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APPROXIMATION OF FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS

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ABSTRACT. Let T be an asymptotically nonexpansive self-mapping of a nonempty closed, bounded, and starshaped (with respect to zero) subset of a smooth reflexive Banach space possessing a duality mapping that is weakly sequentially continuous at zero. Then, if $\mathrm{id}-T$ is demiclosed and T satisfies a strengthened regularity condition, the iteration process $z_{n+1} := \mu_{n+1}T^n(z_n)$ converges strongly to some fixed point of T, provided (μ_n) has certain properties.

0. INTRODUCTION

Motivated by the papers of B. Halpern [4] and P. Vijayaraju [7], we deal with the convergence of almost fixed points $x_n = \mu_n T^n(x_n)$ of an asymptotically nonexpansive mapping T. This class of functions was introduced by K. Goebel and W. A. Kirk [3] in 1972. In §1, convergence of the sequence (x_n) to some fixed point of T is shown, assuming that T is uniformly asymptotically regular and $\mathrm{id}-T$ is demiclosed. These assumptions have been made by P. Vijayaraju [7] to ensure the existence of a fixed point of T.

By strengthening the regularity condition on T in §2, we establish the convergence of the explicit iteration scheme $z_{n+1} := \mu_{n+1}T^n(z_n)$ to some fixed point of T. This iteration method is similar to one that was introduced by B. Halpern [4].

1. CONVERGENCE OF THE SEQUENCE OF APPROXIMATE FIXED POINTS

We shall begin by recalling some definitions needed in the sequel.

Definition 1.1. Let $(E, \|\cdot\|)$ be a normed space, $\varnothing \neq A \subset E$; $\mu: \mathbb{R}^+ \to \mathbb{R}^+$; $J: E \to E^*$; and $x_0 \in E$.

(1) $(E, \|\cdot\|)$ is called *smooth*: $\iff$ $\|\cdot\|$ is Gateaux-differentiable on $\partial\overline{B}(0, 1)$.

(2) μ is said to be a *gauge function*: $\iff$ μ is continuous and strictly increasing with $\mu(0) = 0$ and $\lim_{x\to\infty}\mu(x) = \infty$.

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(3) The set-valued *duality mapping* associated with μ is given by $J_E^\mu(x) := \{u \in E^* | u(x) = ||u||\,||x|| \text{ and } ||u|| = \mu(||x||)\}$ and J is called a duality mapping with respect to μ: $\iff J(x) \in J_E^\mu(x)$ for all $x \in E$. For abbreviation, we set $J_E := J_E^{\mathrm{id}}$.

(4) J is *weakly sequentially continuous at* x_0: $\iff$ for all $(x_n) \in E^{\mathbb{N}}$ with $(x_n) \rightharpoonup x_0$, it follows that $(J(x_n)) \overset{*}{\rightharpoonup} J(x_0)$.

Remark. It is well known that J_E^μ is single-valued if and only if $(E, ||\cdot||)$ is smooth (see, e.g., [1]). In this case, we regard J_E^μ as a mapping from E to E^*. In all our proofs we assume, without loss of generality, that J_E^μ is normalized, i.e., $\mu = \mathrm{id}$. Futhermore, in the sequel all normed spaces are assumed to be real Banach spaces.

Definition 1.2. Let $(E, ||\cdot||)$ be a normed space; $\varnothing \neq A \subset E$; $T: A \to A$; $(k_n) \in [1, \infty)^{\mathbb{N}}$.

(1) T is said to be *asymptotically nonexpansive with sequence* (k_n): $\iff$ $\lim(k_n) = 1$ and $||T^n(x) - T^n(y)|| \leq k_n||x - y||$ for all $n \in \mathbb{N}$ and all $x, y \in A$.

(2) T is called *demiclosed*: $\iff$ for all $(x_n) \in A^{\mathbb{N}}$ and all $x, y \in E$ with $(x_n) \rightharpoonup x$ and $\lim(T(x_n)) = y$, it follows that $x \in A$ and $T(x) = y$.

(3) T is called *uniformly asymptotically regular*: $\iff$ for each $\varepsilon > 0$ there is $n_0 \in \mathbb{N}$, such that $||T^n(x) - T^{n+1}(x)|| \leq \varepsilon$ for all $n \geq n_0$ and all $x \in A$.

