σ-HEREDITARILY CLOSURE-PRESERVING k-NETWORKS
AND g-METRIZABILITY

YOSHIO TANAKA

(Communicated by Dennis Burke)

Dedicated to Professor Ryosuke Nakagawa on his sixtieth birthday

Abstract. We show that a regular space is g-metrizable if and only if it is a
weakly first countable space with a σ-hereditarily closure-preserving k-network.

Introduction

Let $X$ be a space. For each $x \in X$, let $T_x$ be a collection of subsets of $X$ such that any element of $T_x$ contains $x$. Following Arhangel’$skii$ [1], the
collection $T_c = \bigcup \{T_x; x \in X\}$ is a weak base for $X$ if (a) and (b) below are
satisfied (see [4] or [8]).

(a) For each $B_1, B_2 \in T_x$, there exists $B_3 \in T_x$ such that $B_3 \subseteq B_1 \cap B_2$.

(b) $U \subseteq X$ is open if and only if, for each $x \in U$, there exists $B \in T_x$
such that $B \subseteq U$.

We call the $T_x$ a weak neighborhood base for $x$ in $X$, and each element of
$T_x$ a weak neighborhood of $x$. A space $X$ is weakly first countable (= $g$-first
countable [15], or $X$ satisfies the weak first axiom of countability [1]) if $X$ has
a weak base $T_c$ such that each $T_x$ is countable. Every first countable space, or
symmetric space (in the sense of [1, p. 125]) is weakly first countable. Every
weakly first countable σ-space is a symmetric space (cf. [1]).

We recall that a space $X$ is sequential (resp. a $k$-space) if $F \subseteq X$ is closed
in $X$ if and only if $F \cap C$ is closed in $C$ for every convergent sequence $C$
containing its limit point (resp. a compact subset $C$) in $X$. Every weakly first countable space is sequential (for example, [15]), and every sequential space is
a $k$-space.

F. Siwiec [15] introduced the notion of $g$-metrizable spaces as a generalization
of metric spaces. A space $X$ is $g$-metrizable if it has a $σ$-locally finite weak base. A space $X$ is metrizable if and only if it is a Fréchet, $g$-metrizable space.
Here a space $X$ is Fréchet if, whenever $x \in X$, there exists a sequence in $A$ converging to the point $x$. Other interesting properties of $g$-metrizable spaces are obtained by N. Jakovlev [8] (see also [18]).

Let $\mathcal{A} = \{ A_\alpha ; \alpha \in A \}$ be a collection of subsets of a space $X$. We recall that $\mathcal{A}$ is closure-preserving if $\bigcup \{ A_\alpha ; \alpha \in B \} = \bigcup \{ A_\alpha ; \alpha \in B \}$ for any $B \subset A$. $\mathcal{A}$ is hereditarily closure-preserving if $\bigcup \{ B_\alpha ; \alpha \in B \} = \bigcup \{ B_\alpha ; \alpha \in B \}$ whenever $B \subset A$ and $B_\alpha \subset A_\alpha$ for each $\alpha \in A$. A $\sigma$-hereditarily closure-preserving collection is the union of countably many hereditarily closure-preserving collections. We shall use "$\sigma$-CP" (resp. "$\sigma$-HCP") instead of "$\sigma$-closure preserving" (resp. "$\sigma$-hereditarily closure-preserving") in this paper. Obviously, every $\sigma$-locally finite collection is $\sigma$-HCP.

Let $\mathcal{W}$ be a cover of a space $X$. We recall that $\mathcal{W}$ is network (= net) for $X$ if for each $x \in X$ and neighborhood $U$ of $x$, there exists $C \in \mathcal{W}$ such that $x \in C \subset U$. $\mathcal{W}$ is a $k$-network if whenever $K \subset U$ with $K$ compact and $U$ open in $X$, then $K \subset \bigcup \mathcal{W}' \subset U$ for some finite $\mathcal{W}' \subset \mathcal{W}$. A space is an $\kappa$-space [14] if it has a $\sigma$-locally finite $k$-network. Every $\sigma$-HCP weak base is a $k$-network [4].

L. Foged [4] proved that a space is $g$-metrizable if and only if it is a weakly first countable space with a $\sigma$-locally finite $k$-network. (Hence, a $g$-metrizable space is equal to a symmetric, $\kappa$-space.)

