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HIGHER MONOTONICITY PROPERTIES AND INEQUALITIES FOR ZEROS OF BESSEL FUNCTIONS

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ABSTRACT. L. Lorch and P. Szegö have considered the sign-regularity of the higher differences (with respect to the rank k) of the sequence $\{c_{\nu k}\}$ of positive zeros of the Bessel function $\mathscr{C}_\nu(x)$. Our main purpose here is to extend one of their main results to the higher derivatives with respect to κ when $c_{\nu k}$ is appropriately defined as a function of a continuous variable κ rather than the discrete variable k, and the difference operator is replaced by a derivative operator. We also present some inequalities arising from these and other results.

1. INTRODUCTION

L. Lorch and P. Szegö [6] initiated the study of the sign-regularity of higher differences with respect to the rank k, of the sequence $\{c_{\nu k}\} = \{c_{\nu k}(\alpha)\}$ of positive zeros of the cylinder function

$$\mathscr{C}_\nu(x) = \mathscr{C}_\nu(x, \alpha) = \cos\alpha J_\nu(x) - \sin\alpha Y_\nu(x), \qquad 0 \le \alpha < \pi.$$

In particular they showed that for $|\nu| > 1/2$, we have

$$(-1)^n \Delta^{n+1} c_{\nu k} > 0, \qquad n = 0, 1, \dots, k = 1, 2, \dots. \tag{1.1}$$

This result was generalized and extended in [7], [8] and several papers by other authors. The main purpose of this note is to show that the result remains true when $c_{\nu k}$ is appropriately defined as a function of a continuous variable κ rather than the discrete variable k, and the difference operator is replaced by a derivative operator. We also present several other monotonicity properties and inequalities for $c_{\nu k}$.

2. DEFINING THE ZEROS AS CONTINUOUS FUNCTIONS OF k

One can discuss the variation of the positive zeros of $\mathscr{C}_\nu(x, \alpha)$ with respect to any of the three variables ν, α or k (the rank of a zero). However, α

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and k are not really independent; they may, in fact, be subsumed in a single variable $\kappa = k - \alpha/\pi$. To see this, we consider that for $\nu \geq 0$, the zeros of $\mathscr{C}_\nu(x, \alpha)$, $0 < \alpha < \pi$, are the roots of the equation

$$Y_\nu(x)/J_\nu(x) = \cot\alpha. \tag{2.1}$$

The graph of the left-hand side of (2.1) consists of branches which increase from $-\infty$ to $+\infty$ in the intervals $(0, j_{\nu 1})$ and $(j_{\nu k}, j_{\nu, k+1})$, $k = 1, 2, \ldots$, between the positive zeros of $J_\nu(x)$. This is most easily seen by using the relation

$$\frac{d}{dx}\frac{Y_\nu(x)}{J_\nu(x)} = \frac{J_\nu(x)Y_\nu'(x) - Y_\nu(x)J_\nu'(x)}{J_\nu^2(x)} = \frac{2}{\pi x J_\nu^2(x)},$$

where the last equation follows from the Wronskian relation [11, p. 76]. As α decreases from π to 0, $\cot\alpha$ increases from $-\infty$ to ∞. Thus each zero of $\mathscr{C}_\nu(x, \alpha)$ increases from one positive zero $j_{\nu k}$ of $J_\nu(x)$ to the next larger one $j_{\nu, k+1}$. At the same time a new first positive zero appears and increases from 0 to $j_{\nu 1}$. Thus it makes sense to define $j_{\nu\kappa}$ for any real $\kappa \geq 0$, by $j_{\nu 0} = 0$ and $j_{\nu\kappa} = c_{\nu k}(\alpha)$ where k is the largest integer *less than* $\kappa + 1$ and $\alpha = \pi(k - \kappa)$. Thus $j_{\nu\kappa}$ is a continuous increasing function of κ on $[0, \infty)$. The positive zeros of $J_\nu(x)$ correspond to positive integral values of κ and $j_{\nu, k-1/2} = y_{\nu k}$, $k = 1, 2, \ldots$ where $y_{\nu k}$ is the kth positive zero of $Y_\nu(x)$. In [2] it was shown that $j_{\nu\kappa}$ is the unique solution of the differential equation

$$\frac{dj}{d\nu} = 2j\int_0^\infty K_0(2j\sinh t)e^{-2\nu t}dt, \tag{2.2}$$

which satisfies $j(\nu) \to 0$ as $\nu \to -\kappa^+$. This is motivated by the formula [11, p. 408] for the derivative of $c_{\nu k}$ with respect to ν, and the fact that if, for $\nu > 0$, $c_{\nu k}$ is the kth positive zero of $\mathscr{C}_\nu(x)$, then $c_{\nu k}$ may be extended in a continuous way to $\nu < 0$, and $c_{\nu k} \to 0$, as $\nu \to -(k - \alpha/\pi)$. The equation (2.2) may be used to show that $j_{\nu\kappa}$ is an infinitely differentiable function of κ.

