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UNIFORM L^2-WEIGHTED SOBOLEV INEQUALITIES

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ABSTRACT. We prove the inequality (1) for weights w in a class which contains the class J_p, $p > (n-1)/2$, introduced by C. Fefferman and D. H. Phong in studying eigenvalues of Schrödinger operators. In our case, C is independent of the lower order terms of P. As a consequence we prove unique continuation theorem for solutions of $\Delta + V$, V in the same class.

INTRODUCTION

We give conditions on weight functions w so that an inequality

$$\|u\|_{L^2(w)} \leq C\|P(D)u\|_{L^2(w^{-1})} \tag{1}$$

holds for the constant coefficient operator in R^n

$$P(D) = \Delta + \sum a_i \partial/\partial x_i + b,$$

where $n \geq 3$ and the constant C is independent of the lower order terms a_i, b in C.

The weight w must satisfy two conditions. The first is a one-dimensional A_2 Muckenhoupt's requirement, defined at the beginning of §1, which depends only on the direction of the vector a_j. The second is an estimate,

$$\left(r^{-n}\int_{B_r} w^{\alpha}(x)\,dx\right)\left(r^{-n}\int_{B_r} w^{-\alpha}(x)\right)^{-1} \leq Cr^{-4\alpha},$$

where B_r denotes a ball of radius r and C is independent of B_r.

A similar, weaker condition for the particular case of the Laplace operator was given in [CW] (see for instance $(1, 6)'$ in this reference).

Also, an inequality like (1) has been proved in [KRS] for L^p norms; more precisely

$$\|u\|_{L^{p'}} \leq C\|P(D)u\|_{L^p}, \tag{0}$$

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where p and p' satisfy the duality condition and the Sobolev gap $1/p - 1/p' = 2/n$.

The ingredients to prove (1), as in [KRS], are a similar uniform inequality

$$\|u\|_{L^2(w)} \leq C\|(\Delta + z)u\|_{L^2(w^{-1})}, \tag{2}$$

and a weighted restriction theorem for the Fourier transform which we state independently, since it requires a weaker hypothesis.

As a consequence of (1), we prove a version of a result due to Chanillo and Sawyer [CS] on unique continuation of solutions of Schrödinger equations. More precisely, we prove that a solution u of the inequality

$$|\Delta u(x)| \leq |V(x)u(x)|,$$

which is zero in an open set, must vanish everywhere. We assume some minimal conditions on u, and assume V to be in the Fefferman-Phong class

$$J^{\mathrm{loc}}_{p,F} = \left\{ V \in L^p_{\mathrm{loc}}, \text{ such that } \limsup_{r\to 0} \sup_{x\in K} r^2 \left(r^{-n} \int_{B_r(x)} V^p \right)^{1/p} \leq F, \right.$$
$$\left. \text{for any compact } K \subset R^n \right\},$$

for $p > (n-1)/2$ and F some constant depending only on dimension.

Chanillo and Sawyer (see [K]) obtained a stronger unique continuation for V in the Fefferman-Phong class for $F = 0$ and the same p's; their proof relies on an $L^2 \to L^2$ estimate due to Jerison and Kenig very difficult to prove. We take from them the idea of substituting weighted L^2 inequalities for L^p inequalities and also the use of maximal functions in the unique continuation context.

Our proof of (1), and hence a new proof of unique continuation property for potential in the class $J_{p,F}$, relies on restriction theorems for the Fourier transform, stationary phase methods, and real interpolation.

1. Statement of results and consequences

We start with some definitions.

Let w be a nonnegative function. We say that w is in D_β, $\beta < 0$, if there exists a constant $C > 0$ such that for any ball B_t of radius t,

$$\left(t^{-n} \int_{B_t} w(x)\,dx \right) \left(t^{-n} \int_{B_t} w(x)^{-1} \right)^{-1} \leq Ct^\beta,$$

we denote by $|||w|||^2$ the infimum of such constants C.

Let γ in R^n, and let us write for any y in R^n, $y = t\gamma + y'$, where y' is in some hyperplane Π of R^n. We will say that w is in $A_p(\gamma)$ if the function $w_{y'}: R \to R^+$, defined by $w_{y'}(t) = w(y)$ is an A_p weight in R for almost every y' in Π, with A_p norm uniformly bounded.

