ON RINGS FOR WHICH HOMOGENEOUS MAPS ARE LINEAR

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Abstract. Let \( \mathcal{R} \) be the collection of all rings \( R \) such that for every \( R \)-module \( G \), the centralizer near-ring \( M_R(G) = \{ f: G \to G | f(rx) = rf(x) , \ r \in R , \ x \in G \} \) is a ring. We show \( R \in \mathcal{R} \) if and only if \( M_R(G) = \text{End}_R(G) \) for each \( R \)-module \( G \). Further information about \( \mathcal{R} \) is collected and the Artinian rings in \( \mathcal{R} \) are completely characterized.

I. INTRODUCTION

Let \( R \) be a ring with identity and \( G \) a unitary left \( R \)-module. The set \( M_R(G) := \{ f: G \to G | f(rx) = rf(x) , \ r \in R , \ x \in G \} \) is a zero-symmetric near-ring with identity under the operations of function addition and composition. If \( G = R \), \( M_R(R) \cong R \) so \( M_R(R) \) is a ring. If \( R \) is a field and \( G = R^2 \) then it is known that \( M_R(R^2) \) is not a ring [3]. On the other hand, when \( R \) is a finite simple ring, but not a field, it was found in [2] that \( M_R(G) \) is a ring for each finite \( R \)-module \( G \). In this paper we investigate two questions raised by the above remarks; namely, (1) characterize those rings \( R \) such that \( M_R(G) \) is a ring for every \( R \)-module \( G \) and (2) characterize those rings \( R \) such that \( M_R(G) = \text{End}_R(G) \) for every \( R \)-module \( G \).

We let \( \mathcal{E} \) denote the collection of rings satisfying (1) and \( \mathcal{E} \) denote the collection of rings satisfying (2). Of course \( \mathcal{E} \subseteq \mathcal{R} \). We show in the next section that in fact, \( \mathcal{E} = \mathcal{R} \).

The problem then remains to characterize the class \( \mathcal{R} \). It is the objective of this paper to initiate such an investigation. We collect information about \( \mathcal{R} \) and present some classes of rings in \( \mathcal{R} \). In particular we completely characterize the Artinian rings in \( \mathcal{R} \). For a ring \( R \) and an abelian group \( G \) let \( rx = 0 \) for all \( r \in R , \ x \in G \). Then \( M_R(G) = M_0(G) \) which is a nonring whenever
$|G| \geq 3$. Thus we make the following

**Conventions.** All rings have identity 1, all modules are unitary, and all homomorphisms are identity preserving.

## II. General results

Let $R$ be a ring and $G$ an $R$-module. It is well known that $M(G) = G^G = \{f : G \to G\}$ is a near-ring with respect to function addition and function composition. (We refer the reader to the books by Meldrum [4] and Pilz [5] for near-ring information.) The above defined near-ring $M_R(G)$ is a subnear-ring of $M(G)$ with the identity function as identity element. Moreover, $M(G)$ is an $R$-module under the action $(r,f)(x) = r(f(x))$, $r \in R$, $f \in M(G)$, $x \in G$. As above let $\mathcal{E}$ denote the class of all rings $R$ such that $M_R(G) = \text{End}_R(G)$ for each $R$-module $G$, and let $\mathcal{R}$ denote the class of all rings $R$ such that $M_R(G)$ is a ring for each $R$-module $G$.

**Theorem II.1.** $\mathcal{E} = \mathcal{R}$.

**Proof.** Since $\mathcal{E} \subseteq \mathcal{R}$ it suffices to establish the converse. Let $R \in \mathcal{R}$. To each $f \in M_R(G)$ we associate a map $\hat{f} : M(G) \to M(G)$ where $\hat{f}(\varphi) = f \circ \varphi$, $\varphi \in M(G)$. Since $\hat{f}(r\varphi) = f \circ r\varphi = r(f \circ \varphi) = r\hat{f}(\varphi)$ we see that $\hat{f} \in M_R(M(G))$. Now $M_R(M(G))$ is a ring since $R \in \mathcal{R}$; hence $\hat{f}(\alpha + \beta) = \hat{f}(\alpha) + \hat{f}(\beta)$ for each $\alpha, \beta \in M_R(M(G))$. Therefore, for $\varphi \in M(G)$, $\hat{f}(\alpha(\varphi) + \beta(\varphi)) = \hat{f}(\alpha(\varphi)) + \hat{f}(\beta(\varphi))$ which in turn gives $f(\alpha(\varphi)(x) + \beta(\varphi)(x)) = f(\alpha(\varphi)(x)) + f(\beta(\varphi)(x))$ for each $x \in G$.