We shall often use the fact that $j_{1/2,\kappa} = \kappa\pi$.

3. The case $\nu > 1/2$; complete monotonicity

Here we show

Theorem 3.1. *Let $j_{\nu\kappa}$ be defined as in §2. Then, for $\nu > \mu \geq 1/2$,*

$$(-1)^n D_\kappa^n(\log[j_{\nu\kappa}/j_{\mu\kappa}]) > 0, \qquad \kappa > 0\ ,\ n = 0, 1, \ldots, \tag{3.1}$$

and, in particular, for $\nu > 1/2$,

$$(-1)^n D_\kappa^n(\log[j_{\nu\kappa}/(\kappa\pi)]) > 0, \qquad \kappa > 0\ ,\ n = 0, 1, \ldots. \tag{3.2}$$

Theorem 3.2. *Let $j_{\nu\kappa}$ be defined as in §2. Then, for $\nu > \mu \geq 1/2$,*

$$(-1)^n D_\kappa^n(\log[D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}]) > 0, \qquad \kappa > 0\ ,\ n = 0, 1, \ldots, \tag{3.3}$$

and, in particular, for $\nu > 1/2$,

$$(-1)^n D_\kappa^n(\log[D_\kappa j_{\nu\kappa}/\pi]) > 0, \qquad \kappa > 0\ ,\ n = 0, 1, \ldots. \tag{3.4}$$

Corollary 3.3. *Let $j_{\nu\kappa}$ be defined as in §2. Then, for $\nu > 1/2$,*

$$(-1)^n D_\kappa^{n+1}(j_{\nu\kappa}) > 0, \qquad \kappa > 0\ ,\ n = 0, 1, \ldots. \tag{3.5}$$

Proofs. We have, from (2.2),

$$\frac{1}{j_{\lambda\kappa}}\frac{dj_{\lambda\kappa}}{d\lambda} = 2\int_0^\infty K_0(2j_{\lambda\kappa}\sinh t)e^{-2\lambda t}dt.$$

Hence, integrating with respect to λ from μ to ν, we get

$$\log\frac{j_{\nu\kappa}}{j_{\mu\kappa}} = 2\int_\mu^\nu \left\{\int_0^\infty K_0(2j_{\lambda\kappa}\sinh t)e^{-2\lambda t}dt\right\}d\lambda. \tag{3.6}$$

Next we use the result [3, p. 1484] of Elbert and Laforgia that

$$D_\nu \log D_\kappa j_{\nu\kappa} = 2\int_0^\infty K_0(2j_{\nu\kappa}\sinh t)(\tanh^2 t + 2\nu\tanh t)e^{-2\nu t}dt \tag{3.7}$$

whence it follows that

$$\begin{aligned}(3.8)\quad &\log[D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}]\\ &= 2\int_\mu^\nu \left\{\int_0^\infty K_0(2j_{\lambda\kappa}\sinh t)(\tanh^2 t + 2\lambda\tanh t)e^{-2\lambda t}dt\right\}d\lambda,\end{aligned}$$

and, in particular, when $\mu = 1/2$,

$$\begin{aligned}(3.9)\quad &\log[D_\kappa j_{\nu\kappa}]\\ &= \log\pi + 2\int_{1/2}^\nu \left\{\int_0^\infty K_0(2j_{\lambda\kappa}\sinh t)(\tanh^2 t + 2\lambda\tanh t)e^{-2\lambda t}dt\right\}d\lambda.\end{aligned}$$