Theorem 1. *Let* $P(D) = \Delta + \sum a_j \partial/\partial x_j + b$, a_j, b *in* $\mathbb{C}$. *Then there exists a direction* γ *in* R^n, *depending only on the direction of the vector* $(\operatorname{Re} a_j)$, *such that if* w *is in* $A_2(\gamma)$ *and* w^α *is in* D_β *for some* $\alpha > (n-1)/2$, $\beta = -4\alpha$, *the following inequality* (1)

$$\|u\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\|P(D)u\|_{L^2(w^{-1})}$$

holds with an absolute constant C *depending only on the* $A_2(\gamma)$ *constant of* w.

The proof of the above theorem is based on the following

Theorem 2 (Weighted restriction theorem for the Fourier transform). *Let* $d\sigma$ *be the uniform measure on the sphere* S^{n-1} *in* R^n *and* $(d\sigma)^\wedge$ *its Fourier transform; let* w *be a doubling weight* $w > 0$, *i.e. there exists a constant* C' *such that*

$$\int_{B_{2r}} w \le C' \int_{B_r} w .$$

Then for $w^\alpha \in D_\beta$ *for* $1 < \alpha < (1-n-\beta)/2$ *and* $n \ge 3$, *there exists a constant* C *independent of* $|||w^\alpha|||$ *such that*

$$\|(d\sigma)^\wedge * f\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\|f\|_{L^2(w^{-1})}$$

for every $f \in C_0^\infty$.

Theorem 3 (Weighted L^2 restriction theorem for the resolvent). *Let* w *be as in Theorem 2, then there exists a constant* C *such that*

$$\|u\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}|z|^{-1-(\beta/4\alpha)}\|(-\Delta + z)u\|_{L^2(w^{-1})} \tag{3}$$

for any u *in* C_0^∞ *and* z *in* C.

The following method gives weights which satisfy the hypothesis of Theorem 1, with $A_2(\gamma)$ constant depending only on the dimension.

Take $\gamma_1 = \gamma, \gamma_2, \dots, \gamma_n$, an orthogonal basis of R^n and define the strong maximal function

$$Mf(x) = \sup_R |R|^{-1} \int_R |f(y)|\, dy ,$$

where the sup is taken over all the rectangles with edges in the directions $\gamma_1 = \gamma, \gamma_2, \dots, \gamma_n$, which contain x.

Let us define $M_\alpha f = (M(f^\alpha))^{1/\alpha}$. Then we have the following lemma due to Chanillo and Sawyer:

Lemma 1. *Let* f *be in* $J_{p,F}$ *for* $p > (n-1)/2$, *and* $1 < \alpha < p$; *then there exists a constant* C_α, *depending only on the dimension and* α, *such that* $M_\alpha f$ *is in* $J_{p,F'}$, *for* $F' = C_\alpha F$.

Now we are in position to construct the appropriate weights. Following [R], let us take

$$Sf = \sum_{j=0}^{\infty} (2C_\alpha)^{-j} M_\alpha^j f .$$

Then if $V \in J_{p,F}^{\mathrm{loc}}$, define

$$w(x) = S(V\chi_{B_R}). \tag{4}$$

The following properties hold:

(a) $M_\alpha w(x) \le Cw(x)$ for some $C \ge 0$; this is a $A_1^*(\gamma_1, \gamma_2, \dots, \gamma_n)$ condition for w^α, hence w satisfies the same property.

(b) w is in $J_{p,F'}$, if $\alpha < p$, $p > (n-1)/2$;

(c) w^α is in D_β for $\beta = -4\alpha$ and $|||w^\alpha|||^{1/\alpha} \le C_\alpha F$;

(d) w is in $A_2(\gamma)$.

Conditions (a) and (b) are consequences of the results in [GR, page 433] and Lemma 1.

Condition (c) is a consequence of Jensen's inequality.

Condition (d) follows from Theorem 6.2 in [GR].

As a consequence of Theorem 1, we prove the following unique continuation property:

Corollary. *There exists $F \ge 0$, depending only on the dimension, such that for any solution of the differential inequality*

$$|\Delta u(x)| \le V(x)|u(x)|, \tag{5}$$

where V is in $J_{p,F}^{\mathrm{loc}}$ for some $p > (n-1)/2$ and u is in $H_{\mathrm{loc}}^{1,2}$, if u is zero in an open set, u must be zero everywhere.