Now let $\psi \in M(G)$. The map $\overline{\psi}$ defined by $\overline{\psi}(\varphi) = \varphi \circ \psi$, $\varphi \in M(G)$ is in $M_R(M(G))$ since $\overline{\psi}(r\varphi) = r\varphi \circ \psi = r(\varphi \circ \psi) = r\overline{\psi}(\varphi)$. Given any $x_1, x_2$ in $G$, there exist $\psi_1, \psi_2$ in $M(G)$ such that $\psi_1(x) = x_1$, $\psi_2(x) = x_2$ for all $x \in G$. Thus for $\varphi = \text{id}$ in $M(G)$, $\overline{\psi}_i(\varphi)(x) = x_i$, $i = 1, 2$. Hence $f(x_1 + x_2) = f(x_1) + f(x_2)$ so $f \in \text{End}_R(G)$, i.e., $M_R(G) = \text{End}_R(G)$.

We next determine some properties for the class $\mathcal{R}$.

**Theorem II.2.** Let $S \in \mathcal{R}$ and let $\varphi : S \to R$ be a homomorphism. Then $R \in \mathcal{R}$.

**Proof.** Let $G$ be an $R$-module. Then as usual, $G$ is a (unitary) $S$-module via $s \ast g = \varphi(s) \cdot g$, $s \in S$, $g \in G$. For $f \in M_R(G)$, $s \in S$, $g \in G$ we have $f(s \ast g) = f(\varphi(s)g) = \varphi(s)f(g) = s \ast f(g)$ so $f \in M_S(G)$. Since $S \in \mathcal{R}$, $M_S(G)$ is a ring and hence so is $M_R(G)$, i.e., $R \in \mathcal{R}$.

**Corollary II.3.** (i) Let $S \in \mathcal{R}$. If $S$ can be embedded in a ring $R$ (preserving the identity), then $R \in \mathcal{R}$.

(ii) If a subdirect product of rings $R_\alpha$, $\alpha \in A$, is in $\mathcal{R}$ then each $R_\alpha \in \mathcal{R}$.

(iii) If $R \in \mathcal{R}$ then $R^X \in \mathcal{R}$ for each set $X$.

(iv) If $\text{rad}$ is any radical for rings then $R \in \mathcal{R}$ implies that the "semisimple part" $R/\text{rad}(R)$ is in $\mathcal{R}$.

**Proof.** (i), (ii), and (iv) follow from II.2 while (iii) follows from (i) via the identity preserving map $r \mapsto (r, r, \ldots)$. 

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Theorem II.4. Let $R$ be the group direct sum of subrings $R_1, R_2, \ldots, R_n$ which (as rings) are in $\mathcal{R}$. Then $R \in \mathcal{R}$.

Proof. Let $1 = r_1 + r_2 + \cdots + r_n$ be the decomposition of the identity $1 \in R$ and let $1_i$ denote the identity of $R_i$, $i = 1, \ldots, n$. If $G$ is an $R$-module then each $G_i := 1_i G$ is (unitary) $R_i$-module. Also, for $f \in M_R(G)$, $f(G_i) = f(1_i G) \subseteq 1_i f(G) \subseteq G_i$, $i = 1, 2, \ldots, n$. Thus $\Phi: M_R(G) \rightarrow M_{R_1}(G_1) \oplus \cdots \oplus M_{R_n}(G_n)$ defined by $\Phi(f) = (f|_{G_1}, \ldots, f|_{G_n})$ is a near-ring homomorphism. If $f \in \ker \Phi$ then $f$ is the zero map on each $G_i$. Therefore for $x \in G$, $f(x) = f(1 \cdot x) = r_1 f(x) + \cdots + r_n f(x) = f(r_1 x) + \cdots + f(r_n x) = 0$, so $\ker \Phi = \{0\}$ and $\Phi$ is an embedding. Since $M_{R_i}(G_i)$ is a ring, for each $i$, $M_R(G)$ is a ring, so $R \in \mathcal{R}$.

Corollary II.5. Let $R_1, R_2, \ldots, R_n$ be rings and let $R = R_1 \oplus \cdots \oplus R_n$, the direct sum of rings. Then $R \in \mathcal{R}$ if and only if $R_i \in \mathcal{R}$, $i = 1, 2, \ldots, n$.

Proof. If $R \in \mathcal{R}$, each $R_i \in \mathcal{R}$ from II.2 while the converse follows from the previous theorem.

Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of mutually orthogonal idempotents of the ring $R$ with $1 = \sum_{i=1}^n e_i$. We say $E$ is a complete set of orthogonal idempotents. We define a relation $\sim$ on $E$ by $e_i \sim e_j$ if $e_i R$ and $e_j R$ are isomorphic as $R$-modules ($e_i R \cong_R e_j R$). It is clear that $\sim$ is an equivalence relation on $E$. We let $m(E) = \min\{|B| \mid B$ is an equivalence class with respect to $\sim\}$. The following well-known result determines when the $R$-modules $e_i R$ and $e_j R$ are isomorphic.

Lemma II.6 [1, p. 51]. Let $e_1, e_2$ be idempotents of a ring $R$. Then $e_1 R \cong_R e_2 R$ if and only if there exist $e_{12}, e_{21}$ in $R$ such that $e_{12} e_{21} = e_1$, $e_{21} e_{12} = e_2$, $e_i e_{12} e_2 = e_1$, and $e_2 e_{21} e_1 = e_2$. (As pointed out in [1], the first two conditions suffice.)

Our next result gives a very useful criterion for determining many rings in $\mathcal{R}$.

Theorem II.7. Let $R$ be a ring. If $R$ has a complete set $E = \{e_{ij}\}$ of orthogonal idempotents with $m(E) \geq 2$, then $R \in \mathcal{R}$.

Proof. For $e_{ij} \in E$, let $\overline{e_{ij}}$ denote the equivalence class determined by $e_{ij} \in E$. Then without loss of generality we have

$$R = (e_{11} R \oplus \cdots \oplus e_{1j} R) \oplus \cdots \oplus (e_{k1} R \oplus \cdots \oplus e_{kj} R)$$

where $e_{ij} \in \overline{e_{ij}}$, $i = 1, 2, \ldots, k$, $j = e_{11} + \cdots + e_{kj}$, and $j_i \geq 2$ for all $i$. Let $G$ be an $R$-module and let $G_{ij} := e_{ij} G$. Then $G = G_{11} \oplus \cdots \oplus G_{kj}$, a group direct sum. Let $g_{11}, \ldots, g_{kj} \in G$, $f \in M_R(G)$ and consider $g = f(e_{11} g_{11} + \cdots + e_{kj} g_{kj}) - f(e_{11} g_{11}) - \cdots - f(e_{kj} g_{kj})$. Using the orthogonality, we find for each $e_{ij} \in E$, $e_{ij} g = 0$; hence $1 \cdot g = 0$. It remains to show that $f(e_{ij} g_1 + e_{ij} g_2) = f(e_{ij} g_1) + f(e_{ij} g_2)$ for all $e_{ij} \in E$, $g_1, g_2 \in G$. Let $e_{ij}' \in \overline{e_{ij}}$, $e_{ij} \neq e_{ij}'$. For ease of notation we let $e_1 = e_{ij}$, $e_2 = e_{ij}'$. From Lemma II.6,
there exist $e_{12}, e_{21}$ in $R$ with $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$, $e_1e_{12}e_2 = e_{12}$, and $e_2e_{21}e_1 = e_{21}$. Then
\[
f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) = f(e_1g_1 + e_{12}e_2e_{21}g_2 + e_2e_{21}g_2) \]
\[
= (1 + e_{12})f(e_1g_1 + e_2e_{21}g_2) \]
\[
= (1 + e_{12})(f(e_1g_1) + f(e_2e_{21}g_2)) \]
\[
= f(e_1g_1) + f(e_2e_{21}g_2) + f(e_1e_{12}e_2e_{21}g_2),
\]
so we see that $f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) = f(e_1g_1) + f(e_1g_2) + f(e_2e_{21}g_2)$. But
\[
f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) = f(e_1g_1 + e_1g_2) + f(e_2e_{21}g_2)
\]
by the first part of the proof, so the result follows.

As an application of this result we show that $R$ is closed with respect to arbitrary products of matrix rings of size at least two. We remark that it is unknown to the authors if $R$ is closed under arbitrary products of rings in $R$.

To fix some notation we let $M_n(S)$ denote the ring of $n \times n$ matrices over $S$. Further, let $(i, j), (k, l), i, j, k, l \in \{1, 2, \ldots, n\}$ be positions located on some diagonal of the $n \times n$-board for matrices of $M_n(S)$. Then $M((i, j), (k, l))$ will denote the matrix with 1's on this diagonal between and including $(i, j), (k, l)$, and 0's elsewhere. We abbreviate $M((i, i), (j, j))$ by $M(i, j)$ and $M(i, i)$ by $M(i)$.

