NOTE ON A THEOREM OF AVAKUMOVIĆ

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Abstract. A short proof is given of a result due to Avakumović. More specifically the asymptotic behavior of the solution \( y(x) \rightarrow 0 \ (x \rightarrow \infty) \) of the differential equation \( y'' = \phi(x)y^\lambda \) (\( \lambda > 1 \)) in case \( \phi(tx)/\phi(x) \rightarrow t^\sigma \ (x \rightarrow \infty) \), \( \sigma > -2 \) is given.

In a paper published in 1947, Avakumović [1] studies the asymptotic behavior of solutions \( y(x) \rightarrow 0 \ (x \rightarrow \infty) \) of the differential equation

\[
y'' = \phi(x)y^\lambda , \quad \text{with } \lambda > 1 .
\]

If \( \phi \) is regularly varying with exponent \( \sigma > -2 \), notation \( \phi \in RV_\sigma \) (i.e., \( \phi \) is measurable and eventually positive and \( \phi(xy)/\phi(x) \rightarrow y^\sigma \ (x \rightarrow \infty) \) for \( y > 0 \)) and if \( y(x) \) is a solution of (1) satisfying \( y(x) \rightarrow 0 \ (x \rightarrow \infty) \), then

\[
y(x) \sim \left[ \frac{(1+\lambda+\sigma)(\sigma+2)}{2(\lambda-1)^2} \right]^{\frac{1/(\lambda-1)}{(2, s,-l/(\lambda-1)}} \{x^2 \phi(x)\}^{-1/(\lambda-1)} \ (x \rightarrow \infty) .
\]

The above result is generalized to the equation \( y'' = f(x)\phi(y) \) in three papers by Marić and Tomić [5, 6, 7]. A related paper is Omey [8].

Here we present a simple proof of the original result using the following well-known approximation result on regularly varying functions:

Lemma (see [2, Theorem 17]). Suppose \( f \in RV_\alpha \). Then there exist two functions \( f_1 \sim f_2 \) such that \( f_1(t) \leq f(t) \leq f_2(t) \) for \( t \geq t_0 \) and such that the functions \( \psi_i(t) := \log f_i(e^t) \) are \( C^\infty \) on a neighborhood of \( \infty \) and satisfy

\[
\psi_i'(\tau) \rightarrow \alpha \quad (\tau \rightarrow \infty)
\]

and

\[
\psi_i^{(n)}(\tau) \rightarrow 0 \quad (\tau \rightarrow \infty), \ n \geq 2 ,
\]

for \( i = 1, 2 \).

Theorem. If \( y \) is a bounded positive solution of the differential equation \( y'' = \phi(x)y^\lambda \) with \( \phi \in RV_\sigma \), \( \sigma > -2 \), and \( \lambda > 1 \) constant, then

\[
y \in RV_{-(\sigma+2)/(\lambda-1)} .
\]

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Proof. Substitution of $u = y^{1-\lambda}$ and $v(x) = \log u(e^x)$ shows that $v$ satisfies the equation
\begin{equation}
\frac{d^2}{dx^2}v - \frac{dv}{dx} - \beta v = -e^{x-v},
\end{equation}
where $\psi = \psi(x) \log\{(\lambda - 1)e^{2x}\phi(e^x)\}$ and $\beta = (\lambda - 1)^{-1} > 0$.

By the lemma, there exists $\psi_1$ with $\psi - \psi_1 \to 0$, $\psi_1' \to \sigma + 2$, $\psi_1'' \to 0$, and $\psi_1 \leq \psi$ for $x$ sufficiently large. Substituting $v = \psi_1 + c$ in (3) now gives
\begin{equation}
c'' - \gamma c' - \beta c^2 = -(1 + o(1))e^{-c} + (\sigma + 2)(1 + \beta \sigma + 2\beta) + o(1),
\end{equation}
with $\gamma := \gamma(x) \to 2\beta(\sigma + 2) + 1$ ($x \to \infty$). We claim that $c = c(x)$ tends to a finite limit as $x \to \infty$.

The following three cases are possible:

(i) $c' > 0$ for $x > x_0$. Then $c$ is ultimately increasing; i.e., $\lim c(x) \leq \infty$ exists. If $c(x) \to \infty$, then by (4), for $x$ sufficiently large, $\delta' > \gamma \delta + \beta \delta^2 > \frac{1}{2} \delta'$, where $\delta := c'$. This implies $\delta \to \infty$ and by (4), $(-1/\delta') = \delta'/\delta^2 \to \beta$; hence, $-1/c' = -1/\delta \sim \beta x$ ($x \to \infty$). This contradicts the assumption $c' > 0$.

(ii) There is a sequence $x_k \to \infty$ ($k \to \infty$) with $c'(x_k) = 0$. Assume $x_k$ is the sequence of all consecutive zeros of $c$. If $c''(x_k) < 0$ (c attains its maximum in $x_k$) and $e > 0$ arbitrary, then $c(x_k) \to -\log\{(\sigma + 2)(1 + \beta \sigma + 2\beta)\}$ for large $k$, by (4). Similarly, if $c''(x_k) > 0$, we find $c(x_k) > -\log\{(\sigma + 2)(1 + \beta \sigma + 2\beta)\}$; hence, a contradiction.

(iii) $c' < 0$ for $x > x_0$. Then $c$ is ultimately decreasing. If $c(x) \to \infty$ ($x \to \infty$), then, since $\psi_1 \leq \psi$, we have, using (3),
\begin{equation}
-v'' + \frac{dv}{dx} + \beta v = e^{x-v} \geq e^{\psi_1 - v} = e^{-c}.
\end{equation}
Hence there exists a sequence $x_n \to \infty$ ($n \to \infty$) such that $v'(x_n) \to \pm \infty$. If $v'(x_n) \to +\infty$, then $c'(x_n) \to +\infty$; hence, a contradiction. The case $v'(x_n) \to -\infty$ implies $u'(\exp x_n) < 0$; hence, $y'(\exp x_n) > 0$ for $n$ sufficiently large. Since $y'' > 0$, this contradicts the boundedness of $y$.

This finishes the proof, since $\psi - v \to \text{constant}$ implies $x^2\phi(x) \sim cy^{1-\lambda}(x)$; hence, $y$ is regularly varying.

Remark. The conclusion $y \in RV_{-(\sigma+2)/(\lambda-1)}$ implies that $y \to 0$. Moreover, $y''$ is regularly varying as the product of two regularly varying functions. Application of Karamata's theorem (see, e.g., [3, 4]) then gives $x^2y'' \sim c_0y$ with $c_0 = (\sigma + 2)(\sigma + 1 + \lambda)/(\lambda - 1)^2$. Substituting this in (1) gives (2).

References


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