SOME REMARKS ON THE HOMOLOGY OF MODULI SPACE OF INSTANTONS WITH INSTANTON NUMBER 2

YASUHIKO KAMIYAMA

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Dedicated to Professor Akio Hattori on his sixtieth birthday

Abstract. Let $M_2$ be the framed moduli space of SU(2) instantons with instanton number 2. By combining the results of Boyer and Mann and the results of Hattori, we determine the structure of $H^*(M_2; Z_2)$.

1. Introduction

We shall denote by $M_k$ the framed moduli space of SU(2) instantons with instanton number $c_2 = -k$. Recently Boyer and Mann [1] constructed homology operations on $M_k$ for all $k$ and thus constructed new homology classes in $H_*(M_k; Z_p)$. In the case $k = p = 2$, the result is as follows.

Theorem 1 [1]. The elements of $H_*(M_2; Z_2)$ constructed by Boyer and Mann are given by the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>1</th>
<th>2</th>
<th>3</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H_q(M_2; Z_2)$</td>
<td>$z_1 * [1]$</td>
<td>$z_2 * [1]$</td>
<td>$Q_1(z_1) z_2 * z_1 z_3 * [1]$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Q_2(z_1) z_2 * z_1$</td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>6</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z_3 Q_2(z_2)$</td>
</tr>
</tbody>
</table>

5

7

8

9

In another direction Hattori [4] completely determined the homotopy type of $M_2$ and as a result computed $H^*(M_2; Z)$ and $H^*(M_2; Z_2)$. The results are as follows.

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Theorem 2 [4]. The cohomology groups of $M_2$ with $\mathbb{Z}$ coefficients are given by the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^q(M_2;\mathbb{Z})$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
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<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
<td>$\mathbb{Z}_2$</td>
</tr>
<tr>
<td>generators</td>
<td>$\beta$</td>
<td>$\gamma$</td>
<td>$\delta$</td>
<td>$\beta \gamma$</td>
<td>$\delta$</td>
<td>$\xi$</td>
<td>$\gamma \delta$</td>
<td>$\beta \gamma \delta$</td>
<td></td>
</tr>
</tbody>
</table>

Theorem 3 [4]. The cohomology groups of $M_2$ with $\mathbb{Z}_2$ coefficients are given by the following table:

<table>
<thead>
<tr>
<th>$q$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>$H^q(M_2;\mathbb{Z}_2)$</td>
<td>$\mathbb{Z}_2$</td>
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<td>$\mathbb{Z}_2 \oplus \mathbb{Z}_2$</td>
</tr>
<tr>
<td>generators</td>
<td>$u$</td>
<td>$u^2$</td>
<td>$v$</td>
<td>$u^3$</td>
<td>$w$</td>
</tr>
</tbody>
</table>

The choice of the elements $v$ and $w$ will be specified later.

In this paper we combine these results to obtain further homological information about $M_2$.

2. Main results

We first study the following problem. Do the elements of Theorem 1 generate $H_\ast(M_2;\mathbb{Z}_2)$?

Proposition 1. The elements of Theorem 1 generate $H_\ast(M_2;\mathbb{Z}_2)$ and the following relations hold:

1. $Q_1(z_1) + z_2 * z_1 + z_3 * [1] = 0$.
2. $Q_2(z_1) = z_3 * z_1$.
3. $Q_1(z_2) + z_3 * z_2 + Q_3(z_1) = 0$.

Proof. Let $\mathcal{E}_2$ be the orbit space of SU(2) connections with instanton number 2 by the action of the based gauge group and let $i : M_2 \to \mathcal{E}_2$ be the inclusion.

Direct computations show that each element of Theorem 1 is nontrivial in $H_\ast(\mathcal{E}_2;\mathbb{Z}_2)$ and differs in $H_\ast(\mathcal{E}_2;\mathbb{Z}_2)$ except for

$$i_\ast Q_2(z_2) = i_\ast z_3 * z_1.$$ 

Therefore by using Theorem 3 we see that the elements of Theorem 1 generate $H_\ast(M_2;\mathbb{Z}_2)$ and there must be one relation for $q = 3, 4, 5$.