$H_{\mathrm{loc}}^{1,2}$ denotes the Sobolev space of functions in L_{loc}^2 with derivatives in L_{loc}^2.

Proof of the corollary. When w satisfies the hypothesis of Theorem 1, the following Carleman inequality holds, with constant C independent of $v \in R^n$ and $\lambda \in R$:

$$\|e^{\lambda\nu\cdot x}u\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\|e^{\lambda\nu\cdot x}\Delta u\|_{L^2(w^{-1})}. \tag{6}$$

In fact (6) can be reduced, by changing $v = e^{\lambda\nu\cdot x}u$ to

$$\|v\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\|P(D)v\|_{L^2(w^{-1})}, \tag{1'}$$

where $P(D) = \sum |D_j + i\lambda\nu_j|^2$, $\nu = (\nu_j)_{j=1,\dots,n}$, which is a particular case of (1).

By a reflection argument (see [KRS]) we may assume that $u = 0$ out of a compact set, and hence (by translation and dilation) to the case $u = 0$ out of

$$B_1 = \{x \text{ in } R^n,\ x_1^2 + \cdots + x_{n-1}^2 + (x_n + 1)^2 \le 1\};$$

it suffices to prove that $u = 0$ in a neighborhood of the origin.

The above transformations preserve the spaces J_p. Also, by adding to V the function $\varepsilon|x|^{-2}$, we obtain a new potential in (5), which is in $J_{p,F+\varepsilon}^{\mathrm{loc}}$ and is bounded below in compact sets.

We still call V the resulting potential of applying rotations, inversions, and the above perturbation to V in (5).

Take now w as (4) after Lemma 1, with γ given by Theorem 1 when $a_j = 2\nu_j$. From Conditions (a)–(d), w satisfies the hypothesis of this theorem.

Let us choose $\eta \in C_0^\infty(B_\delta)$, $\eta = 1$ in $|x| \le \delta/2$, $\delta > 0$ to be fixed, then (6) with $\nu = (0, \dots, 0, 1)$ gives for δ small

$$\|e^{\lambda x_n} g\|_{L^2(w)} \le 2CF\|e^{\lambda x_n}\Delta g\|_{L^2(w^{-1})},$$

where $g = u\eta$,

$$\le 2CF\|e^{\lambda x_n} uV\|_{L^2(w^{-1}dx, B_{\delta/2})} + C'\|e^{\lambda x_n} g\|_{L^2(w^{-1}dx, B_\delta \setminus B_{\delta/2})}$$

but

$$\|e^{\lambda x_n} uV\|_{L^2(w^{-1}dx, B_{\delta/2})} = \left(\int_{B_{\delta/2}} |e^{\lambda x_n} u(x)|^2 V^2(x) w^{-1}(x)\, dx\right)^{1/2}$$
$$\le \left(\int_{B_{\delta/2}} |e^{\lambda x_n} u(x)|^2 w(x)\, dx\right)^{1/2}$$

since $V(x) \le S(V\chi_{B_R}) = w(x)$ for δ small enough.

Assuming $2CF \le 1/4$, we obtain

$$\|e^{\lambda x_n} g\|_{L^2(wdx, B_{\delta/2})} \le 2C'\|e^{\lambda x_n}\Delta g\|_{L^2(w^{-1}dx, B_\delta \setminus B_{\delta/2})}$$

uniformly in λ; then there exists $\zeta > 0$ such that $g = 0$ on $x_n > -\zeta$, if we prove that the right-hand side is finite.

$$\begin{aligned}
&\|\Delta g\|_{L^2(w^{-1}dx, B_\delta \setminus B_{\delta/2})} \\
&\quad \le \|(\Delta\eta)u\|_{L^2(w^{-1}dx, B_\delta)} + \|\nabla\eta \cdot \nabla u\|_{L^2(w^{-1}, B_\delta)} + \|\eta\Delta u\|_{L^2(w^{-1}, B_\delta)} \\
&\quad \le C\|u\|_{L^2(dx, B_\delta)} + C\|\nabla u\|_{L^2(dx, B_\delta)} + C\|Vu\|_{L^2(w^{-1}dx, B_\delta)}
\end{aligned}$$

since w^{-1} is bounded in B_δ, and u is in $H^{1,2}_{\mathrm{loc}}$. To bound the last term consider that

$$\int |Vu|^2 w^{-1}\, dx \le \int |u|^2 V\, dx \le C\int |\nabla u|^2\, dx,$$

since V is in the Fefferman-Phong class with $p \ge 1$, and Sobolev estimates hold [FP].