Lemma 1.3. *Let* $(E, ||\cdot||)$ *be a smooth normed space*; $\varnothing \neq A \subset E$; $T: A \to A$ *asymptotically nonexpansive with sequence* $(k_n) \in [1, \infty)^{\mathbb{N}}$; $x, y \in A$; $\lambda \in (0, 1)$; $n \in \mathbb{N}$; $x = \lambda T^n(x)$; *and* $y = T^n(y)$. *Then*

$$J_E(y - x)(x) \geq -\frac{k_n - 1}{1/\lambda - 1}||x - y||^2.$$

Proof. Set $\alpha := \frac{1}{\lambda} - 1 > 0$, $z := \alpha x$, and $w := y - x$. Then

$$\begin{aligned} ||z - w|| &= ||(1 + \alpha)x - y|| = ||\tfrac{1}{\lambda}x - y|| \\ &= ||T^n(x) - T^n(y)|| \leq k_n||x - y|| = k_n||w||. \end{aligned}$$

Hence

$$\begin{aligned} J_E(y - x)(x) &= \tfrac{1}{\alpha}J_E(w)(z) = \tfrac{1}{\alpha}J_E(w)(z - w) + \tfrac{1}{\alpha}J_E(w)(w) \\ &\geq -\tfrac{1}{\alpha}||w||\,||z - w|| + \tfrac{1}{\alpha}||w||^2 \geq \tfrac{1}{\alpha}(1 - k_n)||w||^2 \\ &= ((1 - k_n)/((\tfrac{1}{\lambda}) - 1))||x - y||^2. \quad \square \end{aligned}$$

The following lemma can be verified by an easy calculation.

Lemma 1.4. *Let* $\lambda \in (\frac{1}{2}, 1)$. *Then*

(1) $\lambda^2/(2\lambda - 1) > 1$, *and*

(2) $(\lambda k - \lambda)/(k - \lambda) \leq 1 - \lambda$ *for all* $k \in [1, \lambda^2/(2\lambda - 1)]$.

Remark. Part (1) of Lemma 1.4 shows that the assumption $k_n \leq \lambda_n^2/(2\lambda_n - 1)$ and $k_n \in [1, \infty)$ of our next lemma makes sense.

Lemma 1.5. *Let* $(E, \|\cdot\|)$ *be a smooth normed space possessing a duality mapping* $J: E \to E^*$ *that is weakly sequentially continuous at* 0; $\varnothing \neq A \subset E$ *bounded*; $T: A \to A$ *asymptotically nonexpansive with sequence* $(k_n) \in [1, \infty)^{\mathbb{N}}$; $(x_n) \in A^{\mathbb{N}}$; $x \in A$; (x_{φ_n}) *a subsequence of* (x_n) *with* $(x_{\varphi_n}) \rightharpoonup x$; $(\lambda_n) \in (\frac{1}{2}, 1)^{\mathbb{N}}$ *with* $\lim(\lambda_n) = 1$; $x = T(x)$; $x_n = (\lambda_n/k_n)T^n(x_n)$ *for all* $n \in \mathbb{N}$; *and* $k_n \leq \lambda_n^2/(2\lambda_n - 1)$ *for all* $n \in \mathbb{N}$. *Then*

(1) $\lim(x_{\varphi_n}) = x$, *and*

(2) $J(y - x)(x) \geq 0$ *for all* $y \in \mathrm{Fix}(T)$.

$(\mathrm{Fix}(T) := \{z \in A | T(z) = z\})$

Proof. (1) Since $(\lambda_n/k_n) \in (0, 1)^{\mathbb{N}}$, it follows from Lemma 1.3 that

$$J(x - x_n)(x_n) \geq -\frac{k_n - 1}{k_n/\lambda_n - 1}\|x_n - x\|^2 \quad \text{for all } n \in \mathbb{N}.$$

We may choose $M > 0$ such that $\|x_n - z\|^2 \leq M$ for all $n \in \mathbb{N}$ and all $z \in A$ because of the boundedness of A. Obviously, $(k_n - 1)/(k_n/\lambda_n - 1) \geq 0$ and from Lemma 1.4 we know, that

$$\frac{k_n - 1}{k_n/\lambda_n - 1} = \frac{\lambda_n k_n - \lambda_n}{k_n - \lambda_n} \leq 1 - \lambda_n.$$

Hence $J(x - x_n)(x_n) \geq -(1 - \lambda_n)M$, and therefore, for all $n \in \mathbb{N}$,

$$\|x - x_n\|^2 = J(x - x_n)(x) - J(x - x_n)(x_n) \leq J(x - x_n)(x) + (1 - \lambda_n)M,$$

where

(a) $\lim(1 - \lambda_n) = 0$,

(b) $(x_{\varphi_n} - x) \rightharpoonup 0$, hence $\lim J(x - x_{\varphi_n})(x) = 0$.

