HIGHER MONOTONICITY PROPERTIES AND INEQUALITIES
FOR ZEROS OF BESSEL FUNCTIONS

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Abstract. L. Lorch and P. Szegö have considered the sign-regularity of the higher differences (with respect to the rank $k$) of the sequence $\{c_{\nu k}\}$ of positive zeros of the Bessel function $C_\nu(x)$. Our main purpose here is to extend one of their main results to the higher derivatives with respect to $\kappa$ when $c_{\nu k}$ is appropriately defined as a function of a continuous variable $\kappa$ rather than the discrete variable $k$, and the difference operator is replaced by a derivative operator. We also present some inequalities arising from these and other results.

1. Introduction

L. Lorch and P. Szegö [6] initiated the study of the sign-regularity of higher differences with respect to the rank $k$, of the sequence $\{c_{\nu k}\} = \{c_{\nu k}(\alpha)\}$ of positive zeros of the cylinder function

$$C_\nu(x) = C_\nu(x, \alpha) = \cos \alpha J_\nu(x) - \sin \alpha Y_\nu(x), \quad 0 \leq \alpha < \pi.$$  

In particular they showed that for $|\nu| > 1/2$, we have

$$(-1)^n \Delta^{n+1} c_{\nu k} > 0, \quad n = 0, 1, \ldots, k = 1, 2, \ldots.$$  

This result was generalized and extended in [7], [8] and several papers by other authors. The main purpose of this note is to show that the result remains true when $c_{\nu k}$ is appropriately defined as a function of a continuous variable $\kappa$ rather than the discrete variable $k$, and the difference operator is replaced by a derivative operator. We also present several other monotonicity properties and inequalities for $c_{\nu k}$.

2. Defining the zeros as continuous functions of $k$

One can discuss the variation of the positive zeros of $C_\nu(x, \alpha)$ with respect to any of the three variables $\nu$, $\alpha$ or $k$ (the rank of a zero). However, $\alpha$...
and \( k \) are not really independent; they may, in fact, be subsumed in a single variable \( \kappa = k - \alpha/\pi \). To see this, we consider that for \( \nu \geq 0 \), the zeros of \( \mathcal{E}_\nu(x, \alpha) \), \( 0 < \alpha < \pi \), are the roots of the equation

\[
Y_\nu(x)/J_\nu(x) = \cot \alpha.
\]

The graph of the left-hand side of (2.1) consists of branches which increase from \(-\infty\) to \(+\infty\) in the intervals \((0, j_{\nu k})\) and \((j_{\nu k}, j_{\nu, k+1})\), \( k = 1, 2, \ldots \), between the positive zeros of \( J_\nu(x) \). This is most easily seen by using the relation

\[
\frac{d}{dx} \frac{Y_\nu(x)}{J_\nu(x)} = \frac{J_\nu(x)Y'_\nu(x) - Y_\nu(x)J'_\nu(x)}{J_\nu^2(x)} = \frac{2}{\pi x J_\nu^2(x)},
\]

where the last equation follows from the Wronskian relation [11, p. 76]. As \( \alpha \) decreases from \( \pi \) to 0, \( \cot \alpha \) increases from \(-\infty\) to \(+\infty\). Thus each zero of \( \mathcal{E}_\nu(x, \alpha) \) increases from one positive zero \( j_{\nu k} \) of \( J_\nu(x) \) to the next larger one \( j_{\nu, k+1} \). At the same time a new first positive zero appears and increases from 0 to \( j_{\nu 1} \). Thus it makes sense to define \( j_{\nu k} \) for any real \( \kappa \geq 0 \), by \( j_{\nu 0} = 0 \) and \( j_{\nu k} = c_{\nu k}(\alpha) \) where \( k \) is the largest integer less than \( \kappa + 1 \) and \( \alpha = \pi(k - \kappa) \). Thus \( j_{\nu k} \) is a continuous increasing function of \( \kappa \) on \([0, \infty)\).