Proving Theorems 3.1 and 3.2 amounts to showing that

$$\log[j_{\nu\kappa}/j_{\mu\kappa}] \in \mathscr{M}_n(0, \infty), \tag{3.10}$$

and

$$\log[D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}] \in \mathscr{M}_n(0, \infty), \tag{3.11}$$

for $n = 1, 2, \ldots$, where $\mathscr{M}_n(0, \infty)$ is the set of functions f on $(0, \infty)$ satisfying

$$(-1)^j f^{(j)}(\kappa) > 0, \qquad j = 0, \ldots, n.$$

Recall that

$$(-1)^n D_t^n K_0(t) > 0, \qquad t > 0,\ n = 0, 1, \ldots. \tag{3.12}$$

This follows from the integral representation [11, p. 172]

$$K_0(x) = \int_0^\infty e^{-x\cosh t}dt.$$

We will in fact show that, for each $a > 0$, and $\nu \geq 1/2$,

$$K_0(aj_{\nu\kappa}) \in \mathscr{M}_n(0, \infty), \tag{3.13}$$

for $n = 0, 1, \ldots$, and the representations (3.6), (3.8) will then give Theorems 3.1 and 3.2. Now in the case $\nu = 1/2$, (3.13) follows, for $n = 0, 1, \ldots$ from (3.12). We prove (3.13) by induction in the case $\nu > 1/2$. In this case (3.13) is obvious for $n = 0, 1$; this follows from (3.12) and the fact that $j_{\nu\kappa}$ increases as κ increases. Suppose that (3.13) holds for $n = N - 1$. Then, from (3.9), we see that, for $\nu > 1/2$,

$$\log D_\kappa j_{\nu\kappa} \in \mathscr{M}_{N-1}(0, \infty). \tag{3.14}$$

Since the exponential of a function in $\mathscr{M}_n(0, \infty)$ is also in $\mathscr{M}_n(0, \infty)$, we have

$$(-1)^n D_\kappa^{n+1}(j_{\nu\kappa}) > 0, \qquad \kappa > 0,\ n = 0, \ldots, N-1. \tag{3.15}$$

By the formula of Faa di Bruno (see, e.g., [5, §81, pp. 92-93]) for the Nth derivative of a function of a function, we have, for each $a > 0$,

$$(-1)^N D_\kappa^N \{K_0(a j_{\nu\kappa})\} > 0. \tag{3.16}$$

To see this we use (3.12) and (3.15) to show that each term arising from the Faa di Bruno formula is nonnegative and at least one is positive. See [7, Proof of Lemma 2.1] for details. Now (3.16) shows that (3.13) holds for $n = N$. Thus the proof by induction is complete and (3.13) holds for $n = 0, 1, \ldots$. The representations (3.6), (3.8) now give Theorems 3.1 and 3.2.

Corollary 3.3 is a consequence of the fact that (3.15) holds for all N. It recovers and generalizes the result of [3, p. 1485] that $j_{\nu\kappa}$ is concave with respect to κ when $\nu \geq 1/2$. Corollary 3.3 is the continuous version of the result of Lorch and Szegö [6] referred to in the Introduction. That result may be deduced from Corollary 3.3 by using a mean-value theorem for higher derivatives and differences ([10, no. 98] or [5, p. 74]).

The results of this section provide some examples of infinitely divisible distributions. Recall that a probability distribution is *infinitely divisible* if and only if its Laplace transform is of the form $e^{-h(x)}$, where $h(0) = 0$ and $h'(x)$ is completely monotonic [4, p. 425]. Thus, from Theorem 3.1, we can deduce the complete monotonicity of the distribution whose Laplace transform is $j_{\mu, s+\epsilon} j_{\nu, \epsilon} / j_{\nu, s+\epsilon} j_{\mu, \epsilon}$, where $\nu > \mu \geq 1/2$, $\epsilon > 0$ and s is the variable in the Laplace transform.

4. The case $0 \leq \nu < 1/2$; simple monotonicity

It does not seem to be easy to derive *higher* monotonicity results in the case when $0 \leq \nu < 1/2$. In [6] it was conjectured that (1.1) should be replaced by

$$(-1)^n \Delta^{n+2} c_{\nu k} > 0, \qquad n = 0, 1, \ldots, k = 1, 2, \ldots. \tag{4.1}$$

in the case $|\nu| < 1/2$. In [9], some progress was made in this direction when it was shown that (4.1) holds for $1/3 \leq |\nu| < 1/2$. There does not seem to be any easy way to get the analogues of the results of §3 in the case $0 \leq \nu < 1/2$. The condition $\nu > 1/2$ is an essential ingredient in getting (3.14), for example. However, we are able to deduce some *simple* monotonicity results in this case.

