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A CLASS OF ABSOLUTE RETRACTS IN SPACES OF INTEGRABLE FUNCTIONS

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ABSTRACT. We consider a class of subsets of L^1, that are shown to be absolute retracts, that contains at once decomposable sets and sets of solutions to Lipschitzean differential inclusions. In this way we generalize and unify a number of different previous results.

1. INTRODUCTION

Let E be a Banach space and consider a multifunction $\phi : E \to 2^E$ with closed, bounded values, that is contractive with respect to the Hausdorff metric d_H, i.e.,

$$d_H(\phi(u), \phi(v)) \le \alpha \cdot |u - v| \tag{1.1}$$

for some $\alpha < 1$ and all $u, v \in E$.

If the values of each $\phi(u)$ are convex, then the set of fixed points $\mathscr{F} = \{u : \phi \in \phi(u)\}$ is known to be an absolute retract [9]. In this paper we prove that the same is true when $E = L^1(T)$ for some measure space T and the values of $\phi(u)$ are decomposable. This set $\mathscr{F}$ is, in general, not decomposable, while every decomposable set is the set of fixed points of a constant map. The primary motivation for the present study comes from differential inclusions. Let $F : [0, T] \times R^n \to 2^{R^n}$ be a Lipschitz continuous multifunction with compact, not necessarily convex, values. Call S_ξ the set of all Carathéodory solutions of the problem:

$$\dot{x} \in F(t, x), \qquad x(0) = \xi.$$

Then the set of derivatives $\mathscr{F}_\xi = \{\dot{u} : u \in S_\xi\}$ can be represented as the set of fixed points of contractive multifunction with decomposable values, related to the Picard operator. On the other hand, every bounded decomposable set on $[0, T]$ is the set of derivatives of the solutions to $x'(t) \in F(t)$. Our result implies that $\mathscr{F}_\xi$ is an absolute retract of $L^1([0, T], R^n)$. Moreover, one can construct retractions that depend continuously on the initial data ξ. This

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provides a unified and abstract approach to several results concerning approximations and continuous selections for solutions of Lipschitz continuous differential inclusions (see [3–5, 8]). As a corollary we obtain the acyclicity and the fixed point property for every set $\mathcal{F}_\xi$.

2. The main result

Let T be a measure space with a finite, positive, nonatomic measure μ. Given a Banach space E, let $L^1 = L^1(T, E)$ be the Banach space of all Bochner μ-integrable functions $u : T \to E$ with the norm $\|u\| = \int_T |u(t)|\,d\mu$, where $|\cdot|$ stands for the norm in E (see [10]).

We assume that $L^1(T, E)$ is separable. For further notations and definitions we refer to [7]. We recall that d_H stands for the Hausdorff distance and that for a given family V of real functions, the map $T(t) = \operatorname{ess\,inf}\{v(\cdot) : v \in V\}$ is the largest measurable lower bound of V. A subset $K \subset L^1$ is called decomposable if for every u, $v \in K$ and every measurable $A \subset T$,

$$u \cdot \chi_A + v \cdot \chi_{T\setminus A} \in K.$$

Denote by D the family of all nonempty, bounded, closed, and decomposable subsets of L^1 and consider a family of maps

$$\phi_\lambda : L^1 \to D$$

depending on a parameter λ from a separable metric space Λ.

Theorem 1. *Let $(\lambda, u) \to \phi_\lambda(u)$ be a continuous map from $\Lambda \times L^1$ into D, that is contractive with respect to u, i.e.,*

$$d_H(\phi_\lambda(u), \phi_\lambda(v)) \le \alpha \|u - v\| \tag{2.1}$$

for some $\alpha < 1$, all u, $v \in L^1$, and any $\lambda \in \Lambda$. Then each set of fixed points

$$\mathcal{F}_\lambda = \{u : u \in \phi_\lambda(u)\} \tag{2.2}$$

is an absolute retract. A retraction can be chosen that depends continuously on λ; namely, there exists a continuous map $g : \Lambda \times L^1 \to L^1$ such that:

$$g(\lambda, u) \in \mathcal{F}_\lambda \quad \text{for all } u \in L^1, \tag{2.3}$$

$$g(\lambda, u) = u \quad \text{for all } u \in \mathcal{F}_\lambda. \tag{2.4}$$

Proof. Choose $\delta > 0$ so small that

$$\alpha(1 + \delta \cdot \mu(T)) < 1 \tag{2.5}$$

and let

$$\Psi_\lambda(u) = \operatorname{ess\,inf}\{|u(\cdot) - v(\cdot)| : v \in \phi_\lambda(u)\}. \tag{2.6}$$

