) \]

and

\[ \Lambda^- p(\tilde{g}) = p(\Pi_0 \tilde{g} \gamma^{-1}) \]

for all \( \tilde{g} \in \tilde{G} \).

Then

\[ \Lambda^+ p(\tilde{g}) = p \left( \frac{1}{\sqrt{2}} X_0^+ g \right) \quad \text{and} \quad \Lambda^- p(\tilde{g}) = p \left( \frac{1}{\sqrt{2}} X_0^- g \right), \]

for all \( g = \gamma \tilde{g} \gamma^{-1} \in G \). Let us verify that \( \Lambda^+ \) (respectively \( \Lambda^- \)) does indeed map \( H(E^*, (m)) \) into \( V^{(m)^+} \) (respectively \( V^{(m)^-} \)). If

\[ b = \begin{bmatrix} b_{11} & \cdots & \cdots \\ \cdot & \ddots & \cdot \\ \cdot & \cdot & b_{nn} \end{bmatrix} \]

is an element of \( \tilde{B} \) then \( \Pi \tilde{b} = b \Pi \) with

\[ b = \begin{bmatrix} b_{11} & \cdots \\ \cdot & \ddots \\ \cdot & \cdot & b_{nn} \end{bmatrix} \]

in \( B(n) \). So

\[ \Lambda^+ p(\tilde{b} \tilde{g}) = p(\Pi \tilde{b} \tilde{g} \gamma^{-1}) = p(b \Pi \tilde{g} \gamma^{-1}) = \xi^{(m)}(b)p(\Pi \tilde{g} \gamma^{-1}) = \pi^{(m)^+}(\tilde{b}) \Lambda^+ p(\tilde{g}) \]

and obviously \( \Lambda^+ p \) is a holomorphic function on \( \tilde{G} \). So \( \Lambda^+ \) maps \( H(E^*, (m)) \) into \( V^{(m)^+} \). Similarly, \( \Pi_0 \tilde{b} = \Pi(s_0 \tilde{b}s_0)s_0 \) since \( s_0^{-1} = s_0 \), and

\[ s_0 \tilde{b}s_0 = \begin{bmatrix} b_{11} & \cdots & \cdots \\ \cdots & b_{n-1,n-1} & \cdots \\ \cdots & \cdots & b_{nn} \end{bmatrix} \]

\[ \Pi_0 \tilde{b}s_0 = b^- \Pi, \text{ with} \]

\[ b^- = \begin{bmatrix} b_{11} & \cdots & \cdots \\ \cdots & b_{n-1,n-1} & \cdots \\ \cdots & \cdots & b_{nn} \end{bmatrix} \]
Therefore,

\[ \Lambda^+ p(\tilde{b} \tilde{g}) = p(\Pi \tilde{s}_0 \tilde{b} \tilde{g} \gamma^{-1}) \]
\[ = p(\Pi (s_0 \tilde{b} s_0) s_0 \tilde{g} \gamma^{-1}) \]
\[ = p(b^{\gamma} \Pi \tilde{s}_0 \tilde{g} \gamma^{-1}) \]
\[ = b_{11}^{m_1} \cdots b_{n-1,n-1}^{m_{n-1}} b_{nn}^{m_n} p(\Pi \tilde{s}_0 \tilde{g} \gamma^{-1}) \]
\[ = \pi^{(m)}(\tilde{b}) \Lambda^+ p(\tilde{g}). \]

So \( \Lambda^- \) maps \( H(E^*, (m)) \) into \( V^{(m)} \). Now \( \Lambda \) is an intertwining operator since

\[ [\Lambda(R^{(m)}(g_0)p)](\tilde{g}) = [R^{(m)}(g_0)p](\Pi \tilde{g} \gamma^{-1}) + [R^{(m)}(g_0)p](\Pi \tilde{s}_0 \tilde{g} \gamma^{-1}) \]
\[ = p(\Pi \tilde{g} \gamma^{-1} g_0) + p(\Pi \tilde{s}_0 \tilde{g} \gamma^{-1} g_0) \]
\[ = p(\Pi \tilde{g} \tilde{g}_0 \gamma^{-1}) + p(\Pi \tilde{s}_0 \tilde{g} \gamma^{-1} g_0) \]
\[ = \Lambda^+ p(\tilde{g} \tilde{g}_0) + \Lambda^+ p(\tilde{g} \tilde{g}_0) \]
\[ = [R^{(m)}(g_0)\Lambda p](\tilde{g}) \]

for all \( g_0 \in G \) and \( \tilde{g} \in \tilde{G} \).