Theorem 4.1. *Let $j_{\nu\kappa}$ be defined as in §2. Then, if $0 \le \mu < \nu \le 1/2$, the positive function $j_{\nu\kappa}/j_{\mu\kappa}$ decreases to 1 as κ increases on $(0, \infty)$. If $0 \le \mu < 1/2$, then $j_{\mu\kappa}/(\kappa\pi)$ increases with κ, $0 < \kappa < \infty$.*

Theorem 4.2. *Let $j_{\nu\kappa}$ be defined as in §2. Then, if $0 \le \mu < \nu \le 1/2$, the positive function $D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}$ decreases to 1 as κ increases on $(0, \infty)$. If $0 \le \mu < 1/2$, then $D_\kappa j_{\mu\kappa}$ increases with κ, $0 < \kappa < \infty$.*

Remark. It should be noted that the monotonicities recorded here in the case $\mu < 1/2$ are opposite in direction to those which we get, in the case of order $> 1/2$, as a consequence of the theorems in §3.

The main assertions of Theorems 4.1 and 4.2 follow easily from the representations (3.6) and (3.8) once we recall that K_0 is a decreasing function of its argument and that $j_{\nu\kappa}$ increases as κ increases. The final assertions of the theorems follow on putting $\nu = 1/2$.

5. Derivative of a zero with respect to κ

We obviously have $dj_{\nu\kappa}/d\nu > 0$. Here we show

Theorem 5.1.

$$dj_{\nu\kappa}/d\kappa > \pi, \qquad \nu > 1/2, \tag{5.1}$$

$$dj_{\nu\kappa}/d\kappa < \pi, \qquad 0 \le \nu < 1/2. \tag{5.2}$$

Proof. From the MacMahon expansion [11, p. 506],

$$j_{\nu\kappa} = (\kappa + \nu/2 - 1/4)\pi + O(\kappa^{-1}), \qquad \kappa \to \infty, \tag{5.3}$$

we have $j_{\nu\kappa}/\kappa \to \pi$, as $\kappa \to \infty$. Hence, by an indirect application of L'Hospital's rule, we see that $dj_{\nu\kappa}/d\kappa \to \pi$, as $\kappa \to \infty$. Now, for $\nu > 1/2$, $j_{\nu\kappa}$ is a concave function of κ, by Corollary 3.3. Hence, $dj_{\nu\kappa}/d\kappa$ decreases and (5.1) follows. On the other hand, when $0 \le \nu < 1/2$, $dj_{\nu\kappa}/d\kappa$ increases, by Theorem 4.2, and (5.2) follows.

Remark. The same indirect use of L'Hospital's rule shows the known result [1, Corollary 3.4, p. 276], that $dj_{\nu\kappa}/d\nu > 1$, where κ is so large that $j_{\nu\kappa}$ is a concave function of ν.

The principal term of (5.3) gives an upper bound for $j_{\nu\kappa}$ in case $\nu > 1/2$ and a lower bound in case $0 \le \nu < 1/2$. To see this we write

$$f(\kappa) = j_{\nu\kappa} - (\kappa + \nu/2 - 1/4)\pi,$$

and note from [2, Theorem 2.1], or from Theorem 5.1 above, that $f(\kappa)$ is increasing, constant or decreasing according as $\nu > 1/2$, $\nu = 1/2$ or $0 \le \nu < 1/2$. Also $f(\kappa) \to 0$, as $\kappa \to \infty$ by (5.3), so the stated results follow.

We establish now some consequences of Theorem 5.1. Integration of (5.1) between κ_0 and κ, $\kappa > \kappa_0 > 0$, gives

$$j_{\nu\kappa} - j_{\nu\kappa_0} > \pi(\kappa - \kappa_0), \qquad \nu > 1/2 \tag{5.4}$$

where the inequality must be reversed when $0 \le \nu < 1/2$ and becomes equality when $\nu = 1/2$. If $j_{\nu\kappa_0}$ and $j_{\nu\kappa}$ are two consecutive zeros of the same cylinder function, we find the known result that the distance between two consecutive zeros is larger than, equal to or less than π according as $\nu > 1/2$, $\nu = 1/2$ or $\nu < 1/2$.