The positive zeros of \( J_\nu(x) \) correspond to positive integral values of \( \kappa \) and \( j_{\nu, k+1/2} = y_{\nu k} \), \( k = 1, 2, \ldots \) where \( y_{\nu k} \) is the \( k \)th positive zero of \( Y_\nu(x) \). In [2] it was shown that \( j_{\nu k} \) is the unique solution of the differential equation

\[
\frac{dj}{d\nu} = 2j \int_0^\infty K_0(2j \sinh t)e^{-2\nu t}dt,
\]

which satisfies \( j(\nu) \to 0 \) as \( \nu \to -\kappa^+ \). This is motivated by the formula [11, p. 408] for the derivative of \( c_{\nu k} \) with respect to \( \nu \), and the fact that if, for \( \nu > 0 \), \( c_{\nu k} \) is the \( k \)th positive zero of \( \mathcal{E}_\nu(x) \), then \( c_{\nu k} \) may be extended in a continuous way to \( \nu < 0 \), and \( c_{\nu k} \to 0 \), as \( \nu \to -(k - \alpha/\pi) \). The equation (2.2) may be used to show that \( j_{\nu k} \) is an infinitely differentiable function of \( \kappa \).

We shall often use the fact that \( j_{1/2, \kappa} = \kappa \pi \).

3. The case \( \nu > 1/2 \); complete monotonicity

Here we show

**Theorem 3.1.** Let \( j_{\nu k} \) be defined as in §2. Then, for \( \nu > \mu \geq 1/2 \),

\[
(-1)^n D_\kappa^n (\log[j_{\nu k}/j_{\mu k}]) > 0, \quad \kappa > 0, \quad n = 0, 1, \ldots,
\]

and, in particular, for \( \nu > 1/2 \),

\[
(-1)^n D_\kappa^n (\log[j_{\nu k}/(\kappa \pi)]) > 0, \quad \kappa > 0, \quad n = 0, 1, \ldots
\]

**Theorem 3.2.** Let \( j_{\nu k} \) be defined as in §2. Then, for \( \nu > \mu \geq 1/2 \),

\[
(-1)^n D_\kappa^n (\log[D_{\kappa j_{\nu k}}/D_{\kappa j_{\mu k}}]) > 0, \quad \kappa > 0, \quad n = 0, 1, \ldots,
\]

and, in particular, for \( \nu > 1/2 \),

\[
(-1)^n D_\kappa^n (\log[D_{\kappa j_{\nu k}}/\pi]) > 0, \quad \kappa > 0, \quad n = 0, 1, \ldots
\]
Corollary 3.3. Let \( j_{\nu \kappa} \) be defined as in §2. Then, for \( \nu > 1/2 \),

\[
(3.5) \quad (-1)^n D_{\kappa}^{n+1} (j_{\nu \kappa}) > 0, \quad \kappa > 0, \ n = 0, 1, \ldots.
\]

Proofs. We have, from (2.2),

\[
\frac{1}{j_{\nu \kappa}} \frac{d j_{\nu \kappa}}{d \lambda} = 2 \int_0^\infty K_0(2j_{\lambda \kappa} \sinh t) e^{-2\lambda t} dt.
\]

Hence, integrating with respect to \( \lambda \) from \( \mu \) to \( \nu \), we get

\[
(3.6) \quad \log \frac{j_{\nu \kappa}}{j_{\mu \kappa}} = 2 \int_\mu^\nu \left\{ \int_0^\infty K_0(2j_{\lambda \kappa} \sinh t) e^{-2\lambda t} dt \right\} d \lambda.
\]

Next we use the result [3, p. 1484] of Elbert and Laforgia that

\[
(3.7) \quad D_\nu \log D_\kappa j_{\nu \kappa} = 2 \int_0^\infty K_0(2j_{\nu \kappa} \sinh t)(\tanh^2 t + 2\nu \tanh t) e^{-2\nu t} dt
\]

whence it follows that

\[
(3.8) \quad \log[D_{\nu} j_{\nu \kappa} / D_{\kappa} j_{\mu \kappa}] = 2 \int_\mu^\nu \left\{ \int_0^\infty K_0(2j_{\lambda \kappa} \sinh t)(\tanh^2 t + 2\lambda \tanh t) e^{-2\lambda t} dt \right\} d \lambda,
\]

and, in particular, when \( \mu = 1/2 \),

\[
(3.9) \quad \log[D_\kappa j_{\nu \kappa}] = \log \pi + 2 \int_{1/2}^\nu \left\{ \int_0^\infty K_0(2j_{\lambda \kappa} \sinh t)(\tanh^2 t + 2\lambda \tanh t) e^{-2\lambda t} dt \right\} d \lambda.
\]