But [1, Proposition 9.10] shows that there are the following relations.

1. $i_\ast(Q_1(z_1) + z_2 * z_1 + z_3 * [1]) = 0$.
2. $i_\ast(Q_2(z_1) + z_3 * z_1) = 0$.

Using Cartan formula and Adem relation [2] we also see the following relation.
(iii) \( i_*(Q_1(z_2) + z_3 \ast z_2 + Q_3(z_1)) = 0 \).

Now by using Theorem 3 we see that the relations (i)–(iii) imply the relations (1)–(3) in Proposition 1.

Next we shall study the Kronecker products of elements of Theorems 1 and 3. On account of Proposition 1 we can take a basis of \( H_q(M_2; \mathbb{Z}_2) \) for \( q = 3, 4, 5 \) as follows:

\[
\begin{align*}
q = 3 & \quad Q_1(z_1) \quad z_2 \ast z_1 \\
q = 4 & \quad z_2^2 \quad z_3 \ast z_1 \\
q = 5 & \quad Q_1(z_2) \quad z_3 \ast z_2.
\end{align*}
\]

**Theorem 4.** The Kronecker products of elements of Theorems 1 and 3 are given by the following table:

<table>
<thead>
<tr>
<th>Kronecker products</th>
<th>1</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>( \langle u, z_1 \ast {1} \rangle = 1 )</td>
<td>( \langle u^2, z_1^2 \rangle = 0 )</td>
<td>( \langle v, z_1^2 \rangle = 1 )</td>
</tr>
<tr>
<td>( \langle u^2, z_2 \ast {1} \rangle = 1 )</td>
<td>( \langle v, z_2 \ast {1} \rangle = 0 )</td>
<td></td>
</tr>
<tr>
<td>( \langle u^2, z_3 \ast {1} \rangle = 1 )</td>
<td>( \langle v, z_3 \ast {1} \rangle = 0 )</td>
<td></td>
</tr>
</tbody>
</table>

In the above table we define \( v \) by

\[
\langle v, z_1^2 \rangle = 1, \quad \langle v, z_2 \ast \{1\} \rangle = 0.
\]

Note that

\[
\langle u^2, z_1^2 \rangle = 0, \quad \langle u^2, z_2 \ast \{1\} \rangle = 1.
\]

We define \( w \) by

\[
\langle w, z_2^2 \rangle = 1, \quad \langle w, z_3 \ast z_1 \rangle = 0.
\]

Note that

\[
\langle u^2 v, z_2^2 \rangle = 0, \quad \langle u^2 v, z_3 \ast z_1 \rangle = 1.
\]
Proof. Let $\Delta : M_k \to M_k \times M_k$ be the diagonal. Then we can easily show the following relations.

$$
\begin{align*}
\Delta_* z_1 &= z_1 \otimes [1] + [1] \otimes z_1, \\
\Delta_* z_2 &= z_2 \otimes [1] + z_1 \otimes z_1 + [1] \otimes z_2, \\
\Delta_* z_3 &= z_3 \otimes [1] + z_2 \otimes z_1 + z_1 \otimes z_2 + [1] \otimes z_3.
\end{align*}
$$

The following relation is known in [2].

$$
\Delta_* Q_j(a) = \sum_{r,s} Q_{j-r}(a'_r) \otimes Q_r(a''_s),
$$

where $\Delta_* a = \sum_s a'_s \otimes a''_s$. Theorem 4 easily follows from these results.

Next we shall study the integral classes. On account of Theorem 2 there exists an element $\sigma$ that generates $\mathbb{Z}_4$ in $H_3(M_2; \mathbb{Z})$ and there exists an element $\tau$ that generates $\mathbb{Z}$ in $H_1(M_2; \mathbb{Z})$. Let

$$
j_* : H_*(M_2; \mathbb{Z}) \to H_*(M_2; \mathbb{Z}_2)
$$

be mod 2 reduction.

