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ON RINGS FOR WHICH HOMOGENEOUS MAPS ARE LINEAR

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ABSTRACT. Let $\mathscr{R}$ be the collection of all rings R such that for every R-module G, the centralizer near-ring $M_R(G) = \{f: G \to G | f(rx) = rf(x), r \in R, x \in G\}$ is a ring. We show $R \in \mathscr{R}$ if and only if $M_R(G) = \mathrm{End}_R(G)$ for each R-module G. Further information about $\mathscr{R}$ is collected and the Artinian rings in $\mathscr{R}$ are completely characterized.

I. INTRODUCTION

Let R be a ring with identity and G a unitary left R-module. The set $M_R(G) := \{f: G \to G | f(rx) = rf(x), r \in R, x \in G\}$ is a zero-symmetric near-ring with identity under the operations of function addition and composition. If $G = R$, $M_R(R) \cong R$ so $M_R(R)$ is a ring. If R is a field and $G = R^2$ then it is known that $M_R(R^2)$ is not a ring [3]. On the other hand, when R is a finite simple ring, but not a field, it was found in [2] that $M_R(G)$ is a ring for each finite R-module G. In this paper we investigate two questions raised by the above remarks; namely, (1) characterize those rings R such that $M_R(G)$ is a ring for every R-module G and (2) characterize those rings R such that $M_R(G) = \mathrm{End}_R G$, for every R-module G.

We let $\mathscr{R}$ denote the collection of rings satisfying (1) and $\mathscr{E}$ denote the collection of rings satisfying (2). Of course $\mathscr{E} \subseteq \mathscr{R}$. We show in the next section that in fact, $\mathscr{E} = \mathscr{R}$.

The problem then remains to characterize the class $\mathscr{R}$. It is the objective of this paper to initiate such an investigation. We collect information about $\mathscr{R}$ and present some classes of rings in $\mathscr{R}$. In particular we completely characterize the Artinian rings in $\mathscr{R}$. For a ring R and an abelian group G let $rx = 0$ for all $r \in R$, $x \in G$. Then $M_R(G) = M_0(G)$ which is a nonring whenever

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$|G| \geq 3$. Thus we make the following

Conventions. All rings have identity 1, all modules are unitary, and all homomorphisms are identity preserving.

II. General results

Let R be a ring and G an R-module. It is well known that $M(G) = G^G = \{f: G \to G\}$ is a near-ring with respect to function addition and function composition. (We refer the reader to the books by Meldrum [4] and Pilz [5] for near-ring information.) The above defined near-ring $M_R(G)$ is a subnear-ring of $M(G)$ with the identity function as identity element. Moreover, $M(G)$ is an R-module under the action $(rf)(x) = r(f(x))$, $r \in R$, $f \in M(G)$, $x \in G$. As above let $\mathscr{E}$ denote the class of all rings R such that $M_R(G) = \mathrm{End}_R(G)$ for each R-module G, and let $\mathscr{R}$ denote the class of all rings R such that $M_R(G)$ is a ring for each R-module G.

Theorem II.1. $\mathscr{E} = \mathscr{R}$.

Proof. Since $\mathscr{E} \subseteq \mathscr{R}$ it suffices to establish the converse. Let $R \in \mathscr{R}$. To each $f \in M_R(G)$ we associate a map $\hat{f}: M(G) \to M(G)$ where $\hat{f}(\varphi) = f \circ \varphi$, $\varphi \in M(G)$. Since $\hat{f}(r\varphi) = f \circ r\varphi = r(f \circ \varphi) = r\hat{f}(\varphi)$ we see that $\hat{f} \in M_R(M(G))$. Now $M_R(M(G))$ is a ring since $R \in \mathscr{R}$; hence $\hat{f}\circ(\alpha+\beta) = \hat{f}\circ\alpha+\hat{f}\circ\beta$ for each $\alpha, \beta \in M_R(M(G))$. Therefore, for $\varphi \in M(G)$, $\hat{f}(\alpha(\varphi) + \beta(\varphi)) = \hat{f}(\alpha(\varphi)) + \hat{f}(\beta(\varphi))$ which in turn gives $f(\alpha(\varphi)(x) + \beta(\varphi)(x)) = f(\alpha(\varphi)(x)) + f(\beta(\varphi)(x))$ for each $x \in G$.