In view of Foged's result, Y. Kotake [11] asked whether every weakly first countable space with a $\sigma$-HCP $k$-network is $g$-metrizable. Related to this question, he also posed the following problem in [12]: Characterize a space having a $\sigma$-HCP weak base.

In this paper, we give an affirmative answer to the first question and, as for the second question, we show that every $k$-space with a $\sigma$-HCP weak base is precisely a $g$-metrizable space.

We assume that all spaces are regular, and all maps are continuous and onto.

**Results**

For a cover $\mathcal{W}$ of a space $X$, we consider the following condition $(\ast)$, which is labeled $(C_2)$ in [17].

$(\ast)$ If $A$ is a sequence converging to a point $x$ and $U$ is a neighborhood, then there exists $C \in \mathcal{W}$ such that $C \subset U$ and $C$ contains a subsequence of $A$.

Obviously, every $k$-network satisfies $(\ast)$.

**Lemma 1** ([17]). Every $\sigma$-HCP cover satisfying $(\ast)$ is a $k$-network.

**Lemma 2.** Let $X$ be a $k$-space, or a Lindelöf space. If $X$ has a $\sigma$-HCP weak base, then it is weakly first countable.

**Proof.** Let $\mathcal{B} = \bigcup \{ \mathcal{B}_n ; n \in N \}$ be a $\sigma$-HCP weak base for $X$. Let $X$ be a $k$-space. Since $X$ has a $\sigma$-CP network of closed subsets, each point of $X$ is a $G_\delta$-set. Then any compact subset $K$ of $X$ is first countable, for each point of $K$ is a $G_\delta$-set in $K$. Then it is easy to show that a $k$-space $X$ is sequential.
Let $x \in X$ be not isolated. Then there exists a sequence $A$ converging to $x$ with $A \not= x$. Note that any weak neighborhood of $x$ contains $A$ eventually, because $A \cup \{x\}$ is closed, but $A$ is not closed in $X$, but each $\mathcal{B}_n$ is HCP and any infinite subset of $A$ is not closed in $X$. Then for any $n \in N$, $T_x \cap \mathcal{B}_n$ is finite. Hence $T_x \cap \mathcal{B}$ is countable. Then $X$ is weakly first countable. Next, let $X$ be Lindelöf. Then for any nonisolated point $x$ in $X$ and any $n \in N$, $T_x \cap \mathcal{B}_n$ is countable. Indeed, suppose that $T_x \cap \mathcal{B}_n$ is not countable for some $n \in N$. Then there exists a subcollection $\{B_\alpha; \alpha < \omega_1\}$ of $T_x \cap \mathcal{B}_n$. Since $x$ is not isolated in $X$, for any $B_\alpha$ and any neighborhood $G$ of $x$, $B_\alpha \cap G \cap \{x\} \not= \emptyset$. Then for each $\alpha < \omega_1$, there exists $x_\alpha \in B_\alpha \cap \{x\} \cap (X - \{x\})$. Then $\{x_\alpha; \alpha < \omega_1\}$ is a discrete, closed subset of $X$ with cardinality $\omega_1$. But $X$ is Lindelöf. This is a contradiction. Then for any $n \in N$, $T_x \cap \mathcal{B}_n$ is countable. Hence $X$ is weakly first countable.

Lemma 3 ([4]). Every weakly first countable space with a $\sigma$-locally finite $k$-network is $g$-metrizable.

Lemma 4. Let $X$ have a $\sigma$-CP $k$-network. Then, for each closed subset $F$ of $X$, there exists a sequence $\{V_n; n \in N\}$ of open subsets of $X$ such that each $V_n$ contains $F$, and for each sequence $\{x_n; n \in N\}$ converging to a point $x \not= F$, some $V_n$ contains only finitely many $x_n$, but does not contain $x$.

Proof. Let $\mathcal{C} = \bigcup\{\mathcal{C}_n; n \in N\}$ be a $\sigma$-CP $k$-network for $X$ by closed subsets of $X$ such that each $\mathcal{C}_n$ contains $F$, and for each sequence $\{x_n; n \in N\}$ converging to a point $x \not= F$, some $\mathcal{C}_n$ contains only finitely many $x_n$, but does not contain $x$.

Lemma 5. Let $X$ have a weak base $\bigcup\{T_x; x \in X\}$ such that each $T_x$ is countable. Let $D$ be a closed discrete subset of $X$. If $X$ has a $\sigma$-HCP $k$-network, then there exists a $\sigma$-discrete collection $\mathcal{D}$ in $X$ such that, for each $d \in D$, $\mathcal{D} \cap T_d$ is a weak base for $d$ in $X$.