3. The proofs

Proof of Theorem 1. As in [KRS], by rotation, density, and replacing u by $ue^{-ix\cdot c}$ for some c in R^n, we may reduce to the case

$$P(D) = \Delta + \sigma + \tau(\partial/\partial x_n + i\varepsilon).$$

We have to prove (1) with C independent of σ, τ, ε in $R\setminus\{0\}$.

Take

$$m(\xi) = (-|\xi|^2 + \sigma + i\tau(\xi_n + \varepsilon))^{-1};$$

we have to prove

$$|(m(\xi)\hat{f}(\xi))^\wedge\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\|f\|_{L^2(w^{-1})}.$$

By inserting a cutoff function χ such that $\chi(t) = 1$ on $1 < |t| < 2$, we may write

$$(m(\xi)\hat{f}(\xi))^\wedge(x) = \left(\sum_k m_k(\xi)\hat{f}(\xi)\right)^\wedge(x),$$

where $m_k(\xi) = \chi_k(\xi)m(\xi)$ and $\chi_k(\xi) = \chi(2^k(\xi_n + \varepsilon))$. Then (7) is bounded by

$$\left|\sum_k \int |\{m_k\hat{f}\}^\wedge(x)|^2 w(x)\,dx\right|^{1/2}$$
$$\le C|||w^\alpha|||^{1/\alpha}\left(\sum_k \int |\{\chi_k(\cdot)\hat{f}(\cdot)\}^\wedge(x)|^2 w^{-1}(x)\right)^{1/2}$$

and then, by the above argument, bounded by

$$C|||w^\alpha|||^{1/\alpha}\|f\|_{L^2(w^{-1})}.$$

Only (8) remains to be proved; it may be reduced to the boundedness of the Fourier multiplier

$$m_k^*(\xi) = \chi_k(\xi)(-|\xi|^2 + \sigma + i\tau 2^{-k})^{-1},$$

which is bounded by Theorem 3 with $z = \sigma + i\tau 2^{-k}$.

Hence only the difference $m_k - m_k^*$ has to be bounded as a Fourier multiplier; it is given by

$$(Tf)^\wedge(\xi) = \chi_k(\xi_n)f^\wedge(\xi)i\tau(\xi_n + \varepsilon - 2^{-k})$$
$$\times(-|\xi|^2 + \sigma + i\tau(\xi_n + \varepsilon))^{-1}(-|\xi|^2 + \sigma + i\tau 2^{-k})^{-1}.$$

Take polar coordinates $\xi = \rho\xi'$, by Minkowski's inequality

$$\int_0^\infty \left\|\int_{S^{n-1}} (Tf)^\wedge(\rho\xi')e^{i\rho\xi'\cdot x}\,d\sigma(\xi')\right\|_{L^2(w(x)\,dx)} \rho^{n-1}\,d\rho$$
$$= \int_0^\infty \|(d\sigma)^\wedge \cdot_\zeta (m_\rho(\zeta_n)\hat{f}(\rho\zeta))^\wedge(\rho x)\|_{L^2(w)}\rho^{n-1}\,d\rho,$$

where

$$m_\rho(\zeta_n) = \chi_k(\rho\zeta_n)i\tau(\rho\zeta_n + \varepsilon - 2^{-k})$$
$$\times|-\rho^2 + \sigma + i\tau(\rho\zeta_n + \varepsilon)|^{-1}|-\rho^2 + \sigma + i\tau 2^{-k}|^{-1}$$
$$= \int_0^\infty \|(d\sigma)^\wedge \cdot_\zeta (m_\rho(\zeta_n)\hat{f}(\rho\zeta))^\wedge(y)\|_{L^2(w(y/\rho)\,dy)}\rho^{n/2-1}\,d\rho$$
$$\le \int_0^\infty \|(m_\rho(\zeta_n)\hat{f}(\rho\zeta))^\wedge(y)\|_{L^2(w(y/\rho)^{-1}\,dy)}\rho^{n/2-1-\beta/2\alpha}\,d\rho|||w^\alpha|||^{1/\alpha},$$