We shall study $j_* \sigma$ and $j_* \tau$.

**Theorem 5.** The following relations hold.

$$
\begin{align*}
j_* \sigma &= z_3 * [1], \\
j_* \tau &= Q_3(z_2).
\end{align*}
$$

Proof. Let $\{ E^r_+ \}$ be the mod 2 homology Bockstein spectral sequence of $M_2$.

The following Nishida relation is known in [2].

$$
\beta Q^j(a) = (j - 1) Q^{j-1}(a),
$$

where $\beta$ is the Bockstein operation.

By using the Nishida relation we compute $E^2_*$ as follows.

<table>
<thead>
<tr>
<th>$q$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>7</th>
<th>8</th>
<th>9</th>
</tr>
</thead>
<tbody>
<tr>
<td>$E^2_*$</td>
<td>0</td>
<td>0</td>
<td>$z_3 * [1]$</td>
<td>$z^2_2$</td>
<td>0</td>
<td>0</td>
<td>$Q_3(z_2)$</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

From this table Theorem 5 follows.

Next as an application of Proposition 1 and Theorem 4, we prove the following theorem.

**Theorem 6.** The elements of Theorem 2 satisfy the following relations:

1. $\beta^2 = 2\delta$,
2. $\delta^2 = 0$,
3. $\gamma^2 = \beta \delta$.

Note that Theorem 6 completely determines the ring structure of $H^*(M_2; \mathbb{Z})$. 

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Proof. (1) is shown in [4].

As \( H^8(M_2; \mathbb{Z}) = 0 \) holds, (2) follows.

We shall prove (3). Let
\[
j_* : H^*(M_2; \mathbb{Z}) \to H^*(M_2; \mathbb{Z}_2)
\]
be mod 2 reduction.

All we have to show to prove (3) is \( j_* \gamma^2 \neq 0 \). Let \( u, v, w \) be elements in Theorem 3. Either \( j_* \gamma = u^3 \) or \( uv \) or \( u^3 + uv \) holds. We shall show that \( j_* \gamma = u^3 \) cannot occur. Assertion 1. The following relations hold.
\[
\begin{align*}
  & u_4 = 0, \\
  & v^2 = w.
\end{align*}
\]

In fact, in the same way as the proof of Theorem 4, we see the following Kronecker products.
\[
\begin{align*}
  & \langle u^4, z_2^2 \rangle = 0, \\
  & \langle u^4, z_3 * z_1 \rangle = 0, \\
  & \langle v^2, z_2^2 \rangle = 1, \\
  & \langle v^2, z_3 * z_1 \rangle = 0.
\end{align*}
\]

Assertion 2. The following relation holds.
\[
j_* \beta = u^2.
\]

In fact, the following holds.
\[
j_* \beta = Sq^1 u = u^2.
\]

Now suppose \( j_* \gamma = u^3 \). The table in Theorem 2 shows that
\[
j_* (\beta \gamma) \neq 0.
\]

But from Assertions 1 and 2 we have
\[
j_* (\beta \gamma) = (j_* \beta)(j_* \gamma) = u^2 u^3 = 0.
\]

This is a contradiction. Therefore either \( j_* \gamma = uv \) or \( u^3 + uv \) holds. Anyway
\[
(j_* \gamma)^2 = u^2 v^2 = u^2 w \neq 0.
\]

This completes the proof of (3).

Remark. In [4], whether \( \gamma^2 = 0 \) or not is left unknown.

Now by using the above results, we can completely determine \( H^*(M_2; \mathbb{Z}_2) \).