Proof. First we note that, among regular spaces, if $\mathcal{A}$ is HCP, then so is $\overline{\{A; A \in \mathcal{A}\}}$. Thus, $X$ admits a $\sigma$-HCP $k$-network $\mathcal{C} = \bigcup\{\mathcal{C}_n; n \in N\}$ of closed subsets such that $\mathcal{C}_n \subset \mathcal{C}_{n+1}$ for each $n \in N$. Let $D = \{x_\alpha; \alpha \in A\}$. For each $n \in N$ and each $\alpha \in A$, let

$\mathcal{C}_{n, \alpha} = \{C \in \mathcal{C}_n; C \ni x_\alpha$ and $C \cap \{x_\beta; \beta \not= \alpha\} = \emptyset\}$,

let

$C(n, \alpha) = \bigcup\mathcal{C}_{n, \alpha}$,

and let

$C^*(n, \alpha) = C(n, \alpha) - \bigcup\{C(n, \beta); \beta \not= \alpha\}$.

For each $d \in D$, we can assume that $T_d = \{Q_n(d); n \in N\}$ is a decreasing weak neighborhood base for $d$ in $X$ consisting of closed subsets of $X$. For
each $x_\alpha \in D$, there exist $m, n \in N$ such that $Q_n(x_\alpha) \subset C^*(m, \alpha)$. Indeed, there exists $m \in N$ such that $Q_m(x_\alpha) \subset C(m, \alpha)$. To show this, suppose that $Q_n(x_\alpha) \not\subset C(n, \alpha)$ for any $n \in N$. Then there exists a sequence $\mathcal{A} = \{a_n ; n \in N\}$ such that $a_n \in Q_n(x_\alpha) - C(n, \alpha)$. Since the sequence $\mathcal{A}$ converges to $x_\alpha$, and $\mathcal{G}$ is a $k$-network of closed subsets, there exists $C \in \mathcal{G}_n$ for some $n \in N$ such that $C$ contains $x_\alpha$ and a subsequence of $\mathcal{A}$, and $C \cap \{x_\beta ; \beta \neq \alpha\} = \emptyset$. Then $C \subset C(m, \alpha)$ for $m \geq n$. Hence there exists $a_k \in C(k, \alpha)$ for some $k \geq n$. This is a contradiction. Then $Q_m(x_\alpha) \subset C(m, \alpha)$ for some $m \in N$. On the other hand, since $x_\alpha \notin \bigcup\{C(m, \beta) ; \beta \neq \alpha\}$, there exists $i \in N$ such that $Q_i(x_\alpha) \cap \bigcup\{C(m, \beta) ; \beta \neq \alpha\} = \emptyset$. Let $n \geq m, i$. Then $Q_n(x_\alpha) \subset C^*(m, \alpha)$. Now, for each $m, n \in N$, let $\mathcal{D}_{mn} = \{Q_n(x_\alpha) ; Q_n(x_\alpha) \subset C^*(m, \alpha), \alpha \in A\}$. For each $m \in N$, let $\mathcal{D}_m = \bigcup\{\mathcal{D}_{mn} ; n \in N\}$, and let $D_m = \bigcup\mathcal{D}_m$. Then $D \subset \bigcup\{D_m ; m \in N\}$ by the above arguments. We note that for each $m \in N$ and $\alpha, \beta \in A$, if $\alpha \neq \beta$, then $C^*(m, \alpha) \cap C^*(m, \beta) = \emptyset$, and $\mathcal{G}_{m, \alpha} \cap \mathcal{G}_{m, \beta} = \emptyset$. But for each $Q_n(x_\alpha) \in \mathcal{D}_{mn}$, $Q_n(x_\alpha) \subset C^*(m, \alpha)$ and $Q_n(x_\alpha) \subset C(m, \alpha)$. Then, since each $\mathcal{G}_m$ is HCP, each $\mathcal{D}_{mn}$ is a disjoint, CP collection of closed subsets of $X$; hence, it is discrete in $X$. Then each $\mathcal{D}_m$ is a $\sigma$-discrete collection in $X$. Let $\mathcal{D} = \bigcup\{\mathcal{D}_m ; m \in N\}$. Hence $\mathcal{D}$ is a $\sigma$-discrete collection in $X$ such that, for each $d \in D$, $\mathcal{D} \cap T_d$ is a weak base for $d$ in $X$.