Thus we conclude that $\lim \|x - x_{\varphi_n}\| = 0$.

(2) Fix $y \in \mathrm{Fix}(T)$. As already shown in (1), for $y = T(y)$ instead of x, it follows that

$$(*) \qquad J(y - x_n)(x_n) \geq -(1 - \lambda_n)M \quad \text{for all } n \in \mathbb{N}.$$

Since $(E, \|\cdot\|)$ is smooth, J is strong-weak$^{(*)}$ continuous (see e.g., [2]), and so from $\lim(y - x_{\varphi_n}) = y - x$, we conclude that $(J(y - x_{\varphi_n})) \xrightarrow{*} J(y - x)$. Hence $\lim J(y - x_{\varphi_n})(x_{\varphi_n}) = J(y - x)(x)$. This, together with $(*)$ and $\lim(1 - \lambda_n) = 0$, yields $J(y - x)(x) \geq 0$. □

Remark. Lemmas 1.3 and 1.5 are similar to some related results for nonexpansive mappings; namely, [6, Lemmas 4, 5]. The following theorem was demonstrated by P. Vijayaraju in the course of the proof of [7, Theorems 2.1, 2.2].

Theorem 1.6. *Let* $(E, \|\cdot\|)$ *be a Banach space;* $\varnothing \neq A \subset E$ *closed, bounded, and starshaped with respect to* 0; $T: A \to A$ *asymptotically nonexpansive with sequence* $(k_n) \in [1, \infty)^{\mathbb{N}}$ *and uniformly asymptotically regular;* $(\lambda_n) \in [0, 1)^{\mathbb{N}}$; *and* $\lim(\lambda_n) = 1$. *Then*

(1) *for each* $n \in \mathbb{N}$ *there is exactly one* $x_n \in A$ *such that* $x_n = (\lambda_n/k_n)T^n(x_n)$, *and*

(2) $\lim \|x_n - T(x_n)\| = 0$.

Now we are able to prove the main result of this section.

Theorem 1.7. *Let* $(E, \|\cdot\|)$ *be a smooth reflexive Banach space possessing a duality mapping* $J: E \to E^*$ *that is weakly sequentially continuous at* 0; $\varnothing \neq A \subset E$ *closed, bounded, and starshaped with respect to* 0; $T: A \to A$ *asymptotically nonexpansive with sequence* $(k_n) \in [1, \infty)^{\mathbb{N}}$ *and uniformly asymptotically regular;* $\mathrm{id} - T$ *demiclosed;* $(\lambda_n) \in (\frac{1}{2}, 1)^{\mathbb{N}}$; $\lim(\lambda_n) = 1$; $k_n \leq \lambda_n^2/(2\lambda_n - 1)$ *for all* $n \in \mathbb{N}$. *Then*

(1) *for each* $n \in \mathbb{N}$ *there is exactly one* $x_n \in A$ *such that* $x_n = (\lambda_n/k_n)T^n(x_n)$, *and*

(2) (x_n) *converges strongly to some fixed point of* T.

Proof. Part (1) follows from Theorem 1.6, so that it remains to prove part (2). Since $(E, \|\cdot\|)$ is reflexive and A is bounded, there exists $z \in E$ and a subsequence (x_{μ_n}) of (x_n) such that $(x_{\mu_u}) \rightharpoonup z$ (Pettis' theorem). From Theorem 1.6, $\lim \|x_n - T(x_n)\| = 0$, so from the demiclosedness of $\mathrm{id} - T$, $z \in A$ and $T(z) = z$.

Applying Lemma 1.5, we get

$$(*) \qquad J(y - z)(z) \geq 0 \quad \text{for all } y \in \mathrm{Fix}(T).$$

Fix $\varphi: \mathbb{N} \to \mathbb{N}$ and injective.