by Theorem 2, and

$$|||w^\alpha(x/\rho)|||^2 = \rho^{-\beta}|||w^\alpha(x)|||^2,$$
$$= \int_0^\infty \|\{\rho^{-n} m_\rho(\rho^{-1}\zeta_n)\hat{f}(\zeta)\}\hat{}\,(y/\rho)\|_{L^2(w(y/\rho)^{-1}\,dy)}\rho^{n/2-1-\beta/2\alpha}|||w^\alpha|||^{1/\alpha}.$$

But since

$$m_\rho(\rho^{-1}\zeta_n) = \chi_k(\zeta_n)i\tau(\zeta_n+\varepsilon-2^{-k})|-\rho^2+\sigma+i\tau(\zeta_n+\varepsilon)|^{-1}|-\rho^2+\sigma+i\tau 2^{-k}|^{-1}$$

is an $L^2(w^{-1}) \to L^2(w^{-1})$ operator with norm

$$C\frac{\tau 2^{-k}}{(\rho^2-\sigma)^2+(\tau 2^{-k})^2},$$

where C depends only on the $A_2(\gamma)$ constant of w^{-1}, we have

$$\|Tf\|_{L^2(w)} \le C|||w^\alpha|||^{1/\alpha}\int_0^\infty \frac{\rho^{-1-\beta/2\alpha}\tau 2^{-k}\,d\rho}{(\rho^2-\sigma)^2+(\tau 2^{-k})^2}\|f\|_{L^2(w^{-1})}$$
$$= C|||w^\alpha|||^{1/\alpha}\sigma^{-\beta/4\alpha-1}\int_0^\infty \frac{\rho^{-1-\beta/2\alpha}\,d\rho}{(\rho^2-1)^2+\lambda^2}\|f\|_{L^2(w^{-1})},$$

with $\lambda = \tau 2^{-k}/\sigma$; if we take $-\beta/4\alpha-1=0$, the integral is uniformly bounded with respect to λ and the theorem is proved.

Proof of Theorem 2. It is known that

$$(d\sigma)\hat{} = |x|^{-(n/2)+1}J_{(n/2)-1}(|x|), \text{ where } J_t \text{ denotes the Bessel}$$

function of order t.

If $|x| \ge 1$, $J_{(n/2)-1}(|x|)$ is asymptotically like $e^{i|x|}|x|^{-1/2}$.

Let us take a cutoff function $\psi(|x|)$ such that $\operatorname{supp}\psi$ is contained in $[\frac{1}{2}, 2]$ and

$$\sum_{k=1}^\infty \psi(2^{-k}s) = 1 \quad \text{for } |s| > 1;$$

write $d\hat{\sigma} = \sum_{k=0}^\infty F_k(x) = \sum_{k=0}^\infty \psi_k(x)\,d\hat{\sigma}(x)$, where $\psi_k(x) = \psi(2^{-k}|x|)$.

Take T_k the operator of convolution with F_k. We are going to estimate its $L^2(w^{-\theta}) \to L^2(w^\theta)$ mapping norm:

For $\theta = 0$ we use P. Tomas's estimate (see [T]),

$$\|T_k f\|_2 \le C2^k\|f\|_2.$$

For $\theta = \alpha > 1$,

$$|F_k(x)| \le C|\psi(2^{-k}|x|)e^{i|x|}|x|^{-(n-1)/2}|$$
$$\le C2^{-k(n-1)/2}\chi_{\{x\,:\,|x|<2^{k+1}\}}(|x|).$$

Hence

$$|(T_k f)(x)| \le C2^{-k(n-1)/2} \int_{|x-y|\le 2^{k+1}} |f(y)|\,dy$$
$$\le 2^{-k(n-1)/2} M_{2^k} f(x) 2^{nk},$$

where $M_t f(x) = \sup_{|B|>t^n} \int_B |f(x-y)|\,dy$ is the truncated Hardy-Littlewood maximal function.

