ARRAY CONVERGENCE OF FUNCTIONS OF THE FIRST BAIRE CLASS

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Abstract. We show that every array \((x(i, j); 1 \leq i < j < \infty)\) of elements in a pointwise compact subset of the Baire-1 functions on a Polish space, whose iterated pointwise limit \(\lim_i \lim_j x(i, j)\) exists, is converging Ramsey-uniformly. An array \((x(i, j); i < j)\) in a Hausdorff space \(\mathcal{T}\) is said to converge Ramsey-uniformly to some \(x\) in \(\mathcal{T}\), if every subsequence of the positive integers has a further subsequence \((m_i)\) such that every open neighborhood \(U\) of \(x\) in \(\mathcal{T}\) contains all elements \(x(m_i, m_j)\) with \(i < j\) except for finitely many \(i\).

1. Introduction

It is a well-known consequence of Ramsey's theorem that every array \((a_{ij}); 1 \leq i < j < \infty\) of real numbers with \(\lim_i \lim_j a_{ij} = a\) for some \(a \in \mathbb{R}\) has the following property: There is a subsequence \((m_i)\) so that for all \(\epsilon > 0\) there is an \(n \in \mathbb{N}\) such that \(|a_{m_i, m_j} - a| < \epsilon\) for all \(n < m_i < m_j\). This result generalizes easily to Hausdorff spaces which satisfy the first countability axiom.

The purpose of our note is to show that a corresponding result holds for the space of functions of the first Baire class \(\mathcal{B}_1(\Omega)\) on a Polish space \(\Omega\), given the topology of pointwise convergence.

Let us say that an array \((x(i, j); 1 \leq i < j < \infty)\) of elements in a Hausdorff space \(\mathcal{T}\) converges Ramsey-uniformly to some \(x \in \mathcal{T}\), if every subsequence of \(\mathbb{N}\) has a further subsequence \((m_i)\) such that for every open neighborhood \(U\) of \(x\) in \(\mathcal{T}\) there is a \(n \in \mathbb{N}\) so that \(x(m_i, m_j) \in U\) for all \(n < m_i < m_j\).

With this notation we can state our main result as follows:

Theorem 1. Let \(\Omega\) be a Polish space and let \(K\) be a pointwise compact subset of \(\mathcal{B}_1(\Omega)\). If \(x\) and \((x(i, j)); i < j\) are elements in \(K\) with \(\lim_i \lim_j x(i, j) = x\), then \((x(i, j))\) converges Ramsey-uniformly to \(x\).

A topological space \(\Omega\) is Polish, if it is homeomorphic to a complete separable metric space. A real-valued function is of the first Baire class on \(\Omega\), if it is the pointwise limit of a sequence of continuous functions on \(\Omega\).
It is a fundamental result of Bourgain, Fremlin, and Talagrand [2] that $\mathcal{B}_1(\Omega)$ is an angelic space, if $\Omega$ is Polish. A Hausdorff space $\mathcal{T}$ is angelic, if for every relatively compact subset $A$ of $\mathcal{T}$ each point in the closure of $A$ is the limit of a sequence in $A$ and if relatively countably compact sets in $\mathcal{T}$ are relatively compact. In angelic spaces the notions of (relative) compactness, (relative) countable compactness, and (relative) sequential compactness coincide. Further basic results about angelic spaces can be found in [7].

Theorem 1 strengthens—in the case of functions of the first Baire class on a Polish space—a result of Boehme and Rosenfeld [1, Theorem 1], which we phrase for our purposes as follows:

**Lemma 2.** Let $\mathcal{T}$ be an angelic space, and let $x$ and $(x(i, j))_{i<j}$ be elements in a compact subset of $\mathcal{T}$ with $\lim_i \lim_j x(i, j) = x$. Then there is a subsequence $(m_i)$ of $\mathbb{N}$ with $\lim_k x(m_{2k-1}, m_{2k}) = x$.

Lemma 2 was also obtained independently, in the $\mathcal{B}_1(\Omega)$-setting, by Rosenthal [8, Theorem 3.6].

From Theorem 1 and a result by Odell and Rosenthal [6] we obtain the following Banach space corollary:

**Corollary 3.** Let $X$ be a separable Banach space not containing $l_1$. If $x^{**}$ and $(x^{**}(i, j))_{i<j}$ are elements in a bounded subset of $X^{**}$ with $\omega^* \lim_i \omega^* \lim_j x^{**}(i, j) = x^{**}$, then $(x^{**}(i, j))$ converges Ramsey-uniformly to $x^{**}$ in the $\omega^*$-topology.

The proof of Theorem 1 utilizes Lemma 2 to extract “nice” converging subsequences out of the given array $(x(i, j))$. We use Ramsey theory to produce the subarray for which one obtains Ramsey-uniform convergence.

If $M$ is an infinite subset of $\mathbb{N}$, $\mathcal{P}^\infty(M)$ will denote the set of all infinite subsets of $M$. We give $\mathcal{P}^\infty(\mathbb{N})$ the topology, which is inherited by considering $\mathcal{P}^\infty(\mathbb{N})$ as a subspace of $\{0, 1\}^\mathbb{N}$ endowed with the product topology.