Proving Theorems 3.1 and 3.2 amounts to showing that

\[
(3.10) \quad \log[j_{\nu \kappa} / j_{\mu \kappa}] \in \mathcal{M}_n(0, \infty),
\]

and

\[
(3.11) \quad \log[D_\kappa j_{\nu \kappa} / D_{\kappa} j_{\mu \kappa}] \in \mathcal{M}_n(0, \infty),
\]

for \( n = 1, 2, \ldots \), where \( \mathcal{M}_n(0, \infty) \) is the set of functions \( f \) on \( (0, \infty) \) satisfying

\[
(-1)^j f^{(j)}(\kappa) > 0, \quad j = 0, \ldots, n.
\]

Recall that

\[
(3.12) \quad (-1)^n D_t^n K_0(t) > 0, \quad t > 0, \ n = 0, 1, \ldots.
\]

This follows from the integral representation [11, p. 172]

\[
K_0(x) = \int_0^\infty e^{-x \cosh t} dt.
\]

We will in fact show that, for each \( a > 0 \) and \( \nu \geq 1/2 \),

\[
(3.13) \quad K_0(a j_{\nu \kappa}) \in \mathcal{M}_n(0, \infty),
\]
for \( n = 0, 1, \ldots \), and the representations (3.6), (3.8) will then give Theorems 3.1 and 3.2. Now in the case \( \nu = 1/2 \), (3.13) follows, for \( n = 0, 1, \ldots \) from (3.12). We prove (3.13) by induction in the case \( \nu > 1/2 \). In this case (3.13) is obvious for \( n = 0, 1 \); this follows from (3.12) and the fact that \( j_{\nu k} \) increases as \( \kappa \) increases. Suppose that (3.13) holds for \( n = N - 1 \). Then, from (3.9), we see that, for \( \nu > 1/2 \),

\[
(3.14) \quad \log D_k j_{\nu k} \in M_{N-1}(0, \infty).
\]

Since the exponential of a function in \( M_\infty(0, \infty) \) is also in \( M_\infty(0, \infty) \), we have

\[
(3.15) \quad (-1)^n D^{n+1}_k (j_{\nu k}) > 0, \quad \kappa > 0, \quad n = 0, \ldots, N - 1.
\]

By the formula of Faa di Bruno (see, e.g., [5, §81, pp. 92-93]) for the \( N \)th derivative of a function of a function, we have, for each \( a > 0 \),

\[
(3.16) \quad (-1)^N D^K_k (K_0(a j_{\nu k})) > 0.
\]

To see this we use (3.12) and (3.15) to show that each term arising from the Faa di Bruno formula is nonnegative and at least one is positive. See [7, Proof of Lemma 2.1] for details. Now (3.16) shows that (3.13) holds for \( n = N \). Thus the proof by induction is complete and (3.13) holds for \( n = 0, 1, \ldots \).

The representations (3.6), (3.8) now give Theorems 3.1 and 3.2.

Corollary 3.3 is a consequence of the fact that (3.15) holds for all \( N \). It recovers and generalizes the result of [3, p. 1485] that \( j_{\nu k} \) is concave with respect to \( \kappa \) when \( \nu \geq 1/2 \). Corollary 3.3 is the continuous version of the result of Lorch and Szegö [6] referred to in the Introduction. That result may be deduced from Corollary 3.3 by using a mean-value theorem for higher derivatives and differences ([10, no. 98] or [5, p. 74]).

The results of this section provide some examples of infinitely divisible distributions. Recall that a probability distribution is infinitely divisible if and only if its Laplace transform is of the form \( e^{-h(x)} \), where \( h(0) = 0 \) and \( h'(x) \) is completely monotonic [4, p. 425]. Thus, from Theorem 3.1, we can deduce the complete monotonicity of the distribution whose Laplace transform is \( j_{\mu, s + \epsilon j_{\nu, s + \epsilon j_{\mu, s + \epsilon}}} \), where \( \nu > \mu \geq 1/2 \), \( \epsilon > 0 \) and \( s \) is the variable in the Laplace transform.