An interesting particular case of (5.4) is when $\kappa_0 = n - 1/2$ and $\kappa = n$. In this case we find that

$$j_{\nu,n} > y_{\nu,n} + \pi/2, \qquad \nu > 1/2,\ n = 1, 2, \ldots.$$

Clearly other inequalities of this kind can be found taking into consideration, in (5.4), different particular values of κ_0 and κ.

6. Inequalities arising from convexity or concavity with respect to κ

The function $j_{\nu\kappa}$ is known to be concave (convex) with respect to κ, $0 < \kappa < \infty$, for $\nu > 1/2$ $(0 \le \nu < 1/2)$ [3, p. 1485]. By considering the chord joining two points on the graph of this function, we thus obtain:

$$j_{\nu\kappa} > \frac{\kappa - k}{K - k}(j_{\nu K} - j_{\nu k}) + j_{\nu k}, \qquad k < \kappa < K,\ \nu > 1/2, \tag{6.1}$$

and

$$j_{\nu\kappa} < \frac{\kappa - k}{K - k}(j_{\nu K} - j_{\nu k}) + j_{\nu k}, \qquad k < \kappa < K,\ 0 \le \nu < 1/2. \tag{6.2}$$

These become equalities when $\nu = 1/2$.

If we put $k = n$, $K = n + 1$, we get

$$j_{\nu\kappa} > (\kappa - n)(j_{\nu,n+1} - j_{\nu n}) + j_{\nu n}, \qquad \nu > 1/2, \tag{6.3}$$

$$j_{\nu\kappa} < (\kappa - n)(j_{\nu,n+1} - j_{\nu n}) + j_{\nu n}, \qquad 0 \le \nu < 1/2, \tag{6.4}$$

In the case where $\kappa = n + 1/2$, we get

$$y_{\nu,n+1} > (j_{\nu,n+1} + j_{\nu n})/2, \qquad \nu > 1/2, \tag{6.5}$$

$$y_{\nu,n+1} < (j_{\nu,n+1} + j_{\nu n})/2, \qquad 0 \le \nu < 1/2. \tag{6.6}$$

For example, taking $\nu = 0$, $n = 39$ in (6.6) gives $y_{0,40} < 123.3085255$, where the values of $j_{0,39}$, $j_{0,40}$ are taken from [11, p. 748]. The "exact" value of $y_{0,40}$, from the same source, is 123.3085253.

On the other hand, if in (6.2) we take $k = n - 1/2$, $\kappa = n$, $K = n + 1/2$, we get

$$j_{\nu,n} < (y_{\nu,n+1} + y_{\nu n})/2, \qquad \nu < 1/2. \tag{6.7}$$

In particular, with $n = 39$, $\nu = 0$, we get $y_{0,40} > 2j_{0,39} - y_{0,39} = 123.308525$.

The upper and lower bounds

$$123.308525 < y_{0,40} < 123.3085255$$

show the sharpness of these inequalities.

We have found the inequalities to be similarly sharp in the case $\nu = 1$ and even more so in the case $\nu = 1/3$, as may be expected from the fact that they become equalities when $\nu = 1/2$.

Inequalities of a similar kind may be obtained by using, for example, the increase of $j_{\nu\kappa}/\kappa$ with respect to κ $(0 < \kappa < \infty)$ for $0 \leq \nu < 1/2$ (Theorem 4.1). Thus, for example, with $\nu = 1/3$, we get

$$\frac{y_{1/3,40}}{39.5} = \frac{j_{1/3,39.5}}{39.5} > \frac{j_{1/3,39}}{39}$$

or

$$y_{1/3,40} > (79/78)j_{1/3,39} = 123.8283293;$$

the "actual" value of $y_{1/3,40}$ is 123.8316712. The numerical values are from [11, p. 75].

An advantage of the results enunciated here is that the knowledge of the zeros of one cylinder function ($J_\nu(x)$, for example) enables us to establish approximations for the zeros of any other cylinder function, those of $Y_\nu(x)$ being but one example. Also, some of the inequalities found in this section are more informative than some of those which can be obtained by a direct application of the Sturm comparison theorem. For example, when $\nu > 1/2$, one can use the Sturm theorem to show that $y_{\nu,n+1}$ belongs to the interval $(j_{\nu,n}, j_{\nu,n+1})$, but formula (6.5) asserts that it belongs to the second half of this interval.

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