Now let $\psi \in M(G)$. The map $\overline{\psi}$ defined by $\overline{\psi}(\varphi) = \varphi \circ \psi$, $\varphi \in M(G)$ is in $M_R(M(G))$ since $\overline{\psi}(r\varphi) = r\varphi \circ \psi = r(\varphi \circ \psi) = r\overline{\psi}(\varphi)$. Given any x_1, x_2 in G, there exist ψ_1, ψ_2 in $M(G)$ such that $\psi_1(x) = x_1$, $\psi_2(x) = x_2$ for all $x \in G$. Thus for $\varphi = \mathrm{id}$ in $M(G)$, $\overline{\psi}_i(\varphi)(x) = x_i$, $i = 1, 2$. Hence $f(x_1 + x_2) = f(x_1) + f(x_2)$ so $f \in \mathrm{End}_R(G)$, i.e., $M_R(G) = \mathrm{End}_R(G)$.

We next determine some properties for the class $\mathscr{R}$.

Theorem II.2. *Let $S \in \mathscr{R}$ and let $\varphi: S \to R$ be a homomorphism. Then $R \in \mathscr{R}$.*

Proof. Let G be an R-module. Then as usual, G is a (unitary) S-module via $s * g = \varphi(s) \cdot g$, $s \in S$, $g \in G$. For $f \in M_R(G)$, $s \in S$, $g \in G$ we have $f(s * g) = f(\varphi(s)g) = \varphi(s)f(g) = s * f(g)$ so $f \in M_S(G)$. Since $S \in \mathscr{R}$, $M_S(G)$ is a ring and hence so is $M_R(G)$, i.e., $R \in \mathscr{R}$.

Corollary II.3. (i) *Let $S \in \mathscr{R}$. If S can be embedded in a ring R (preserving the identity), then $R \in \mathscr{R}$.*

(ii) *If a subdirect product of rings R_α, $\alpha \in A$, is in $\mathscr{R}$ then each $R_\alpha \in \mathscr{R}$.*

(iii) *If $R \in \mathscr{R}$ then $R^X \in \mathscr{R}$ for each set X.*

(iv) *If* rad *is any radical for rings then $R \in \mathscr{R}$ implies that the "semisimple part" $R/\mathrm{rad}(R)$ is in $\mathscr{R}$.*

Proof. (i), (ii), and (iv) follow from II.2 while (iii) follows from (i) via the identity preserving map $r \mapsto (r, r, \ldots)$.

Theorem II.4. *Let R be the group direct sum of subrings $R_1, R_2, \ldots, R_n$ which (as rings) are in $\mathscr{R}$. Then $R \in \mathscr{R}$.*

Proof. Let $1 = r_1 + r_2 + \cdots + r_n$ be the decomposition of the identity $1 \in R$ and let 1_i denote the identity of R_i, $i = 1, \ldots, n$. If G is an R-module then each $G_i := 1_i G$ is (unitary) R_i-module. Also, for $f \in M_R(G)$, $f(G_i) = f(1_i G) \subseteq 1_i f(G) \subseteq G_i$, $i = 1, 2, \ldots, n$. Thus $\Phi\colon M_R(G) \to M_{R_1}(G_1) \oplus \cdots \oplus M_{R_n}(G_n)$ defined by $\Phi(f) = (f|_{G_1}, \ldots, f|_{G_n})$ is a near-ring homomorphism. If $f \in \ker \Phi$ then f is the zero map on each G_i. Therefore for $x \in G$, $f(x) = 1 \cdot f(x) = r_1 f(x) + \cdots + r_n f(x) = f(r_1 x) + \cdots + f(r_n x) = 0$, so $\ker \Phi = \{0\}$ and Φ is an embedding. Since $M_{R_i}(G_i)$ is a ring, for each i, $M_R(G)$ is a ring, so $R \in \mathscr{R}$.

Corollary II.5. *Let $R_1, R_2, \ldots, R_n$ be rings and let $R = R_1 \oplus \cdots \oplus R_n$, the direct sum of rings. Then $R \in \mathscr{R}$ if and only if $R_i \in \mathscr{R}$, $i = 1, 2, \ldots, n$.*

Proof. If $R \in \mathscr{R}$, each $R_1 \in \mathscr{R}$ from II.2 while the converse follows from the previous theorem.

