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A REMARK ON INCOMPARABLE ULTRAFILTERS IN THE RUDIN-KEISLER ORDER

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ABSTRACT. If $2^{<\mathfrak{c}} > \mathfrak{c}$ and p is an ultrafilter on ω of character $\mathfrak{c}$ then there exist many ultrafilters that are incomparable with p in the Rudin-Keisler order.

The Rudin-Keisler order $\leq_{\mathbf{RK}}$ on ω^* is defined as $p \leq_{\mathbf{RK}} q$ iff there exists a function $f: \omega \to \omega$ such that $\beta f(q) = p$, hence $A \in p \equiv f^{-1}[A] \in q$. Problem 19 from [vM] and problem 48 from [HvM] asks: does there exist for every $p \in \omega^*$ a $q \in \omega^*$ such that p and q are incomparable in the Rudin-Keisler order? A partial answer is given by Hindman in [H]: if p has character $\mathfrak{c}$ and either $\mathfrak{c}$ is singular or all predecessors of p are of character $\mathfrak{c}$ then such a q exists.

We shall obtain a stronger result as a corollary to the following theorem.

Theorem. *If $\kappa < \mathfrak{c}$ and $p \in \omega^*$ has character $\mathfrak{c}$ then there exist 2^κ ultrafilters which are not $\leq_{\mathbf{RK}}$-greater than p.*

Before we give the proof we fix some notation.

Let F denote the Fréchet (cofinite) filter on ω.

For $\mathscr{F} \subseteq \mathscr{P}(\omega)$ put

$$\langle\langle \mathscr{F} \rangle\rangle = \left\{ A \subseteq \omega : \exists H \in [\mathscr{F}]^{<\omega} \bigcap H \subseteq A \right\}.$$

We call the system $\mathscr{F}$ centered if $|\bigcap H| = \omega$ for every finite $H \subseteq \mathscr{F}$.

Clearly, if $\mathscr{F}$ is centered then $\langle\langle \mathscr{F} \rangle\rangle$ is the filter generated by $\mathscr{F}$.

Let ${}^\kappa 2$ denote the product of κ copies of 2, or equivalently the set of functions from κ to 2. Of course $|{}^\kappa 2| = 2^\kappa$.

Proof. We find 2^κ distinct ultrafilters that are not $\leq_{\mathbf{RK}}$-greater than p.

Let $\{f_\gamma : \gamma < \mathfrak{c}\}$ be an enumeration of the set of functions from ω to ω.

We shall construct by induction on $\gamma < \mathfrak{c}$ filters $\mathscr{F}_g^\gamma$ for $g \in {}^\kappa 2$ such that

(1) $\chi(\mathscr{F}_g^\gamma) \leq \kappa + \gamma$,

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(2) there is a set $G \in \mathscr{F}_g^{\gamma+1}$ such that $f_\gamma[G] \notin p$, and

(3) if $\gamma < \delta < \mathbf{c}$ then $\mathscr{F}_g^\gamma \subseteq \mathscr{F}_g^\delta$.

We shall see that (1) ensures (2) can be satisfied. Note that (2) implies any ultrafilter q_g extending $\bigcup_{\gamma<\mathbf{c}} \mathscr{F}_g^\gamma$ will not be $\leq_{\mathbf{RK}}$-above p.

Let $\{A_\beta^i : \beta < \kappa,\ i \in 2\}$ be an independent family of sets. This means that for every $\beta < \kappa$ we have $A_\beta^0 \cap A_\beta^1 = \varnothing$ and $A_\beta^0 \cup A_\beta^1 = \omega$ and that for every finite partial function $\varphi: \kappa \to 2$ the intersection $\bigcap_{\beta \in \operatorname{dom}\varphi} A_\beta^{\varphi(\beta)}$ is infinite. Such families exist; see e.g., [vM, Lemma 3.3.2].

For every $g \in {}^\kappa 2$ we consider the family $\mathscr{F}_g = \{A_\beta^{g(\beta)} : \beta < \kappa\}$. Each family $\mathscr{F}_g$ is centered since we only took sets from the independent family. We start with the distinct filters

$$\mathscr{F}_g^0 = \langle\langle F \cup \mathscr{F}_g \rangle\rangle \quad \text{for } g \in {}^\kappa 2.$$

Now suppose we are at stage $\gamma < \mathbf{c}$ and take $g \in {}^\kappa 2$. Here we use part of Hindman's construction. Since $\chi(p) = \mathbf{c}$ the system $\{f_\gamma[B] : B \in \mathscr{F}_g^\gamma\}$ does not generate p. Hence there is a set E such that $\omega - E \in p$ and $\{f_\gamma[B] : B \in \mathscr{F}_g^\gamma\} \cup \{E\}$ is centered. It follows that $\mathscr{F}_g^\gamma \cup \{f_\gamma^{-1}[E]\}$ is also centered and we can add $G = f_\gamma^{-1}[E]$ to $\mathscr{F}_g^\gamma$, i.e., we let $F_g^{\gamma+1} = \langle\langle \mathscr{F}_g^\gamma \cup \{f_\gamma^{-1}[E]\}\rangle\rangle$.

In case γ is a limit we let $\mathscr{F}_g^\gamma = \bigcup_{\delta<\gamma} \mathscr{F}_g^\delta$.

If at any moment $\mathscr{F}_g^\gamma$ is an ultrafilter then we can stop. It cannot possibly be $\leq_{\mathbf{RK}}$-above p since its character is smaller than that of p.

Since clearly every ultrafilter has at most $\mathbf{c}$ $\leq_{\mathbf{RK}}$-predecessors, the following is an immediate consequence of our theorem.

Corollary. *If $\kappa < \mathbf{c}$ is such that $2^\kappa > \mathbf{c}$ then for every $p \in \omega^*$ of character $\mathbf{c}$ there exist 2^κ ultrafilters that are $\leq_{\mathbf{RK}}$-incomparable with p.*

Remarks. 1. Since $2^{cf\,\mathbf{c}} > \mathbf{c}$ our corollary also covers the case when $\mathbf{c}$ is singular.

2. Another easy consequence of our corollary is that if one assumes that $2^\kappa > \mathbf{c}$ for some $\kappa < \mathbf{c}$ *and* that there is a $p \in \omega^*$ such that $\chi(r) = \mathbf{c}$ for all $r \leq_{\mathbf{RK}} p$ then for *every* $u \in \omega^*$ there is a $v \in \omega^*$ that is $\leq_{\mathbf{RK}}$-incomparable with it. (Given u, if the character of u is $\mathbf{c}$, the corollary applies. Otherwise p is not $\leq_{\mathbf{RK}}$-comparable to u.)

References

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