Denote by $O \subset \Lambda \times L^1$ the set

$$O = \{(\lambda, u) : u \notin \phi_\lambda(u)\} \tag{2.7}$$

and observe that O is open. For each $(\lambda, u) \in O$ define

$$k_\lambda(u)(t) = \Psi_\lambda(u)(t) + \delta\left(1 \wedge \int_T \Psi_\lambda(u)(s)\, d\mu(s)\right) \tag{2.8}$$

and

$$K_\lambda(u) = \mathrm{cl}\{v \in \phi_\lambda(u) : |u(t) - v(t)| < k_\lambda(u)(t) \text{ a.e. in } T\}, \tag{2.9}$$

where cl denotes the closure in L^1.

Notice that from Proposition 2 in [2] it follows that $K_\lambda(u) \in D$. Thus from Proposition 3 in [2] the map $(\lambda, u) \to K_\lambda(u)$ is l.s.c. from O into D, since $(\lambda, u) \to k_\lambda(u)$ is continuous on O. By the selection theorem due to Bressan and Colombo [2, Theorem 3] there exists a continuous selection $f(\lambda, u) \in K_\lambda(u)$ defined on O. Extend f to $\Lambda \times L^1$ by setting

$$f(\lambda, u) = u \quad \text{if } u \in \phi_\lambda(u).$$

By construction we have

$$f(\lambda, u) \in \phi_\lambda(u) \tag{2.9}$$

and

$$\|f(\lambda, u) - u\| \leq (1 + \delta \cdot \mu(T)) \cdot d(u, \phi_\lambda(u)). \tag{2.10}$$

We claim that $f(\lambda, u)$ is continuous.

Clearly, it is enough to check it at $(\lambda, u) \notin O$. Fix (λ, u) such that

$$u \in \phi_\lambda(u)$$

and let $\lambda_n \to \lambda$, $u_n \to u$. Then by continuity

$$\lim_{n\to\infty} d(u_n, \phi_{\lambda_n}(u_n)) = d(u, \phi_\lambda(u)) = 0.$$

Therefore from (2.10) and (2.9) we obtain $f(\lambda_n, u_n) \to u = f(\lambda, u)$, proving the claim.

Set $g_1(\lambda, u) = f(\lambda, u)$ and by induction,

$$g_{n+1}(\lambda, u) = f(\lambda, g_n(\lambda, u)). \tag{2.11}$$

Clearly each $g_n(\lambda, u)$ is continuous and by (2.9),

$$g_{n+1}(\lambda, u) \in \phi_\lambda[g_n(\lambda, u)]. \tag{2.12}$$

We shall show that g_n converges locally uniformly to a continuous function $g(\lambda, u)$ satisfying (2.3) and (2.4).

Indeed, from (2.11) and (2.12) we see that

$$\begin{aligned} \|g_{n+1}(\lambda, u) - g_n(\lambda, u)\| &\leq (1 + \delta\mu(T)) d(g_n(\lambda, u), \phi_\lambda(g_n(\lambda, u))) \\ &\leq (1 + \delta\mu(T)) \cdot d_H(\phi_\lambda(g_n(\lambda, u)), \phi_\lambda(g_{n-1}(\lambda, u))) \\ &\leq \alpha(1 + \delta\mu(T)) \|g_n(\lambda, u) - g_{n-1}(\lambda, u)\|. \end{aligned} \tag{2.14}$$

Therefore

$$\|g_{n+1}(\lambda, u) - g_n(\lambda, u)\| \leq [\alpha(1 + \delta\mu(T)]^n \cdot d(u, \phi_\lambda(u)). \tag{2.15}$$

Since the right-hand side of (2.15) is locally bounded, $g_n(\lambda, u)$ converges locally uniformly. This gives the continuity of the map

$$g(\lambda, u) = \lim_{n\to\infty} g_n(\lambda, u).$$

Moreover, for $u \in \phi_\lambda(u)$, we have $g(\lambda, u) = u$, for every n, since $g_n(\lambda, u) = u$.