**Theorem II.8.** Let $\{R_\alpha | \alpha \in A\}$ be a collection of rings, $\{n_\alpha | \alpha \in A\}$ a collection of integers with $n_\alpha \geq 2$, and let $R = \prod_\alpha M_{n_\alpha}(R_\alpha)$. Then $R \in R$. In particular, for any ring $R$, if $n \geq 2$, $M_n(R) \in R$.

**Proof.** We define a complete set $E = \{e_1, \ldots, e_5\}$ of orthogonal idempotents as follows. If $n_\alpha$ is odd let $e_k(\alpha) = M(k)$ for $k \in \{1, 2, 3\}$, $e_4(\alpha) = 0 = e_5(\alpha)$ if $n_\alpha = 3$ and $e_4(\alpha) = M(4, (n_\alpha + 3)/2)$ and $e_5(\alpha) = M((n_\alpha + 5)/2, n_\alpha)$ for $n_\alpha > 3$. If $n_\alpha$ is even, let $e_k(\alpha) = 0$ for $k \in \{1, 2, 3\}$, $e_4(\alpha) = M(1, n_\alpha/2)$, and $e_5(\alpha) = M((n_\alpha + 2)/2, n_\alpha)$. One then verifies that $E$ is a complete set of orthogonal idempotents. Let $f, g \in R$ be defined by
\[
f(\alpha) = M\left(\left(4, \frac{n_\alpha + 5}{2}\right), \left(\frac{n_\alpha + 3}{2}, n_\alpha\right)\right),
\]
\[
g(\alpha) = M\left(\left(\frac{n_\alpha + 5}{2}, 4\right), \left(n_\alpha, \frac{n_\alpha + 3}{2}\right)\right) \quad \text{if } n_\alpha \text{ is odd and } n_\alpha > 3,
\]
\[
f(\alpha) = 0 = g(\alpha) \quad \text{if } n_\alpha = 3,
\]
and
\[
f(\alpha) = M\left(\left(1, \frac{n_\alpha + 2}{2}\right), (n_\alpha/2, n_\alpha)\right),
\]
\[
g(\alpha) = M\left(\left(\frac{n_\alpha + 2}{2}, 1\right), (n_\alpha, n_\alpha/2)\right) \quad \text{if } n_\alpha \text{ is even.}
\]

From this we see that $fg = e_4$ and $gf = e_5$; hence $e_4R \cong_R e_5R$ by the remark in II.6. Moreover, $e_1R \cong_R e_2R \cong_R e_3R$ so $m(E) = 2$, unless $n_\alpha = 3$ for all $\alpha \in A$ in which case $m(E) = 3$. Therefore $R \in R$. 

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Corollary II.9. Let $R$ be a ring. Then $m(E) \geq 2$ for some complete set $E$ of orthogonal idempotents if and only if $R$ contains a subring $S$ such that $1 \in S$ and $S$ is a direct sum of ideals $I_k$ which (as rings) are isomorphic to full matrix rings of size at least 2.

Proof. Let $E = \{e_1, \ldots, e_n\}$ be a complete set of orthogonal idempotents such that $m(E) \geq 2$ and let $E_1, \ldots, E_t$ denote the equivalence classes with respect to $\sim$. If $I_k = \sum_{e_j \in E_k} e_j$, $k \in \{1, 2, \ldots, t\}$, then $S = I_1 \oplus \cdots \oplus I_t$ is a subring of $R$ and $1 \in S$. We note that $\sum_{e_j \in E_k} e_j$ is the identity for $I_k$. Since $I_k I_s = \{0\}$ for $r \neq s$, each $I_k$ is an ideal of $S$. Let $e_k \in E_k$. Since $e_k R \cong_R e_j R$ for each $e_j \in E_k$, there exist $e_{kj}, e_{jk}$ with the properties of Lemma II.6. We define $e_{ij} = e_{ik} e_{kj}$ and observe that $e_{ij} = e_{ik} e_{ij} e_{ij}$ and $e_{ji} = e_{j} e_{ji} e_{i}$; hence $e_{ij}, e_{ji} \in I_k$. As in [1, p. 52] $\{e_{ij}\}$ is a set of matrix units for $I_k$ so $I_k$ is a matrix ring of size at least 2 since $|E_k| \geq 2$. For the converse, it follows from Theorem II.8 that $m(E) \geq 2$ for some complete set $E$ of orthogonal idempotents of $S$. Since $1 \in S$ our statement follows.