Let \( p_{\xi}(X) = \Delta_1^{m_1-m_2}(X) \Delta_2^{m_2-m_3}(X) \cdots \Delta_{n-1,n-1}^{m_{n-1,m_n}}(X) \Delta_n^{m_n}(X) \) and set \( p_0^{(m)^+}(X) = p_{\xi}(X \gamma), \ p_0^{(m)^-}(X) = p_{\xi}(X \gamma s_0) \). Then an easy computation analogous to the one in Lemma 3.4 of [T1] shows that both \( p_0^{(m)^+} \) and \( p_0^{(m)^-} \) belong to \( H(E^*, (m)) \). Moreover, from Equation (3.4) it follows that

\[ \Lambda^+ p_0^{(m)^+}(\tilde{g}) = p_0^{(m)^+}(\Pi \tilde{g} \gamma^{-1}) = p_{\xi}(\Pi \tilde{g} \gamma^{-1} \gamma) \]
\[ = p_{\xi}(\Pi \tilde{g}) = f(\tilde{m}^+)(\tilde{g}). \]

Now it can be shown that the conjugation \( \tilde{g} \rightarrow s_0 \tilde{g} s_0 \) preserves the Gauss decomposition (see [Z, Chapter XVI, §114]) and, if

\[ \tilde{b}[\tilde{g}] = \begin{bmatrix} b_{11} & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{n-1,n-1}^{m_{n-1}} & b_{nn}^{m_n} & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ \cdots & \cdots & \cdots & \cdots \\ b_{11}^{-1} & \cdots & \cdots & \cdots \end{bmatrix}, \]
then

\[
\bar{b}[s_0\bar{g}s_0] = \begin{bmatrix}
  b_{11} & & & \\
  & \ddots & & \\
  & & b_{n-1,n-1} & \frac{b_{n-1}}{b_{n,n}} \\
  & & & b_{n,n} \frac{b_{n-1}}{b_{n,n}}
\end{bmatrix}
\]

so that

\[
\Lambda^{-} p_{(m)}(\bar{g}) = p_{(m)}((\Pi s_0\bar{g}\gamma)^{-1}) = p_{\varsigma}(\Pi s_0\bar{g}\gamma^{-1}s_0) = p_{\varsigma}(\Pi s_0\bar{g}s_0) \\
= p_{\varsigma}(\Pi \bar{b}[s_0\bar{g}s_0]z[s_0\bar{g}s_0]) = p_{\varsigma}(\Pi \bar{z}[s_0\bar{g}s_0]) = f^{(m)}(\bar{g}),
\]

for

\[
b^v = \begin{bmatrix}
  b_{11} & & & \\
  & \ddots & & \\
  & & b_{n-1,n-1} & \frac{b_{n-1}}{b_{n,n}} \\
  & & & b_{n,n} \frac{b_{n-1}}{b_{n,n}}
\end{bmatrix}
\]

and \(p(\Pi \bar{z}[s_0\bar{g}s_0]) = 1\). Also,

\[
\Lambda^{+} p_{(m)}(\bar{g}) = p_{(m)}((\Pi \bar{g}\gamma)^{-1}) = p_{\varsigma}(\Pi \bar{g}\gamma^{-1}s_0) = p_{\varsigma}(\Pi \bar{g}s_0) = p_{\varsigma}(\Pi \bar{b}[\bar{g}]z[\bar{g}]s_0) = p_{\varsigma}(\Pi \bar{z}[\bar{g}]s_0) = f^{(m)}(\bar{g}),
\]

Since the conjugation \(\bar{g} \to s_0\bar{g}s_0\) preserves the Gauss decomposition of \(\tilde{G}\),

\[
s_0\bar{z}[\bar{g}]s_0\]