A subset $\mathcal{A} \subset \mathcal{P}^\infty(\mathbb{N})$ is called a Ramsey set, if for all $L \in \mathcal{P}^\infty(\mathbb{N})$ there is an $M \in \mathcal{P}^\infty(L)$ such that either $\mathcal{P}^\infty(M) \subset \mathcal{A}$ or $\mathcal{P}^\infty(M) \cap \mathcal{A} = \emptyset$. It is known that analytic (and coanalytic) subsets of $\mathcal{P}^\infty(\mathbb{N})$ are Ramsey sets [3, 9]. For a proof of this result, some history and more general results, see [5].

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2. Proof

**Proof of Theorem 1.** Let $x$ and $x(i, j)$ with $1 \leq i < j < \infty$ be elements in $K$ such that $\lim_i \lim_j x(i, j) = x$. We let

$$\mathcal{A} = \{M = (m_i) \in \mathcal{P}^\infty(\mathbb{N}) : (x(m_{2k-1}, m_{2k}))_{k=1}^{\infty} \text{ is pointwise convergent}\}.$$

**Lemma 4.** $\mathcal{A}$ is coanalytic.
We postpone the proof of the lemma and proceed with the proof of the theorem. Since $\mathcal{A}$ is coanalytic, $\mathcal{A}$ is a Ramsey set. Let $L \in \mathcal{P}^\infty(N)$. We can thus find $M = (m_i)_{i=1}^\infty \in \mathcal{P}^\infty(L)$ so that $\mathcal{P}^\infty(M) \subset \mathcal{A}$ or $\mathcal{P}^\infty(M) \cap \mathcal{A} = \emptyset$. Lemma 2 shows that the first alternative holds. Moreover, Lemma 2 asserts that $\lim_k x(m_{2k-1}', m_{2k}') = x$ for some $M' = (m_i') \in \mathcal{P}^\infty(M)$.

Suppose now the conclusion of Theorem 1 fails for $M'$. Then there is an open neighborhood $U$ of $x$ and a subsequence $M'' \subset M'$ with $x(m_{2k-1}', m_{2k}') \notin U$ for all $k \in \mathbb{N}$.

Since $M'' \in \mathcal{P}^\infty(M)$, we have $M'' \in \mathcal{A}$ and thus $\lim_k x(m_{2k-1}', m_{2k}') = y$ for some $y \in \mathcal{P}^\infty(L)$. Note that $y \neq x$.

We now construct a subsequence $N = (n_i) \in \mathcal{P}^\infty(M)$ inductively as follows:

Let $n_1 = m_1'$ and $n_2 = m_2'$. Once $n_1, n_2, \ldots, n_{2k}$ have been chosen, we define $n_{2k+1}$ and $n_{2k+2}$ as follows: If $k$ is odd, we choose an $l \in \mathbb{N}$ so that $m_{2l-1}' > n_{2k}$ and let $n_{2k+1} = m_{2l-1}', n_{2k+2} = m_{2l}'$. If $k$ is even, we can find an $l \in \mathbb{N}$ with $m_{2l-1}' > n_{2k}$ and then let $n_{2k+1} = m_{2l-1}', n_{2k+2} = m_{2l}'$. On the one hand the sequence $(x(n_{2k-1}', n_{2k}))$ is pointwise convergent, on the other hand it contains two subsequences converging to $x$ and $y$ respectively. This yields a contradiction.

**Proof of Lemma 4.** The proof of Lemma 4 uses techniques similar to those employed in [10].

Let $Y$ be the set of all real-valued arrays $(a(i, j))_{i \leq j}$, endowed with the topology of pointwise convergence. We set $Z = \mathcal{P}^\infty(N) \times Y$ and denote by $\phi: \Omega \to Y$ the canonical map defined by $\phi(\omega) = (x(i, j)(\omega))_{i \leq j}$. Since $\phi$ is a Borel-measurable map and $\Omega$ is Polish, $\phi(\Omega)$ is analytic in $Y$ (see [4, §38]). Consequently $Z_1 := \mathcal{P}^\infty(N) \times \phi(\Omega)$ is analytic in $Z$.

We define a set $Z_2 \subset \mathcal{P}^\infty(N) \times Y$ as follows:

$$Z_2 = \{(M, (a(i, j))): (a(m_{2k-1}', m_{2k}'))_{k=1}^\infty \text{ is not Cauchy}\}.$$ 

Observing that the set

$$Z_{2}^{l, N} := \{(M, (a(i, j))): \text{there are } k_1, k_2 > N \text{ with } |a(m_{2k_1-1}', m_{2k_1}) - a(m_{2k_2-1}', m_{2k_2})| > 2^{-l}\}$$

is open, and that

$$Z_2 = \bigcup_{l \in \mathbb{N}} \bigcap_{N \in \mathbb{N}} Z_2^{l, N},$$

we obtain that $Z_2$ is a $G_{\sigma}$-set in $Z$.

Consequently $Z_1 \cap Z_2$ is analytic in $Z$. We let $P: Z \to \mathcal{P}^\infty(N)$ be the projection of $Z$ onto its first coordinate. One can see easily that the complement of $\mathcal{A}$ is equal to $P(Z_1 \cap Z_2)$. Thus $\mathcal{P}^\infty(N) \setminus \mathcal{A}$ is analytic in $\mathcal{P}^\infty(N)$ as the continuous image of an analytic set in $Z$ (see [4, §38]).
Problem. Does Theorem 1 hold for arbitrary angelic spaces? Lemma 2 reduces this problem to the apparently open question, whether the set $\mathcal{A} \subset \mathcal{B}^\infty(\mathbb{N})$, defined at the beginning of the proof, is still a Ramsey set for arbitrary angelic spaces.

References


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