Let $E = \{e_1, e_2, \ldots, e_n\}$ be a set of mutually orthogonal idempotents of the ring R with $1 = \sum_{i=1}^n e_i$. We say E is a *complete set of orthogonal idempotents.* We define a relation $\sim$ on E by $e_i \sim e_j$ if $e_i R$ and $e_j R$ are isomorphic as R-modules ($e_i R \cong_R e_j R$). It is clear that $\sim$ is an equivalence relation on E. We let $m(E) = \min\{|B| \,|\, B$ is an equivalence class with respect to $\sim\}$. The following well-known result determines when the R-modules $e_i R$ and $e_j R$ are isomorphic.

Lemma II.6 [1, p. 51]. *Let e_1, e_2 be idempotents of a ring R. Then $e_1 R \cong_R e_2 R$ if and only if there exist e_{12}, e_{21} in R such that $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$, $e_1 e_{12} e_2 = e_{12}$, and $e_2 e_{21} e_1 = e_{21}$. (As pointed out in* [1], *the first two conditions suffice.)*

Our next result gives a very useful criterion for determining many rings in $\mathscr{R}$.

Theorem II.7. *Let R be a ring. If R has a complete set $E = \{e_{ij}\}$ of orthogonal idempotents with $m(E) \geq 2$, then $R \in \mathscr{R}$.*

Proof. For $e_{ij} \in E$, let $\bar{e}_{ij}$ denote the equivalence class determined by $e_{ij} \in E$. Then without loss of generality we have

$$R = (e_{11}R \oplus \cdots \oplus e_{1j_1}R) \oplus \cdots \oplus (e_{k1}R \oplus \cdots \oplus e_{kj_k}R)$$

where $e_{ij} \in \bar{e}_{i1}$, $i = 1, 2, \ldots, k$, $1 = e_{11} + \cdots + e_{kj_k}$, and $j_i \geq 2$ for all i. Let G be an R-module and let $G_{ij} := e_{ij}G$. Then $G = G_{11} \oplus \cdots \oplus G_{kj_k}$, a group direct sum. Let $g_{11}, \ldots, g_{kj_k} \in G$, $f \in M_R(G)$ and consider $g = f(e_{11}g_{11} + \cdots + e_{kj_k}g_{kj_k}) - f(e_{11}g_{11}) - \cdots - f(e_{kj_k}g_{kj_k})$. Using the orthogonality, we find for each $e_{ij} \in E$, $e_{ij}g = 0$; hence $1 \cdot g = 0$. It remains to show that $f(e_{ij}g_1 + e_{ij}g_2) = f(e_{ij}g_1) + f(e_{ij}e_2)$ for all $e_{ij} \in E$, $g_1, g_2 \in G$. Let $e'_{ij} \in \bar{e}_{ij}$, $e_{ij} \neq e'_{ij}$. For ease of notation we let $e_1 = e_{ij}$, $e_2 = e'_{ij}$. From Lemma II.6, there exist e_{12}, e_{21} in R with $e_{12}e_{21} = e_1$, $e_{21}e_{12} = e_2$, $e_1e_{12}e_2 = e_{12}$, and $e_2e_{21}e_1 = e_{21}$. Then

$$\begin{aligned} f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) &= f(e_1g_1 + e_{12}e_2e_{21}g_2 + e_2e_{21}g_2) \\ &= (1 + e_{12})f(e_1g_1 + e_2e_{21}g_2) \\ &= (1 + e_{12})(f(e_1g_1) + f(e_2e_{21}g_2)) \\ &= f(e_1g_1) + f(e_2e_{21}g_2) + f(e_{12}e_2e_{21}g_2), \end{aligned}$$

so we see that $f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) = f(e_1g_1) + f(e_1g_2) + f(e_2e_{21}g_2)$. But $f(e_1g_1 + e_1g_2 + e_2e_{21}g_2) = f(e_1g_1 + e_1g_2) + f(e_2e_{21}g_2)$ by the first part of the proof, so the result follows.

As an application of this result we show that $\mathscr{R}$ is closed with respect to arbitrary products of matrix rings of size at least two. We remark that it is unknown to the authors if $\mathscr{R}$ is closed under arbitrary products of rings in $\mathscr{R}$.

To fix some notation we let $\mathbf{M}_n(S)$ denote the ring of $n \times n$ matrices over S. Further, let $(i, j), (k, l), i, j, k, l \in \{1, 2, \dots, n\}$ be positions located on some diagonal of the $n \times n$-board for matrices of $\mathbf{M}_n(S)$. Then $M((i, j), (k, l))$ will denote the matrix with 1's on this diagonal between and including $(i, j), (k, l)$, and 0's elsewhere. We abbreviate $M((i, i), (j, j))$ by $M(i, j)$ and $M(i, i)$ by $M(i)$.