Now, we prove the main theorem.

**Theorem 6.** The following are equivalent:

(a) $X$ is $g$-metrizable.

(b) $X$ is a $k$-space with a $\sigma$-HCP weak base.

(c) $X$ is a weakly first countable space with a $\sigma$-HCP $k$-network.

**Proof.** Every $g$-metrizable space is sequential [15]; hence, it is a $k$-space. Then (a) $\Rightarrow$ (b) holds. We note that, for any sequence converging to a point $x$, any weak neighborhood of $x$ contains the sequence eventually. Then (b) $\Rightarrow$ (c) follows from Lemmas 1 and 2. We prove (c) $\Rightarrow$ (a). By Lemmas 1 and 3, it suffices to show that $X$ has a $\sigma$-locally finite cover satisfying $(*)$. Let $\mathcal{G} = \{\mathcal{G}_n ; n \in N\}$ be a $\sigma$-HCP $k$-network for $X$ by closed subsets of $X$ such that $\mathcal{G}_n \subset \mathcal{G}_{n+1}$ for each $n \in N$. For each $n \in N$, let $D_n = \{x \in X ; \mathcal{G}_n$ is not point-finite at $x\}$. Since $X$ is weakly first countable, it is sequential. Then, since each $\mathcal{G}_n$ is HCP, it follows that each $D_n$ is closed, and discrete in $X$. By Lemma 4, for each $n \in N$, there exists a collection $\{V_{mn} ; m \in N\}$ of open subsets of $X$ satisfying the condition in Lemma 4 with respect to the closed subset $D_n$. For each $m, n \in N$, let $\mathcal{G}_{mn} = \{C - V_{mn} ; C \in \mathcal{G}_n\}$. Let $\mathcal{G}' = \bigcup\{\mathcal{G}_{mn} ; m, n \in N\}$, and each $\mathcal{G}_{mn}$ is a point-finite and CP collection of closed subsets of $X$, it is locally finite in $X$. Hence $\mathcal{G}'$ is a $\sigma$-locally finite collection of closed subsets of $X$. Let $\bigcup\{T_x ; x \in X\}$ be a weak base for $X$ such that each $T_x$ is countable. Since each $D_n$ is a closed discrete subset of $X$, by Lemma 5 there exists a $\sigma$-discrete collection $\mathcal{D}_n$ in $X$ such that, for each $d \in D_n$, $\mathcal{D}_n \cap T_d$ is a weak base for $d$ in $X$. Let $\mathcal{D} = \bigcup\{\mathcal{D}_n ; n \in N\}$. 

License or copyright restrictions may apply to redistribution; see https://www.ams.org/journal-terms-of-use
Let $\mathcal{P} = \mathcal{C}' \cup \mathcal{D}$. Then $\mathcal{P}$ is a $\sigma$-locally finite cover of $X$. We show that $\mathcal{P}$ satisfies $(\ast)$. Let $U$ be open in $X$ and $A$ be a sequence converging to a point $x$ in $U$. Let $S = X - \bigcup \{D_n; n \in \mathbb{N}\}$. If $x \in S$, then there exists an element $C$ of $\mathcal{C}'$ such that $C \subset U$ and $C$ contains a subsequence of $A$. If $x \notin S$, then there exists an element $D$ of $\mathcal{D}$ such that $D \subset U$ and $D$ contains $A$ eventually. Then $\mathcal{P}$ satisfies $(\ast)$. Hence $X$ has a $\sigma$-locally finite cover $\mathcal{P}$ satisfying $(\ast)$.

**Lemma 7 ([15]).** Every Fréchet, $g$-metrizable space is metrizable.

**Remark 8.** (1) As for networks (= nets), the following (a) holds by [16]: (a) $X$ has a $\sigma$-HCP network $\iff X$ has a $\sigma$-CP network $\iff X$ has a $\sigma$-discrete network ($\iff X$ is a $\sigma$-space). In (a), we cannot replace “network” by “k-network.” Indeed, not every space with a $\sigma$-HCP $k$-network has a $\sigma$-locally finite $k$-network; see Theorem 14. However, among weakly first countable spaces, we have the following (b) in view of [8, Theorem 5] (or the proof of [4, Theorem 2.4]) and Theorem 6: (b) Let $X$ be weakly first countable. Then $X$ has a $\sigma$-HCP $k$-network $\iff X$ has a $\sigma$-discrete $k$-network $\iff X$ has a $\sigma$-discrete weak base ($\iff X$ is a $g$-metrizable space).