Repeating the above argument, we find a subsequence $(x_{\varphi_{\psi_n}})$ of (x_{φ_n}) converging weakly to some $x \in A$ such that $x = T(x)$ and

$$(**) \qquad J(y - x)(x) \geq 0 \quad \text{for all } y \in \mathrm{Fix}(T).$$

Substituting x into (*) and z into (**), it follows that $J(x - z)(z) \geq 0$ and $J(z - x)(x) \geq 0$. Hence $0 \leq J(x - z)(-x) + J(x - z)(z) = -\|x - z\|^2 \leq 0$, so that $x = z$, and therefore $(x_{\varphi_{\psi_n}}) \rightharpoonup z$. This shows that $(x_n) \rightharpoonup z$, and we now apply Lemma 1.5 (with $\varphi := \mathrm{id}$) to obtain $\lim \|x_n - z\| = 0$. □

Remark. For $(k_n) \in (1, \infty)^{\mathbb{N}}$ with $\lim(k_n) = 1$, the conditions $(\lambda_n) \in (\frac{1}{2}, 1)^{\mathbb{N}}$, $\lim(\lambda_n) = 1$, and $k_n \leq \lambda_n^2/(2\lambda_n - 1)$ for all $n \in \mathbb{N}$ are fulfilled, for example, if one chooses $\lambda_n := k_n - \sqrt{k_n^2 - k_n}$ for all $n \in \mathbb{N}$.

2. An explicit iteration scheme

Now we give an explicit iteration scheme for the approximation of a fixed point of T. In analogy to B. Halpern's work [4], we use the fact that the

implicit iteration process $x_n = (\lambda_n/k_n)T^n(x_n)$ already converges to some fixed point of T to show that the same holds for the explicit iteration procedure $z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n)$.

In addition to several technical assumptions on the values λ_n and k_n, we require that T fulfills a certain strengthened regularity condition (assumption (b) of Theorem 2.2). Although parts of the proof of Theorem 2.2 are parallel to those of the proof of [4, Theorem 3], they are included for the sake of completeness.

Definition 2.1. Let $(\varepsilon_n) \in (0, \infty)^{\mathbb{N}}$ and $(\mu_n) \in (0, 1)^{\mathbb{N}}$. $((\varepsilon_n), (\mu_n))$ is called *admissible* $:\Longleftrightarrow$

(1) (ε_n) is decreasing;
(2) (μ_n) is strictly increasing with $\lim(\mu_n) = 1$;
(3) there exists $(\beta_n) \in \mathbb{N}^{\mathbb{N}}$ such that
  (i) (β_n) is increasing,
  (ii) $\lim(\beta_n(1 - \mu_n)) = \infty$,
  (iii) $\lim((1 - \mu_{n+\beta_n})/(1 - \mu_n)) = 1$, and
  (iv) $\lim((\varepsilon_n \beta_n \mu_{n+\beta_n})/(1 - \mu_n)) = 0$.

Remark. (a) Parts (2) and (3) (i)–(iii) are due to B. Halpern [4]. (b) $\lim(\beta_n) = \infty$ because of (3)(ii) and (2), so that it follows from (3)(iv) that $\lim(\varepsilon_n) = 0$.

Theorem 2.2. *Let $(E, \|\cdot\|)$ be a normed space; $\varnothing \neq A \subset E$ bounded and starshaped with respect to 0; $T: A \to A$ asymptotically nonexpansive with sequence $(k_n) \in [1, \infty)^{\mathbb{N}}$; $(\lambda_n) \in (0, 1)^{\mathbb{N}}$ with $\lim(\lambda_n) = 1$; $(x_n) \in A^{\mathbb{N}}$; $\mu_n := \lambda_n/k_n$ for all $n \in \mathbb{N}$; $z_0 \in A$; $x_n = (\lambda_n/k_n)T^n(x_n)$ for all $n \in \mathbb{N}$, and $z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n)$ for all $n \in \mathbb{N}_0$; $(\varepsilon_n) \in (0, \infty)^{\mathbb{N}}$ such that $((\varepsilon_n), (\mu_n))$ is admissible;*

(a) *$((1 - \mu_n)/(1 - \lambda_n))$ bounded;*
(b) *$\|T^n(x) - T^{n+1}(x)\| \leq \varepsilon_n$ for all $n \in \mathbb{N}$ and all $x \in A$.*

Assume further that $q := \lim(x_n)$ exists. Then $\lim(z_n) = q$.