We now turn to a characterization for a rather large class of rings, which includes Artinian rings, to be in $\mathcal{R}$. We need first a lemma which gives a necessary condition for a ring to be in $\mathcal{R}$.

Lemma II.10. Let $\varphi: R \to S$ be a homomorphism such that $S$ is integral, i.e., $S$ has no divisors of zero. Then $R \notin \mathcal{R}$.

Proof. From Theorem II.2, it suffices to show $S \notin \mathcal{R}$. Let $G$ denote the $S$-module $S \oplus S$ and let $X = (S \oplus \{0\}) \setminus \{(0, 0)\}$. Then $s(s_1, s_2) \in X$ implies $s s_2 = 0$. Since $(0, 0) \notin X$, $s \neq 0$ so $(s_1, s_2) \in X$. It is straightforward to verify that $X$ satisfies the conditions of Theorem II.2 of [3]. Hence $M_S(G)$ is not a ring, so $S \notin \mathcal{R}$.

As a corollary we obtain further necessary conditions for a ring $R$ to be in $\mathcal{R}$.

Corollary II.11. (i) If there exists a homomorphism $\psi: R \to S$ where $S$ is commutative then $R \notin \mathcal{R}$.

(ii) If $R$ has no nonzero nilpotent elements then $R \notin \mathcal{R}$.

Proof. (i) Since $S$ is commutative, $S$ has a nonzero integral homomorphic image. Since $S \notin \mathcal{R}$, $R \notin \mathcal{R}$.

(ii) If $R$ has no nonzero nilpotent elements then again we find that $R$ has a nonzero integral homomorphic image [6, p. 202].

We recall [6, p. 217] that a ring $R$ with Jacobson radical $J(R)$ is semiperfect if $R/J(R)$ is semisimple Artinian and $J(R)$ is idempotent lifting. In particular every Artinian ring is semiperfect. We use Theorem II.7 to completely characterize those semiperfect rings in $\mathcal{R}$.

Theorem II.12. Let $R$ be a semiperfect ring. The following are equivalent:

(i) $R \in \mathcal{R}$;
(ii) $R/J(R) \in \mathcal{S}$;

(iii) $R/J(R)$ is the direct product of $n_i \times n_i$ matrix rings over division rings $D_i$ with $n_i \geq 2$ for each $i$.

Proof. (i) $\Rightarrow$ (ii) follows from Corollary II.3 (iv). Since idempotents in $R/J(R)$ can be lifted to $R$ and since each $n_i \geq 2$, there is a complete set $E$ of idempotents in $R$ with $m(E) \geq 2$. Hence $R \in \mathcal{S}$ and (iii) $\Rightarrow$ (i). Suppose now $R/J(R)$ is in $\mathcal{S}$. Since $R$ is semiperfect, $R/J(R)$ is the direct product of a finite number of $n_i \times n_i$-matrix rings over division rings $D_i$. Since $\overline{R} = R/J(R)$ is in $\mathcal{S}$ by hypothesis, $\overline{R}$ has no nonzero integral homomorphic images. Thus we must have $n_i \geq 2$ for all $i$.

III. Miscellaneous remarks

In this section we collect a few remarks about rings in $\mathcal{S}$. We start out with an example which shows that the converse of Corollary II.3 (ii) does not hold, i.e., we show that a subdirect product of rings in $\mathcal{S}$ need not be in $\mathcal{S}$.

Example III.1. Let $R := \{(A_1, A_2, \ldots) \in \prod_{n=1}^{\infty} M_2(Z) | A_n \text{ is a diagonal matrix except for finitely many n}\}$. Then $R$ is a subdirect product of the rings $M_2(Z)$ which are in $\mathcal{S}$. But $R \notin \mathcal{S}$ since $I := \{(A_1, A_2, \ldots) \in R | A_n = 0 \text{ for all but finitely many n}\}$ is an ideal in $R$ and $R/I$ is commutative.