We invoke the following lemma (see, for instance, [GR]):

Lemma 2. *Let V be a measure in R^{n+1}, and u a doubling measure in R^n, such that for any cube I_h of side length $h > 0$, there exists a constant $K > 0$ such that*

$$V(I_h \times [0, h]) \le K u(I_h);$$

then

$$\left| \int_{R_+^{n+1}} |M_s f(x)|^2\, dV(x, s) \right|^{1/2} \le K^{1/2} \left(\int_{R^n} |f(x)|^2\, du \right)^{1/2}.$$

We take $dV(x, t) = w^\alpha \otimes \delta_t$, where δ_t is the singular measure on $s = t$ in R^n, and $du(x) = w^{-\alpha}(x)\,dx$; in this case the Carleson constant K is bounded by $|||w^\alpha|||^2 t^\beta$, in fact:

$$\begin{aligned} w^\alpha \otimes \delta_t(I_h \times [0, h]) &= \delta_t([0, h]) w^\alpha(I_h) \\ &= \chi_{h>t}(h) w^\alpha(I_h) = \int_{I_h} w^\alpha(x)\,dx \chi_{(t,\infty)}(h) \\ &\le \int_{I_h} w^{-\alpha}(x)\,dx |||w^\alpha|||^2 h^\beta \chi_{(t,\infty)}(h) \\ &\le w^{-\alpha}(I_h) |||w^\alpha|||^2 t^\beta \ (\beta < 0). \end{aligned}$$

Hence

$$\left(\int |M_t f(x)|^2\, dw^\alpha(x) \right)^{1/2} \le |||w^\alpha|||^{\beta/2} \left(\int |f(x)|^2 w^{-\alpha}(x)\,dx \right)^{1/2}$$

and

$$\|T_k\|_{L^2(w^{-1})\to L^2(w)} \le C2^{k(1-1/\alpha)} 2^{(-k(n-1)/2+nk+k\beta/2)1/\alpha} |||w^\alpha|||^{1/\alpha}.$$

We apply interpolation and

$$\begin{aligned} \|T_k\|_{L^2(w^{-1})\to L^2(w)} &\le C2^{k(1-1/\alpha)} 2^{(-k(n-1)/2+nk+k\beta/2)1/\alpha} |||w^\alpha|||^{1/\alpha} \\ &= C2^{k(1-(1-n-\beta)/(2\alpha))} |||w^\alpha|||^{1/\alpha}, \end{aligned}$$

and the sum is convergent for $2 < 2\alpha < (1 - n - \beta)$.

Proof of Theorem 3. We follow along the lines of the proof of Theorem 2.

Take $m(\xi) = (|\xi|^2 + z)^{-1}$ as a Fourier multiplier; we may assume $\operatorname{Im} z \ne 0$ by a density argument.

The Fourier transform of m is given by

$$K(x) = C\left(\frac{z}{|x|^2}\right)^{1/2(n/2-1)} K_{n/2-1}((z|x|^2)^{1/2}),$$

where $K_{n/2-1}$ is a modified Bessel function (see [KRS] and [GS]) whose behavior is

$$|K_{n/2-1}(t)| \le C|t|^{-n/2+1}, \quad \text{for } |t| \le 1, \ \operatorname{Re} t > 0,$$

$$K_{n/2-1}(t) = b(t)t^{-1/2}e^{-t}, \quad \text{for } |t| > 1, \ \operatorname{Re} t > 0,$$

where $b(t)$ satisfies

$$|(d/d\rho)^j b(\rho t/|t|)| \le C_j|\rho|^{-j}.$$

We take a determination of $(z|x|^2)^{1/2}$ with positive real part. Now,

$$|K(x)| \le C|x|^{-n+2} \quad \text{for } |z^{1/2}|x|| \le 1,$$

with C independent of z, and we may consider the convolution operator with $\psi_k(r|x|)K(x)$. One can see that the worst case occurs when $\operatorname{Im} z$ is small. In this case the operator behaves like the restriction operator dilated by $|z|$. This is the idea in the following calculation:

(a) $T_k\colon L^2 \to L^2$ with norm $2^k/|z|$. In fact, if we denote $s = |x|$ and $z^{1/2} = r\cos\alpha + ir\operatorname{sen}\alpha$, the multiplier is

$$\begin{aligned}
&z^{(n-2)/2}\int_{R^n} \psi_k(r|x|)K_{(n/2)-1}(z^{1/2}|x|)|z|x|^2|^{-(n-2)/4}e^{ix\cdot\xi}dx \\
&\quad = z^{(n-2)/2}\int_{2^{k-1}<rs<2^k} \psi_k(rs)b(z^{1/2}s)(zs^2)^{-(n-1)/4}e^{-irs\operatorname{sen}\alpha - rs\cos\alpha} \\
&\qquad \cdot \int_{s^{n-1}} e^{isw\cdot\xi}d\sigma(w)s^{n-1}\,ds \\
&\quad = z^{(n-2)/2}\int_{2^{k-1}<rs<2^k} \psi_k(rs)b(z^{1/2}s)(zs^2)^{-(n-1)/4}e^{-irs\operatorname{sen}\alpha - rs\cos\alpha} \\
&\qquad \cdot a(s|\xi|)(s|\xi|)^{-(n-1)/2}e^{is|\xi|}s^{n-1}\,ds,
\end{aligned}$$

where a is a function with the same type of estimates as b.

Since $\cos\alpha > 0$, if $\cos\alpha > \eta > 0$ the above integral is bounded by $C\exp(-2^k\eta r)$. Hence the worst case is when $0 < \cos\alpha < \eta$; we use the stationary phase method and obtain the bound

$$|z|^{(n-2)/2}|r\operatorname{sen}\alpha|^{-(n-1)/2}2^k r^{-1} \approx C2^k r^{-2}.$$

(b) $T_k\colon L^2(w^{-\alpha}) \to L^2(w^\alpha)$ with norm $r^{-2-(\beta/2)}2^{k(n+1+\beta)/2}|||w^\alpha|||$, by a repetition of the argument in Theorem 2.

Using interpolation we obtain the range of convergence $2 < 2\alpha < (1-\beta-n)$ and the operator norm $L^2(w^{-1}) \to L^2(w)$ of T_k

$$Cr^{-2-(\beta/2\alpha)}|||w^\alpha|||^{1/\alpha}.$$

This is independent of r for $\beta = -4\alpha$.

Proof of Lemma 1. We may assume, without any restriction, that the γ's are in the coordinates' directions.

Define

$$M_{j,\alpha}f(x) = \left(\sup_{a<x_j<b} \frac{1}{b-a} \int_{[a,b]} |f(x_1, \dots, x_{j-1}, t, x_{j+1}, \dots, x_n)|^\alpha \, dt \right)^{1/\alpha}.$$

We see that $M_\alpha f(x) \le M_{1,\alpha} \cdots M_{n,\alpha} f(x)$; hence if we prove that for any $j = 1, \dots, n$, there exists a constant C'_α such that for any $g \in J_{p,F}$, $M_{j,\alpha} g \in J_{p,F'}$, for $F' = C'_\alpha F$, a recurrent application of this fact would prove our statement with $C_\alpha = C'^n_\alpha$.

Let Q be a cube centered at $z = (z_1, \dots, z_n)$ of side length δ; in order to evaluate the average of $M_{1,\alpha}$ in Q, we may assume that f is supported on the strip

$$S = \{x \text{ in } R^n, \text{ there exists } t \text{ with } (t, x_2, \dots, x_n) \text{ in } Q\}.$$

Decompose S in pair of rectangles $R_k \cup R_{-k}$ in Q, where

$$R_k \cup R_{-k} = \{y \in R^n, \ |y_j - z_j| < \delta, \ j = 2, \dots, n, \ 2^k\delta < |y_1 - y_n| < 2^{k+1}\delta\}, \quad k = 1, 2, \dots.$$

Let us denote

$$f_k = f\chi_{R_k}, \qquad f_0 = f\chi_Q, \qquad f_{-k} = f\chi_{R_{-k}}.$$

Then

$$\left(\delta^{-n} \int_Q (M_{1,\alpha}(f)^p\right)^{1/p} \le \left(\sum_{k=-\infty}^{\infty} \delta^{-n} \int_Q (M_{1,\alpha} f_k)^p \right)^{1/p}.$$