Proof. Observe that $(z_n) \in A^{\mathbb{N}}$ is well defined, because $T(A) \subset A$, $(\mu_n) \in (0, 1)^{\mathbb{N}}$, and A is starshaped with respect to 0. Since A is bounded and T is Lipschitzian, there is a $d > 0$ such that $\|z\| \leq d$ for all $z \in A$ and there is a $c > 0$ such that $\|T(w)\| \leq c$ for all $w \in A$. Additionally, (a) implies the existence of a number $\kappa > 0$, such that $(1 - \mu_n)/(1 - \lambda_n) \leq \kappa$ for all $n \in \mathbb{N}$, and we may choose (β_n) according to Definition 2.1. For $n > p \geq i$, we have

$$\|z_{p+1} - x_i\| \leq \|\mu_{p+1}T^p(z_p) - \mu_i T^i(z_p)\| + \|\mu_i T^i(z_p) - \mu_i T^i(x_i)\|,$$

where

$$\|\mu_i T^i(z_p) - \mu_i T^i(x_i)\| \leq \mu_i k_i \|z_p - x_i\| = \lambda_i \|z_p - x_i\|,$$

and

$$\begin{aligned}&\|\mu_{p+1}T^p(z_p) - \mu_i T^i(z_p)\| \\ &\quad \le \|\mu_{p+1}T^p(z_p) - \mu_{p+1}T^i(z_p)\| + \|\mu_{p+1}T^i(z_p) - \mu_i T^i(z_p)\| \\ &\quad \le \mu_{p+1}\|T^p(z_p) - T^i(z_p)\| + |\mu_{p+1} - \mu_i|c.\end{aligned}$$

Since $n > p \ge i$, it follows that $|\mu_{p+1} - \mu_i| \le \mu_n - \mu_i$. Additionally,

$$\begin{aligned}\|T^p(z_p) - T^i(z_p)\| &\le \sum_{j=i}^{p-1} \|T^{j+1}(z_p) - T^j(z_p)\| \\ &\le \sum_{j=i}^{p-1} \varepsilon_j \le \sum_{j=i}^{p-1} \varepsilon_i = (p-i)\varepsilon_i \le (n-i)\varepsilon_i.\end{aligned}$$

Combining the above inequalities, we get

(1) $\|z_{p+1} - x_i\| \le \lambda_i \|z_p - x_i\| + (\mu_n - \mu_i)c + \mu_n(n-i)\varepsilon_i$ for all $n > p \ge i$,

from which it follows by induction that

$$\|z_{m+j} - x_i\| \le \lambda_i^j \|z_m - x_i\| + (\mu_n - \mu_i)c \sum_{\nu=0}^{j-1} \lambda_i^\nu + \mu_n(n-i)\varepsilon_i \sum_{\nu=0}^{j-1} \lambda_i^\nu,$$

for all $n > m \ge i$ and $j \le n - m$. Since $\sum_{\nu=0}^{j-1} \lambda_i^\nu \le 1/(1-\lambda_i)$, we obtain, defining $j = n - m$,

(2) $\|z_n - x_i\| \le \lambda_i^{n-m} \|z_m - x_i\| + (\mu_n - \mu_i)(1-\lambda_i)^{-1}c + \mu_n(n-i)\varepsilon_i(1-\lambda_i)^{-1}$

for all $n > m \ge i$.

Set $m := i$ and $n := i + \beta_i > m$ to get

$$\|z_{i+\beta_i} - x_i\| \le 2d\lambda_i^{\beta_i} + (\mu_{i+\beta_i} - \mu_i)(1-\lambda_i)^{-1}c + \mu_{i+\beta_i}\beta_i\varepsilon_i(1-\lambda_i)^{-1},$$

for all $i \in \mathbb{N}$. Observe that

(i) $|\mu_{i+\beta_i} - \mu_i||1-\lambda_i|^{-1} = |1 - (1-\mu_{i+\beta_i})(1-\mu_i)^{-1}||(1-\mu_i)/(1-\lambda_i)|$ $\le \kappa|1 - (1-\mu_{i+\beta_i})(1-\mu_i)^{-1}|$,

from which we conclude that $\lim((\mu_{i+\beta_i} - \mu_i)/(1-\lambda_i)) = 0$, taking into account the admissibility of $((\varepsilon_n), (\mu_n))$. Also,