As we have seen, no division ring is in $\mathcal{S}$. One next investigates which simple rings are in $\mathcal{S}$. If $R$ is a simple ring with a minimal left ideal then from [1, p. 88] or [6, p. 157] $R$ is a matrix ring of size at least 2 over a division ring. Thus $R \in \mathcal{S}$. However, not every simple ring which is not a division ring is in $\mathcal{S}$. For example, we let $R$ be the ring of differential polynomials over a field. Then $R$ is a simple ring with no minimal left ideals, but $R$ is integral so $R \notin \mathcal{S}$. On the other hand, $\mathcal{S}$ does contain some simple rings without minimal left ideals. In fact, let $V$ be any vector space of countable dimension over a division ring $D$ and let $I$ be the ideal of $\text{End}_D V$ consisting of those linear transformations of $V$ of finite dimensional range. We show $\text{End}_D V \in \mathcal{S}$. We actually show that for any vector space $W$ over $D$ for which $\dim_D W \geq 2$, $\text{End}_D W \in \mathcal{S}$. If $W$ is finite dimensional then the result follows from Theorem II.8. Therefore we take $W$ to be infinite dimensional over $D$ with basis $B$. Since $B$ is infinite, there exist disjoint subsets $B_1, B_2$ of $B$ with $B = B_1 \cup B_2$ and a bijection $\sigma : B_1 \rightarrow B_2$. For $x \in B_1$, let $e_1(x) = x$ and for $x \in B_2$, let $e_1(x) = 0$. Extend $e_1$ linearly to obtain an endomorphism $e_1 \in \text{End}_D W$. In the same manner we get $e_2 \in \text{End}_D W$, $e_2(x) = 0$, $x \in B_1$ and $e_2(x) = x$, $x \in B_2$. Then $1_W = e_1 + e_2$, $e_1$ and $e_2$ are idempotents and $e_i e_j = 0$ for $i \neq j$.

Similarly, define $e_{12} \in \text{End}_D W$ by $e_{12}(x) = 0$, $x \in B_1$, $e_{12}(x) = \sigma^{-1}(x)$ for $x \in B_2$, and $e_{21} \in \text{End}_D W$ by $e_{21}(x) = \sigma(x)$, $x \in B_1$, and $e_{21}(x) = 0$, $x \in B_2$. Then $e_{12}e_{21} = e_1$ and $e_{21}e_{12} = e_2$. From Theorem II.7 we see that $\text{End}_D W \in \mathcal{S}$.
We return to our special case and note that since $\text{End}_D(V) \in \mathcal{R}$ so does $\text{End}_D(V)/I$. But this is a simple ring with no minimal left ideals.

In our final result we present an interesting characterization of $2 \times 2$ matrix rings. It is unknown to the authors if this result is new, but we have not been able to locate it in the literature.

**Theorem III.2.** For a ring $R$ the following are equivalent:

(i) $R$ is a ring of $2 \times 2$ matrices over some ring $S$.

(ii) There exist elements $x, y \in R$ such that $x^2 = y^2 = 0$ and $x + y$ is invertible.

**Proof.** (i) $\Rightarrow$ (ii). If $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a set of matrix units for $R$, then $e_{12}^2 = e_{21}^2 = 0$ and $(e_{12} + e_{21})^2 = 1$.

(ii) $\Rightarrow$ (i). Suppose that $(x + y)r = r(x + y) = 1$. Then $xyr = x$ and $rxy = y$, so $rx = yr$. Also, $rxy = x$, $yxr = y$, hence $ry = xr$. Consequently $xr + rx = 1$. But then $xrx = x$ and $(rx)^2 = rx$. Further $rx \neq 1$ and $rx \neq 0$ since $r$ is invertible and $x \neq 0$. Therefore $rx$ is a nontrivial idempotent. Similarly $ry = xr$ is a nontrivial idempotent. Now let $e_{11} = rx$, $e_{22} = ry$, $e_{12} = r^2 y$, and $e_{21} = x$. Then $e_{12}^2 = r^2 yrry = r^2 rxxr = 0$, $e_{12} e_{21} = rryx = rxxr = rx = e_{11}$, $e_{21} e_{12} = xrry = (ry)^2 = ry = e_{22}$, and $e_{11} e_{22} = rxy = rxxr = 0$. In fact, one verifies that $e_{ij} e_{kl} = \delta_{jk} e_{il}$. Thus $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a set of matrix units for $R$. Our statement now follows from [1, p. 52].

Thus a ring satisfying condition (ii) of the above theorem must be in $\mathcal{R}$. We also note that all of our examples of rings in $\mathcal{R}$ have nontrivial idempotents, hence the following question.

**Question A.** Are there rings in $\mathcal{R}$ with no nontrivial idempotents?

We conclude with a related question.

**Question B.** If $R \in \mathcal{R}$, is $m(E) \geq 2$ for some complete set $E$ of orthogonal idempotents in $R$?

**References**


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