(2) In Theorem 6, we cannot weaken “$\sigma$-HCP $k$-network” (resp. “$\sigma$-HCP weak base”) to “$\sigma$-CP $k$-network” (resp. “$\sigma$-CP weak base”), by (a) and (b) below: (a) There exists a separable, first countable space $X$ with a $\sigma$-CP open base, but $X$ is not $g$-metrizable; (b) There exists a weakly first countable, CW-complex $X$, but $X$ is not $g$-metrizable. As for (a), the nonmetrizable, separable, first countable $M_1$-space $X$ in [3, Example 9.2] is the desired space by Lemma 7.

As for (b), let $X$ be the CW-complex in [17, p. 336] such that $X$ is the quotient, finite-to-one image of a metric space, but $X$ has no $\sigma$-HCP $k$-networks. Note that every CW-complex has a $\sigma$-CP open base by [3, Corollary 8.6], and that every quotient, finite-to-one image of a metric space is weakly first countable in view of [1, p. 125]. Then $X$ is the desired space.

In the following corollary, (a) $\iff$ (b) is due to [2], and (a) $\iff$ (c) is Guthrie's unpublished result.

**Corollary 9.** The following are equivalent:

(a) $X$ is metrizable.
(b) $X$ has a $\sigma$-HCP open base.
(c) $X$ is a first countable space with a $\sigma$-HCP $k$-network.

**Proof.** As for (b), note that every space with a $\sigma$-HCP open base is first countable by [2, Lemma 4]. Then the corollary follows from Theorem 6 and Lemma 7.

A space is $g$-second countable [15] if it has a countable weak base. The following Corollary is obtained from Theorem 6 and Lemma 2.
Corollary 10. The following are equivalent:

(a) $X$ is a $g$-second countable space.
(b) $X$ is a Lindelöf space with a $\sigma$-HCP weak base.
(c) $X$ is a Lindelöf, weakly first countable space with a $\sigma$-HCP $k$-network.

The author has the following question in view of Theorem 6 and Corollary 10.

Question. Is every space with a $\sigma$-HCP weak base $g$-metrizable?

Let us give some partial answers to this question. We recall that a space $X$ is a c-space (or $X$ has countable tightness) if, whenever $x \in \overline{A}$, then $x \in \overline{C}$ for some countable $C \subset A$. Every $g$-metrizable (more generally, every sequential) space is a c-space; see [13, p. 123], for example. A space is meta-Lindelöf if every open cover has a point-countable open refinement.

Proposition 11. Let $X$ have a $\sigma$-HCP weak base. Then (a) or (b) below implies that $X$ is $g$-metrizable.

(a) (CH). $X$ is a separable space, or a c-space.
(b) $X$ is a meta-Lindelöf c-space.

Proof. For case (a), let $X$ be a separable space. Then, as is well known, $X$ has an open base of cardinality $\leq 2^\omega$, hence of cardinality $\leq \omega_1$. Let $\mathcal{B} = \bigcup \{\mathcal{B}_n; n \in N\}$ be a $\sigma$-HCP weak base for $X$. Let $x \in X$ be not isolated in $X$. Then for any $n \in N$ and for any $A \in T_x \cap \mathcal{B}_n$, $x$ is not isolated in $A$. Hence in view of [2, Corollary 2], $T_x \cap \mathcal{B}_n$ is countable for each $n \in N$. Then $T_x \cap \mathcal{B}$ is countable. Thus $X$ is weakly first countable, hence a $k$-space. Next, let $X$ be a c-space. Then it is easy to show that $F \subset X$ is closed in $X$ if $F \cap S$ is closed for every separable closed subset $S$ of $X$. Let $S$ be any separable closed subset of $X$. Since $S$ is closed in $X$, it has a $\sigma$-HCP weak base. Since $S$ is separable, it is a k-space, as seen above. Thus any separable closed subset of $X$ is a $k$-space. To show that $X$ is a $k$-space, for $F \subset X$, let $F \cap C$ be closed for every compact subset $C$ of $X$. Let $S$ be any separable closed subset of $X$. Then $F \cap K$ is closed for every compact subset $K$ of $S$. Since $S$ is a $k$-space, $F \cap S$ is closed. Then $F$ is closed in $X$. Thus $X$ is a $k$-space. Then for case (a), $X$ is a $k$-space. Hence it is $g$-metrizable, by Theorem 6.