(ii) $\log(\lambda_i^{\beta_i}) \le \beta_i(\lambda_i - 1) \le -\beta_i(1-\mu_i)1/\kappa$. Since $((\varepsilon_n), (\mu_n))$ is admissible, it follows that $\lim(\log(\lambda_i^{\beta_i})) = -\infty$, and thus $\lim(\lambda_i^{\beta_i}) = 0$. In addition,

(iii) $0 \le \mu_{i+\beta_i}\beta_i\varepsilon_i(1-\lambda_i)^{-1} \le \kappa\mu_{i+\beta_i}\beta_i\varepsilon_i(1-\mu_i)^{-1}$, from which it follows that $\lim((\mu_{i+\beta_i}\beta_i\varepsilon_i)/(1-\lambda_i)) = 0$, again using the admissibility of $((\varepsilon_n), (\mu_n))$. Hence

(3) $$\lim \|z_{i+\beta_i} - x_i\| = 0.$$

As already noticed in the proof of [4, Theorem 3], there exists $j\colon \mathbb{N} \to \mathbb{N}$ and $n_0 \in \mathbb{N}$ such that $\lim(j_n) = \infty$ and $j_n + \beta_{j_h} < n \le j_n + 1 + \beta_{j_{n+1}}$ for all $n \ge n_0$. Now let $n \ge n_0$. With $n > m := j_n + \beta_{j_n} \ge i := j_n + 1$, it follows from (2) that

$$\|z_n - x_{j_n+1}\| \le \|z_{(j_n+\beta_{jn})} - x_{j_n+1}\| + (\mu_n - \mu_{j_n+1})(1-\lambda_{j_n+1})^{-1}c$$
$$+ \mu_n(n-(j_n+1))\varepsilon_{j_n+1}(1-\lambda_{j_n+1})^{-1} \quad \text{for all } n \in \mathbb{N},$$

where

(i) $0 \le (\mu_n - \mu_{j_n+1})(1-\lambda_{j_n+1})^{-1} \le (\mu_{(j_n+1+\beta_{j_n+1})} - \mu_{j_n+1})(1-\lambda_{j_n+1})^{-1}$, which tends to 0 by 2(i).

(ii) $0 \le \mu_n(n-(j_n+1))\varepsilon_{j_n+1}(1-\lambda_{j_n+1})^{-1} \le \mu_{(j_n+1+\beta_{j_n+1})}\beta_{j_n+1}\varepsilon_{j_n+1}(1-\lambda_{j_n+1})^{-1}$, which tends to 0 by (2)(iii).

(iii) $\|z_{(j_n+\beta_{j_n})} - x_{j_n+1}\| \le \|z_{(j_n+\beta_{j_n})} - x_{j_n}\| + \|x_{j_n} - q\| + \|q - x_{j_n+1}\|$, which also tends to 0 because $\lim\|x_n - q\| = 0$ and $\lim\|z_{(j_n+\beta_{j_n})} - x_{j_n}\| = 0$ by (3). Hence $\lim\|z_n - x_{j_n+1}\| = 0$. Finally, since $\|z_n - q\| \le \|z_n - x_{j_n+1}\| + \|x_{j_n+1} - q\|$, it follows that $\lim\|z_n - q\| = 0$. □

We now prove the main result of this section.

Theorem 2.3. *Let $(E, \|\cdot\|)$ be a reflexive smooth Banach space possessing a duality mapping that is weakly sequentially continuous at 0; $\varnothing \ne A \subset E$ closed, bounded, and starshaped with respect to 0; $T\colon A \to A$ asymptotically nonexpansive with sequence $(k_n) \in [1, \infty)^{\mathbb{N}}$; $\mathrm{id} - T$ demiclosed; $(\lambda_n) \in (\frac{1}{2}, 1)^{\mathbb{N}}$; $z_0 \in A$; $z_{n+1} := (\lambda_{n+1}/k_{n+1})T^n(z_n)$ for all $n \in \mathbb{N}_0$;*