For $k = 1, 2, \dots,$ and x in Q

$$M_{1,\alpha} f_k(x) \le \left(\frac{1}{2^{k+1}\delta} \int_{|z_1 - t| \le 2^{k+1}\delta} |f_k|^\alpha \, dt \right)^{1/\alpha};$$

hence since $\alpha < p$,

$$\begin{aligned} \delta^2 \left(\delta^{-n} \int_Q |M_{1,\alpha} f_k|^p\right)^{1/p} &\le \delta^2 \left(\frac{1}{|R_k|} \int_{R_k} |f|^p \, dx \right)^{1/p} \\ &\le \delta^2 \left(2^{k(n-1)} |Q_k|^{-1} \int_{Q_k} |f|^p \right)^{1/p}, \end{aligned}$$

where Q_k is a cube of side length $2^{k+1}\delta$.

The above expression is bounded by $2^{k(n-1)/p-2k}F$. For f_0, since $p/\alpha > 1$ and M_1 is bounded in $L^{p/\alpha}$, we have

$$\begin{aligned}\delta^2\left(|Q|^{-1}\int_Q |M_{1,\alpha}f_0|^p\right)^{1/p} &= \delta^2|Q|^{-1/p}\left(\int_Q |M_1|f_0|^\alpha|^{p/\alpha}\right)^{1/p}\\ &\le C\delta^2|Q|^{-1/p}(\|f_0\|_{p/\alpha})^{1/2}\\ &= C\delta^2\left(1/|Q|\int_Q |f|^p\right)^{1/p} \le C_\alpha F.\end{aligned}$$

For $k = -1, -2, \dots$, we obtain similar bounds (with $|k|$ in the exponent). The sum is convergent if $(n-1)/2 < p$.

Remark and open questions. The condition u in $H^{1,2}$ in the corollary is chosen to avoid a more complicated condition, which states that $\nabla u \in L^2(w^{-1})$ and $u \in L^2(w)$, for the A_1 weights constructed in the proof. In [KRS] the similar requirement is $u \in H^{2,p}$, $p = 2n/(n+2)$.

In [KRS] the inequality (0) is proved for the operator $\partial^2/\partial t^2 - \Delta_n +$ lower order; our proof involves the geometry of the level sets of the kernel; in the Klein-Gordon case this geometry is more complicated. We wonder if there is an appropriate class of potentials V adapted to this case.

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References

[CS] S. Chanillo and E. Sawyer, *Unique continuation for $\Delta + V$ and the C. Fefferman-Phong class*, preprint.

[CW] S. Chanillo and R. L. Wheeden, *L^p estimates for fractional integrals and Sobolev inequalities with applications to Schrodinger operators*, Comm. Partial Differential Equations **10** (1985), 1077–1116.

[F] C. Fefferman, *A note on spherical summation multipliers*, Israel J. Math.

[FP] C. Fefferman and D. H. Phong, *Lower bounds for Schrodinger equations*, J. Eqs. aux dérivées partielle. Saint Jean de Monts, 1982, pp. 129–205.

[GR] J. Garcia-Cuerva and J. L. Rubio de Francia, *Weighted norms inequalities and related topics*, North-Holland, Amsterdam, 1985.

[GS] I. M. Gelfand and G. E. Shilov, *Generalized functions*, vol. 1, New York, Academic Press, 1964.

[K] C. Kenig, *Restriction theorems, Carleman estimates, uniform Sobolev inequalities and unique continuation*, Harmonic Analysis and PDES (Proceedings, el Escorial, 1987), Lecture Notes in Math., vol. 1384, Springer-Verlag, New York, 1989.

[KRS] C. Kenig, A. Ruiz, and C. Sogge, *Uniform Sobolev inequalities and unique continuation for second order constant coefficients differential operators*, Duke Math. J. **55** (1987), 329–347.

[KW] D. Kurtz and R. L. Wheeden, *Results on weighted norm inequalities for multipliers*, Trans. Amer. Math. Soc. **255** (1979), 343–362.

[R] J. L. Rubio de Francia, *Factorization theory and A_p weights*, Amer. J. Math. **106** (1984), 533–543.

[T] P. Tomas, *A restriction theorem for the Fourier transform*, Bull. Amer. Math. Soc. **81** (1975), 477–478.

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