For case (b), note that every separable closed subset of a meta-Lindelöf space $X$ is Lindelöf. But every Lindelöf closed subset of $X$ is a $k$-space by Lemma 2. Since $X$ is a c-space, $X$ is $g$-metrizable in view of the proof for case (a).

Lemma 12. Let $f : X \to Y$ be a closed map. If $X$ has a $\sigma$-HCP $k$-network, then so does $Y$.

Proof. Let $\mathcal{B}$ be a $\sigma$-HCP $k$-network. Then $f(\mathcal{B})$ is a $\sigma$-HCP cover of $Y$. Then, by Lemma 1, it suffices to show that $f(\mathcal{B})$ satisfies $(\ast)$. To show this, let $A$ be a sequence converging to $y$ in $Y$, and let $U$ be a neighborhood of $y$. Since $f$ is a closed map such that each point of $X$ is a $G_\delta$-set, in view of
[17, Lemma 1.6] there exists a convergent sequence $B$ in $X$ such that $f(B)$ is a subsequence of $A$. Thus there exists $C \in \mathcal{G}$ such that $f(C) \subset U$ and $f(C)$ contains a subsequence of $A$. Hence $f(\mathcal{G})$ satisfies $(*)$.

As is well known, every perfect image of a metric space is metrizable. However, not every perfect image of a countable, $g$-metrizable space is $g$-metrizable [18]. We recall that every closed image of a metric space is metrizable if and only if it is first countable. As for $g$-metrizability, we have the following analogue by Lemma 12 and Theorem 6:

**Theorem 13.** Let $f: X \to Y$ be a closed map such that $X$ is $g$-metrizable. Then $Y$ is $g$-metrizable if and only if it is weakly first countable.

Finally, among Fréchet spaces, let us give a survey of characterizations for nice properties by means of $k$-networks or weak bases. We recall that a map $f: X \to Y$ is an $s$-map if $f^{-1}(y)$ is separable for each $y \in Y$. For $M_3$-spaces, see [3].

**Theorem 14.** Let $X$ be a Fréchet space. Then we have the following:

(a) $X$ is a metric space $\iff$ $X$ has a $\sigma$-locally finite weak base $\iff$ $X$ has a $\sigma$-HCP weak base.

(b) $X$ is the closed $s$-image of a metric space $\iff$ $X$ has a $\sigma$-locally finite $k$-network.

(c) $X$ is the closed image of a metric space $\iff$ $X$ has a $\sigma$-HCP $k$-network.

(d) $X$ is an $M_3$-space $\iff$ $X$ has a $\sigma$-CP $k$-network $\iff$ $X$ has a $\sigma$-CP weak base.

**Proof.** (a) follows from Theorem 6 and Lemma 7. (This is also shown in [11]. Indeed, every Fréchet space with a $\sigma$-HCP weak base has a $\sigma$-HCP open base. Then $X$ is metrizable by Corollary 9.) (b) is due to [6], (c) is due to [5], and the equivalence in the first part of (d) is due to [10]. Let $X$ be an $M_3$-space. By [7] or [9], $X$ has a $\sigma$-CP quasi-base $\mathcal{B}$; that is, for $x \in X$ and a neighborhood $U$ of $x$, there exists $B \in \mathcal{B}$ such that $x \in \text{int}B \subset B \subset U$. For $x \in X$, let $T_x = \{B \in \mathcal{B} \mid x \in \text{int}B\}$. Then $\bigcup\{T_x \mid x \in X\}$ is a $\sigma$-CP weak base for $X$. Thus $X$ has a $\sigma$-CP weak base. Conversely, let $\mathcal{B}$ have a $\sigma$-CP weak base. Since $X$ is Fréchet, for any $x \in X$ and for any $B \in T_x \cap \mathcal{B}$, $x \in \text{int}B$. Hence $\mathcal{B}$ is a $\sigma$-CP quasi-base for $X$. Hence $X$ is an $M_3$-space by [3, Theorem 2.2].

**References**

11. Y. Kotake, personal communication.

DEPARTMENT OF MATHEMATICS, TOKYO GAKUGEI UNIVERSITY, KOGANEI-SHI, TOKYO (184), JAPAN