(a) *$\lim(\lambda_n) = 1$; $\mu_n := \lambda_n/k_n$ for all $n \in \mathbb{N}$; $k_n \le \lambda_n^2/(2\lambda_n - 1)$ for all $n \in \mathbb{N}$; $((1-\mu_n)/(1-\lambda_n))$ bounded; $(\varepsilon_n) \in (0, \infty)^{\mathbb{N}}$, such that $((\varepsilon_n), (\mu_n))$ is admissible;*

(b) *$\|T^n(x) - T^{n+1}(x)\| \le \varepsilon_n$ for all $n \in \mathbb{N}$ and all $x \in A$.*

Then (z_n) converges strongly to some fixed point of T.

Proof. Since $((\varepsilon_n), (\mu_n))$ is admissible, it follows that $\lim(\varepsilon_n) = 0$, so that assumption (b) implies that T is uniformly asymptotically regular. The result follows by a combination of Theorems 1.7, 2.2. □

Remark. (1) For a related result concerning the *weak* convergence of the successive approximations $x_{n+1} := T(x_n)$ to some fixed point of an asymptotically nonexpansive mapping T in an Opial space we refer the reader to [5].

(2) Theorem 2.3, of course, is applicable to nonexpansive mappings as well (set $k_n := 1$ for all $n \in \mathbb{N}$). In this special case, however, there is a much better result: [6, Theorem 10] allows us to drop the assumptions that $\mathrm{id} - T$ is demiclosed and that $\|T^n(x) - T^{n+1}(x)\| \le \varepsilon_n$ for all $n \in \mathbb{N}$ and all $x \in A$, provided use is made of the iteration scheme $z_{n+1} := \lambda_{n+1}T(z_n)$, T is nonexpansive, and $(E, \|\cdot\|)$ possesses a duality mapping that is weakly sequentially continuous on the whole of E.

We close this section with a special case of Theorem 2.3, which shows that it is really possible to fulfill the assumptions made on (λ_n), (μ_n), (ε_n), and (k_n).

Theorem 2.4. *Let $(E, \|\cdot\|)$ be a reflexive smooth Banach space possessing a duality mapping that is weakly sequentially continuous at 0; $\emptyset \neq A \subset E$ closed, bounded, and starshaped with respect to 0; $T: A \to A$ asymptotically nonexpansive with sequence $(1+1/(\sqrt{n+1}-1))$; $\mathrm{id}-T$ demiclosed; $\mu_n := 1-(n+1)^{-1/4}$ for all $n \in \mathbb{N}$; $z_0 \in A$; $z_{n+1} := \mu_{n+1}T^n(z_n)$ for all $n \in \mathbb{N}_0$; and*

$$(*) \qquad \|T^n(x) - T^{n+1}(x)\| \leq (n+1)^{-7/8} \quad \textit{for all } n \in \mathbb{N} \textit{ and all } x \in A.$$

Then (z_n) converges strongly to some fixed point of T.

Proof. For $n \in \mathbb{N}$ define $k_n := 1 + 1/(\sqrt{n+1} - 1)$, $\lambda_n := k_n - \sqrt{k_n^2 - k_n}$, $\varepsilon_n := (n+1)^{-7/8}$, and $\beta_n := [(n+1)^{1/2}]_G$ (cf. Definition 2.1), where $[x]_G := \max\{n \in \mathbb{N}_0 | n \leq x\}$. Then easy calculations show that all the assumptions of Theorem 2.3 are fulfilled, especially that $\mu_n = \lambda_n/k_n$ for all $n \in \mathbb{N}$ and that $((\varepsilon_n), (\mu_n))$ is admissible. □

Remark. For $x \in A$ and $n, m \in \mathbb{N}$ with $n > m$, the following holds:

$$\|T^n(x) - T^m(x)\| \leq \sum_{j=m}^{n-1} \|T^{j+1}(x) - T^j(x)\| \leq \sum_{j=m}^{n-1} (j+1)^{-7/8}.$$

Nevertheless, since $\sum_{j=1}^{\infty}(j+1)^{-7/8} = \infty$, $(*)$ does *not* imply that $(T^n(x))$ is a Cauchy sequence (from which it would follow that $(T^n(x))$ converges strongly to some fixed point of T).

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