Passing to the limit in (2.12) we obtain

$$g(\lambda, u) \in \phi_\lambda(g(\lambda, u)),$$

proving (2.3).

Remark. For any $\varepsilon > 0$ by choosing $\delta > 0$ suitably small the previous construction yields a continuous family of retractions $g(\lambda, \cdot)$ such that

$$\|g(\lambda, u) - u\| \le ((1+\varepsilon)/(1-\alpha))[d(u, \phi_\lambda(u)) + \varepsilon].$$

3. Applications to differential inclusions

Let E be a separable Banach space and consider a map $F : [0, T] \times E \to E$ with nonempty, closed, bounded values such that

(i) $F(\cdot, \cdot)$ is jointly measurable;
(ii) there exists an integrable $m \in L^1([0, T], R)$ such that for all $(x, y) \in E$,

$$d_H(F(t, x), F(t, y)) \le m(t)|x - y|; \tag{3.1}$$

(iii) there exist α, β in $L^1([0, T], R)$ such that

$$|F(t, x)| \le \alpha(t) + \beta(t)|x|.$$

For each $\xi \in E$ denote by S_ξ the set of Carathéodory solutions of the Cauchy problem

$$\dot{x} \in F(t, x), \qquad x(0) = \xi \tag{3.2}$$

and define

$$\mathscr{F}_\xi = \{\dot{u} : u \in S_\xi\} \subset L^1([0, T], E).$$

Theorem 2. *There exists a continuous* $g : E \times L^1 \to L^1$ *such that*

(i) $g(\xi, u) \in \mathscr{F}_\xi$ *for all* u;
(ii) $g(\xi, u) = u$ *if* $u \in \mathscr{F}_\xi$.

Proof. Consider the space L^1 of Bochner integrable functions $u : [0, T] \to E$ with the norm

$$\|u\|_* = \int_0^T |u(t)|\, d\mu(t),$$

where μ is the absolutely continuous measure on $[0, T]$ defined by:

$$d\mu = e^{-2r(t)}\, dt, \qquad r(t) = \int_0^t m(s)\, ds.$$

Set

$$\phi_\xi(u) = \left\{ v \in L^1 : v(t) \in F(t, \xi + \int_0^t u(s)\,ds) \text{ a.e. in } [0, T] \right\}$$

and observe that $\mathscr{F}_\xi$ is precisely the set of fixed points of the map $u \to \phi_\xi(u)$. We claim that the map $(\xi, u) \to \phi_\xi(u)$ satisfies all hypotheses of Theorem 1.

Indeed, to check (2.1) given $v_1 \in \phi_\xi(u_1)$ and $\varepsilon > 0$, take $v_2 \in \phi_\xi(u_2)$ such that, almost everywhere,

$$|v_1(t) - v_2(t)| < d\left(v_1(t), F\left(t, \xi + \int_0^t u_2(s)\,ds\right)\right) + \varepsilon.$$

Then

$$\begin{aligned}
\|v_1 - v_2\|_* &= \int_0^T e^{-2r(t)} \cdot |v_1(t) - v_2(t)|\,dt \\
&\le \int_0^T e^{-2r(t)} d\left(v_1(t), F\left(t, \xi + \int_0^t u_2(s)\,ds\right)\right) dt + \varepsilon \cdot \int_0^t e^{-2r(t)}\,dt \\
&\le \int_0^T e^{-2r(t)} m(t) \cdot \left| \int_0^T (u_2(s) - u_1(s))\,ds \right| + \varepsilon T \\
&\le -\frac{1}{2} e^{-2r(t)} \cdot \int_0^T |u_2(t) - u_1(t)|\,dt \\
&\quad + \frac{1}{2} \int_0^T e^{-2r} |u_2(t) - u_1(t)|\,dt + \varepsilon T \\
&\le \frac{1}{2} \|u_2 - u_1\|_* + \varepsilon T.
\end{aligned}$$

Since ε is arbitrarily small

$$d(v_1, \phi_\xi(u_1)) \le \tfrac{1}{2} \|u_2 - u_1\|_*$$

and, therefore,

$$d_H(\phi_\xi(u_1), \phi_\xi(u_2)) \le \tfrac{1}{2} \|u_1 - u_2\|_*.$$

Now Theorem 1 applied to the space $[0, T]$ with measure $d\mu = e^{-2r(t)}\,dt$ gives the desired result.

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