is of the form

\[
\begin{bmatrix}
  1 & & & * \\
  & \ddots & & \\
  & & 1 & * \\
  & & & 1
\end{bmatrix}
\]
and
\[ \Pi s_0(s_0 \bar{Z}[\bar{g}]s_0) = \begin{bmatrix} 1 & 1 & \cdots & \cdots & * \\ & & & & \\ 0 & & & & \\ \end{bmatrix} \]
so that \( \Delta_n(\Pi s_0(s_0 \bar{Z}[\bar{g}]s_0)) = 0 \) and hence \( p_\xi(\Pi s_0(s_0 \bar{Z}[\bar{g}]s_0)) = 0 \). It follows that \( \Lambda^+ p_0^{(m)} = 0 \). Similarly,
\[
\Lambda^- p_0^{(m)}(\bar{g}) = p_0^{(m)}(\Pi s_0 \bar{g} \gamma^{-1}) = p_\xi(\Pi s_0 \bar{g})
\]
\[
= p_\xi(\Pi s_0 \bar{b}[\bar{g}] \bar{Z}[\bar{g}])
\]
\[
= p_\xi((s_0 \bar{b}[\bar{g}]s_0) s_0 \bar{Z}[\bar{g}]).
\]
Again, since the conjugation \( \bar{g} \to s_0 \bar{g} s_0 \) preserves the Gauss decomposition of \( \bar{G} \), \( s_0 \bar{b}[\bar{g}]s_0 \) is of the form
\[
\begin{bmatrix}
 b_{11} \\
 & \ddots \\
 & & b_{n-1, n-1} \\
 & & & b_{n-1} \\
 & & & & b_{n,n} \\
 & & & & & \cdots \\
 & & & & & & b_{n,n} \\
& & & & & & & b_{11}^{-1}
\end{bmatrix}
\]
so that \( \Pi(s_0 \bar{b}[\bar{g}]s_0) = b^v \Pi \). It follows that
\[
p_\xi((s_0 \bar{b}[\bar{g}]s_0) s_0 \bar{Z}[\bar{g}]) = \xi^{(m)}(b^v) p_\xi((s_0 \bar{Z}[\bar{g}]).
\]
As above we see that \( p_\xi(s_0 \bar{Z}[\bar{g}]) = 0 \) and infer that \( \Lambda^- p_0^{(m)} = 0 \). Since \( \Lambda^+ p_0^{(m)} = \Lambda^+ p_0^{(m)} = f^{(m)} \) (resp. \( \Lambda^- p_0^{(m)} = \Lambda^- p_0^{(m)} = f^{(m)} \)) is a cyclic vector of the simple \( G \)-module \( V^{(m)} \) (resp. \( V^{(m)} \)) and \( \Lambda \) is an intertwining operator it follows that \( \Lambda \) is a \( G \)-module epimorphism. If \( p \in H(E^*, (m)) \) and \( \Lambda p = \Lambda^+ p + \Lambda^- p = 0 \) then Equation (3.5) implies that \( p((1/\sqrt{2}) X_0^+ g) = p((1/\sqrt{2}) X_0^- g) = 0 \) for all \( g \in G \), that is, \( p \) vanishes on both \( P^+ \) and \( P^- \). Since \( P^+ \cup P^- = P \), it follows that \( p = 0 \) on \( P \) and, by Corollary 2.2(iii), \( p \) is the 0 polynomial function. Thus \( \Lambda \) is a \( G \)-module monomorphism, and, hence, a \( G \)-module isomorphism. Let \( H^+(E^*, (m)) = \text{Ker} \Lambda^- \) (resp. \( H^-(E^*, (m)) = \text{Ker} \Lambda^+ \)) be the subspace of all elements of \( H(E^*(m)) \) which vanish on \( P^- \) (resp. \( P^+ \)). Then clearly \( H^+(E^*, (m)) \) and \( H^-(E^*(m)) \) are \( G \)-submodules of \( H(E^*, (m)) \) and \( \Lambda_{|H^+(E^*, (m))} = \Lambda^+ \), \( \Lambda_{|H^-(E^*, (m))} = \Lambda^- \). Moreover, \( p_0^{(m)} \in H^+(E^*, (m)) \) and \( p_0^{(m)} \in H^-(E^*, (m)) \). So both \( H^+(E^*(m)) \) and \( H^-(E^*, (m)) \) are nonzero, and it follows that \( \Lambda^+: H^+(E^*, (m)) \to V^{(m)} \) and \( \Lambda^-: H^-(E^*, (m)) \to V^{(m)} \) are isomorphisms of simple \( G \)-modules. The
fact that the inner product $\langle \cdot, \cdot \rangle$ defined earlier on $S(E^*)$ is invariant under the restriction of the action of $G$ to $G_0 = SO(k)$ and that $H^+(E^*, (m))$ and $H^-(E^*(m))$ are inequivalent simple $G$-modules implies immediately that $H(E^*, (m)) = H^+(E^*, (m)) \oplus H^-(E^*, (m))$ is an orthogonal direct sum. □

REFERENCES


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