Theorem II.8. *Let $\{R_\alpha | \alpha \in A\}$ be a collection of rings, $\{n_\alpha | \alpha \in A\}$ a collection of integers with $n_\alpha \geq 2$, and let $R = \prod_\alpha \mathbf{M}_{n_\alpha}(R_\alpha)$. Then $R \in \mathscr{R}$. In particular, for any ring R, if $n \geq 2$, $M_n(R) \in \mathscr{R}$.*

Proof. We define a complete set $E = \{e_1, \dots, e_5\}$ of orthogonal idempotents as follows. If n_α is odd let $e_k(\alpha) = M(k)$ for $k \in \{1, 2, 3\}$, $e_4(\alpha) = 0 = e_5(\alpha)$ if $n_\alpha = 3$ and $e_4(\alpha) = M(4, (n_\alpha + 3)/2)$ and $e_5(\alpha) = M((n_\alpha + 5)/2, n_\alpha)$ for $n_\alpha > 3$. If n_α is even, let $e_k(\alpha) = 0$ for $k \in \{1, 2, 3\}$, $e_4(\alpha) = M(1, n_\alpha/2)$, and $e_5(\alpha) = M((n_\alpha + 2)/2, n_\alpha)$. One then verifies that E is a complete set of orthogonal idempotents. Let $f, g \in R$ be defined by

$$\begin{aligned} &f(\alpha) = M\left(\left(4, \frac{n_\alpha + 5}{2}\right), \left(\frac{n_\alpha + 3}{2}, n_\alpha\right)\right), \\ &g(\alpha) = M\left(\left(\frac{n_\alpha + 5}{2}, 4\right), \left(n_\alpha, \frac{n_\alpha + 3}{2}\right)\right) \quad \text{if } n_\alpha \text{ is odd and } n_\alpha > 3, \\ &f(\alpha) = 0 = g(\alpha) \quad \text{if } n_\alpha = 3, \end{aligned}$$

and

$$\begin{aligned} &f(\alpha) = M\left(\left(1, \frac{n_\alpha + 2}{2}\right), (n_\alpha/2, n_\alpha)\right), \\ &g(\alpha) = M\left(\left(\frac{n_\alpha + 2}{2}, 1\right), (n_\alpha, n_\alpha/2)\right) \qquad \text{if } n_\alpha \text{ is even}. \end{aligned}$$

From this we see that $fg = e_4$ and $gf = e_5$; hence $e_4R \cong_R e_5R$ by the remark in II.6. Moreover, $e_1R \cong_R e_2R \cong_R e_3R$ so $m(E) = 2$, unless $n_\alpha = 3$ for all $\alpha \in A$ in which case $m(E) = 3$. Therefore $R \in \mathscr{R}$.

Corollary II.9. *Let R be a ring. Then $m(E) \geq 2$ for some complete set E of orthogonal idempotents if and only if R contains a subring S such that $1 \in S$ and S is a direct sum of ideals I_k which (as rings) are isomorphic to full matrix rings of size at least 2.*

Proof. Let $E = \{e_1, \dots, e_n\}$ be a complete set of orthogonal idempotents such that $m(E) \geq 2$ and let $E_1, \dots, E_t$ denote the equivalence classes with respect to $\sim$. If $I_k = \sum\{e_i R e_j | e_i, e_j \in E_k\}$, $k \in \{1, 2, \dots, t\}$, then $S = I_1 \oplus \cdots \oplus I_t$ is a subring of R and $1 \in S$. We note that $\sum_{e_i \in E_k} e_i$ is the identity for I_k. Since $I_r I_s = \{0\}$ for $r \neq s$, each I_k is an ideal of S. Let $e_k \in E_k$. Since $e_k R \cong_R e_j R$ for each $e_j \in E_k$, there exist e_{kj}, e_{jk} with the properties of Lemma II.6. We define $e_{ij} = e_{ik} e_{kj}$ and observe that $e_{ij} = e_i e_{ij} e_j$ and $e_{ji} = e_j e_{ji} e_i$; hence $e_{ij}, e_{ji} \in I_k$. As in [1, p. 52] $\{e_{ij}\}$ is a set of matrix units for I_k so I_k is a matrix ring of size at least 2 since $|E_k| \geq 2$. For the converse, it follows from Theorem II.8 that $m(E) \geq 2$ for some complete set E of orthogonal idempotents of S. Since $1 \in S$ our statement follows.