Theorem 7. \( H^*(M_2; \mathbb{Z}_2) = \mathbb{Z}_2[u, v]/(u^4, v^4) \) and \( Sq^1 v = uv \) hold. Note that the \( \mathcal{A} \) (2)-module structure of \( H^*(M_2; \mathbb{Z}_2) \) is completely determined.

Proof. The ring structure follows from Theorem 3 and Assertion 1 in Theorem 6. By using Theorem 4 and the following Kronecker products we can easily prove \( Sq^1 v = uv \).
\[
\begin{align*}
  & \langle Sq^1 v, Q_1(z_1) \rangle = 1, \\
  & \langle Sq^1 v, z_2 * z_1 \rangle = 1.
\end{align*}
\]

3. Appendix

The proof of Proposition 9.5 seems incomplete in [1]. By using Theorem 3, we shall give an explicit proof of this proposition.
Proposition 9.5 [1]. \( z_i * [1] = Q_i[1] \) for \( i = 1, 2, 3 \).

Proof. The proof of \( z_i * [1] = Q_i[1] \) is given in [1].

(i) Proof of \( z_2 * [1] = Q_2[1] \). Let \( i : M_2 \to \Omega^3 S^3 \) be the inclusion. Clearly \( H_2(\Omega^3 S^3; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \) and the basis of \( Q_i[1]^2 * [-2] \) and \( Q_2[1] \). By Theorem 3, \( H_2(M_2; \mathbb{Z}_2) = \mathbb{Z}_2 \oplus \mathbb{Z}_2 \). Note that \( z_i^2, Q_2[1], z_2 * [1] \) are elements of \( H_2(M_2; \mathbb{Z}_2) \). But

\[
\begin{cases}
i_z z_1 = Q_1[1] * [-1], \\
i_z z_2 = Q_2[1] * [-1]
\end{cases}
\]

are given in [1, Theorem 8.6]. Hence \( i_z z_1^2 = (Q_1[1] * [-1])^2 = Q_1[1]^2 * [-2] \) and \( i_z Q_2[1] = Q_2[1] \). Therefore \( i_z : H_2(M_2; \mathbb{Z}_2) \to H_2(\Omega^3 S^3; \mathbb{Z}_2) \) is an isomorphism. But

\[
i_z (z_2 * [1]) = (Q_2[1] * [-1]) * [1] = Q_2[1] \text{ by (A). Therefore } \]


(ii) Proof of \( z_3 * [1] = Q_3[1] \). Let \( f : SO(3) \to M_2 \) be the composite of \( SO(3) \to M_1 \times 1 \to M_1 \times M_1 \to M_2 \) and \( g : SO(3) \to M_2 \) be the composite of

\[
SO(3) \to S^3 \times z_1 1 \times 1 \to S^3 \times z_2 M_1 \times M_1 \to M_2.
\]

Clearly \( f_z z_i = z_i * [1] \) and \( g_z z_i = Q_i[1] \) hold for \( i = 1, 2, 3 \). But we have shown the following.

\[
f_z z_1 = g_z z_1, \quad f_z z_2 = g_z z_2.
\]

By Theorem 3, all we need is to prove the following equalities.

\[
\langle u^3, f_z z_3 \rangle = \langle u^3, g_z z_3 \rangle, \quad \langle u v, f_z z_3 \rangle = \langle u v, g_z z_3 \rangle.
\]

Let \( \Delta \) be the diagonal; then we easily see the following.

\[
\Delta_z z_3 = z_3 \otimes 1 + z_2 \otimes z_1 + z_1 \otimes z_2 + 1 \otimes z_3.
\]

Then

\[
\langle u^3, f_z z_3 \rangle = \langle u^2, f_z z_2 \rangle \langle u, f_z z_1 \rangle = \langle u^2, g_z z_2 \rangle \langle u, g_z z_1 \rangle = \langle u^3, g_z z_3 \rangle.
\]

\[
\langle u v, f_z z_3 \rangle = \langle u v, g_z z_3 \rangle \quad \text{is similarly proved.}
\]

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References


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