We now turn to a characterization for a rather large class of rings, which includes Artinian rings, to be in $\mathscr{R}$. We need first a lemma which gives a necessary condition for a ring to be in $\mathscr{R}$.

Lemma II.10. *Let $\varphi: R \to S$ be a homomorphism such that S is integral, i.e., S has no divisors of zero. Then $R \notin \mathscr{R}$.*

Proof. From Theorem II.2, it suffices to show $S \notin \mathscr{R}$. Let G denote the S-module $S \oplus S$ and let $X = (S \oplus \{0\}) \setminus \{(0, 0)\}$. Then $s(s_1, s_2) \in X$ implies $ss_2 = 0$. Since $(0, 0) \notin X$, $s \neq 0$ so $(s_1, s_2) \in X$. It is straightforward to verify that X satisfies the conditions of Theorem II.2 of [3]. Hence $M_S(G)$ is not a ring, so $S \notin \mathscr{R}$.

As a corollary we obtain further necessary conditions for a ring R to be in $\mathscr{R}$.

Corollary II.11. (i) *If there exists a homomorphism $\psi: R \to S$ where S is commutative then $R \notin \mathscr{R}$.*

(ii) *If R has no nonzero nilpotent elements then $R \notin \mathscr{R}$.*

Proof. (i) Since S is commutative, S has a nonzero integral homomorphic image. Since $S \notin \mathscr{R}$, $R \notin \mathscr{R}$.

(ii) If R has no nonzero nilpotent elements then again we find that R has a nonzero integral homomorphic image [6, p. 202].

We recall [6, p. 217] that a ring R with Jacobson radical $J(R)$ is *semiperfect* if $R/J(R)$ is semisimple Artinian and $J(R)$ is idempotent lifting. In particular every Artinian ring is semiperfect. We use Theorem II.7 to completely characterize those semiperfect rings in $\mathscr{R}$.

Theorem II.12. *Let R be a semiperfect ring. The following are equivalent:*

(i) $R \in \mathscr{R}$;

(ii) $R/J(R) \in \mathscr{R}$;

(iii) *$R/J(R)$ is the direct product of $n_i \times n_i$ matrix rings over division rings D_i with $n_i \geq 2$ for each i.*

Proof. (i) $\Rightarrow$ (ii) follows from Corollary II.3 (iv). Since idempotents in $R/J(R)$ can be lifted to R and since each $n_i \geq 2$, there is a complete set E of idempotents in R with $m(E) \geq 2$. Hence $R \in \mathscr{R}$ and (iii) $\Rightarrow$ (i). Suppose now $R/J(R)$ is in $\mathscr{R}$. Since R is semiperfect, $R/J(R)$ is the direct product of a finite number of $n_i \times n_i$-matrix rings over division rings D_i. Since $\overline{R} = R/J(R)$ is in $\mathscr{R}$ by hypothesis, $\overline{R}$ has no nonzero integral homomorphic images. Thus we must have $n_i \geq 2$ for all i.

III. Miscellaneous remarks

In this section we collect a few remarks about rings in $\mathscr{R}$. We start out with an example which shows that the converse of Corollary II.3 (ii) does not hold, i.e., we show that a subdirect product of rings in $\mathscr{R}$ need not be in $\mathscr{R}$.

Example III.1. Let $R := \{(A_1, A_2, \dots) \in \prod_{\mathbf{N}} \mathbf{M}_2(\mathbf{Z}) | A_n$ is a diagonal matrix except for finitely many $n\}$. Then R is a subdirect product of the rings $\mathbf{M}_2(\mathbf{Z})$ which are in $\mathscr{R}$. But $R \notin \mathscr{R}$ since $I := \{(A_1, A_2, \dots) \in R | A_n = 0$ for all but finitely many $n\}$ is an ideal in R and R/I is commutative.

As we have seen, no division ring is in $\mathscr{R}$. One next investigates which simple rings are in $\mathscr{R}$. If R is a simple ring with a minimal left ideal then from [1, p. 88] or [6, p. 157] R is a matrix ring of size at least 2 over a division ring. Thus $R \in \mathscr{R}$. However, not every simple ring which is not a division ring is in $\mathscr{R}$. For example, we let R be the ring of differential polynomials over a field. Then R is a simple ring with no minimal left ideals, but R is integral so $R \notin \mathscr{R}$. On the other hand, $\mathscr{R}$ does contain some simple rings without minimal left ideals. In fact, let V be any vector space of countable dimension over a division ring D and let I be the ideal of $\mathrm{End}_D V$ consisting of those linear transformations of V of finite dimensional range. We show $\mathrm{End}_D V \in \mathscr{R}$. We actually show that for any vector space W over D for which $\dim_D W \geq 2$, $\mathrm{End}_D W \in \mathscr{R}$. If W is finite dimensional then the result follows from Theorem II.8. Therefore we take W to be infinite dimensional over D with basis B. Since B is infinite, there exist disjoint subsets B_1, B_2 of B with $B = B_1 \cup B_2$ and a bijection $\sigma: B_1 \to B_2$. For $x \in B_1$, let $e_1(x) = x$ and for $x \in B_2$, let $e_1(x) = 0$. Extend e_1 linearly to obtain an endomorphism $e_1 \in \mathrm{End}_D W$. In the same manner we get $e_2 \in \mathrm{End}_D W$, $e_2(x) = 0$, $x \in B_1$ and $e_2(x) = x$, $x \in B_2$. Then $1_W = e_1 + e_2$, e_1 and e_2 are idempotents and $e_i e_j = 0$ for $i \neq j$.

Similarly, define $e_{12} \in \mathrm{End}_D W$ by $e_{12}(x) = 0$, $x \in B_1$, $e_{12}(x) = \sigma^{-1}(x)$ for $x \in B_2$, and $e_{21} \in \mathrm{End}_D W$ by $e_{21}(x) = \sigma(x)$, $x \in B_1$, and $e_{21}(x) = 0$, $x \in B_2$. Then $e_{12}e_{21} = e_1$ and $e_{21}e_{12} = e_2$. From Theorem II.7 we see that $\mathrm{End}_D W \in R$.

We return to our special case and note that since $\mathrm{End}_D V \in \mathscr{R}$ so does $\mathrm{End}_D(V)/I$. But this is a simple ring with no minimal left ideals.

In our final result we present an interesting characterization of 2×2 matrix rings. It is unknown to the authors if this result is new, but we have not been able to locate it in the literature.

Theorem III.2. *For a ring R the following are equivalent:*

(i) *R is a ring of 2×2 matrices over some ring S.*

(ii) *There exist elements $x, y \in R$ such that $x^2 = y^2 = 0$ and $x + y$ is invertible.*

Proof. (i) $\Rightarrow$ (ii). If $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a set of matrix units for R, then $e_{12}^2 = e_{21}^2 = 0$ and $(e_{12} + e_{21})^2 = 1$.

(ii) $\Rightarrow$ (i). Suppose that $(x + y)r = r(x + y) = 1$. Then $xyr = x$ and $rxy = y$, so $rx = yr$. Also, $ryx = x$, $yxr = y$, hence $ry = xr$. Consequently $xr + rx = 1$. But then $xrx = x$ and $(rx)^2 = rx$. Further $rx \neq 1$ and $rx \neq 0$ since r is invertible and $x \neq 0$. Therefore rx is a nontrivial idempotent. Similarly $ry = xr$ is a nontrivial idempotent. Now let $e_{11} = rx$, $e_{22} = ry$, $e_{12} = r^2 y$, and $e_{21} = x$. Then $e_{12}^2 = r^2 yrry = r^2 rxxr = 0$, $e_{12}e_{21} = rryx = rxrx = rx = e_{11}$, $e_{21}e_{12} = xrry = (ry)^2 = ry = e_{22}$, and $e_{11}e_{22} = rxry = rxxr = 0$. In fact, one verifies that $e_{ij}e_{kl} = \delta_{jk}e_{il}$. Thus $\{e_{ij} \mid 1 \leq i, j \leq 2\}$ is a set of matrix units for R. Our statement now follows from [1, p. 52].

Thus a ring satisfying condition (ii) of the above theorem must be in $\mathscr{R}$. We also note that all of our examples of rings in $\mathscr{R}$ have nontrivial idempotents, hence the following question.

Question A. Are there rings in $\mathscr{R}$ with no nontrivial idempotents?

We conclude with a related question.

Question B. If $R \in \mathscr{R}$, is $m(E) \geq 2$ for some complete set E of orthogonal idempotents in R?

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