4. THE CASE \( 0 \leq \nu < 1/2 \); SIMPLE MONOTONICITY

It does not seem to be easy to derive higher monotonicity results in the case when \( 0 \leq \nu < 1/2 \). In [6] it was conjectured that (1.1) should be replaced by

\[
(4.1) \quad (-1)^n c_{\nu k} > 0, \quad n = 0, 1, \ldots, k = 1, 2, \ldots.
\]

in the case \( |\nu| < 1/2 \). In [9], some progress was made in this direction when it was shown that (4.1) holds for \( 1/3 \leq |\nu| < 1/2 \). There does not seem to be any easy way to get the analogues of the results of §3 in the case \( 0 \leq \nu < 1/2 \). The condition \( \nu > 1/2 \) is an essential ingredient in getting (3.14), for example. However, we are able to deduce some simple monotonicity results in this case.
Theorem 4.1. Let $j_{\nu\kappa}$ be defined as in §2. Then, if $0 \leq \mu < \nu \leq 1/2$, the positive function $j_{\nu\kappa}/j_{\mu\kappa}$ decreases to 1 as $\kappa$ increases on $(0, \infty)$. If $0 \leq \mu < 1/2$, then $j_{\mu\kappa}/(\kappa\pi)$ increases with $\kappa$, $0 < \kappa < \infty$.

Theorem 4.2. Let $j_{\nu\kappa}$ be defined as in §2. Then, if $0 \leq \mu < \nu \leq 1/2$, the positive function $D_\kappa j_{\nu\kappa}/D_\kappa j_{\mu\kappa}$ decreases to 1 as $\kappa$ increases on $(0, \infty)$. If $0 \leq \mu < 1/2$, then $D_\kappa j_{\mu\kappa}$ increases with $\kappa$, $0 < \kappa < \infty$.

Remark. It should be noted that the monotonicities recorded here in the case $\mu < 1/2$ are opposite in direction to those which we get, in the case of order $> 1/2$, as a consequence of the theorems in §3.

The main assertions of Theorems 4.1 and 4.2 follow easily from the representations (3.6) and (3.8) once we recall that $K_0$ is a decreasing function of its argument and that $j_{\nu\kappa}$ increases as $\kappa$ increases. The final assertions of the theorems follow on putting $\nu = 1/2$.

5. Derivative of a zero with respect to $\kappa$

We obviously have $dj_{\nu\kappa}/d\nu > 0$. Here we show

Theorem 5.1. \begin{align*}
(5.1) & \frac{dj_{\nu\kappa}}{d\kappa} > \pi, \quad \nu > 1/2, \\
(5.2) & \frac{dj_{\nu\kappa}}{d\kappa} < \pi, \quad 0 < \nu < 1/2.
\end{align*}

Proof. From the MacMahon expansion [11, p. 506],

\begin{equation}
(5.3) \quad j_{\nu\kappa} = (\kappa + \nu/2 - 1/4)\pi + O(\kappa^{-1}), \quad \kappa \to \infty,
\end{equation}

we have $j_{\nu\kappa}/\kappa \to \pi$, as $\kappa \to \infty$. Hence, by an indirect application of L'Hospital's rule, we see that $dj_{\nu\kappa}/d\kappa \to \pi$, as $\kappa \to \infty$. Now, for $\nu > 1/2$, $j_{\nu\kappa}$ is a concave function of $\kappa$, by Corollary 3.3. Hence, $dj_{\nu\kappa}/d\kappa$ decreases and (5.1) follows. On the other hand, when $0 \leq \nu < 1/2$, $dj_{\nu\kappa}/d\kappa$ increases, by Theorem 4.2, and (5.2) follows.

Remark. The same indirect use of L'Hospital's rule shows the known result [1, Corollary 3.4, p. 276], that $dj_{\nu\kappa}/d\nu > 1$, where $\kappa$ is so large that $j_{\nu\kappa}$ is a concave function of $\nu$.

The principal term of (5.3) gives an upper bound for $j_{\nu\kappa}$ in case $\nu > 1/2$ and a lower bound in case $0 \leq \nu < 1/2$. To see this we write

\begin{equation}
(5.4) \quad f(\kappa) = j_{\nu\kappa} - (\kappa + \nu/2 - 1/4)\pi,
\end{equation}

and note from [2, Theorem 2.1], or from Theorem 5.1 above, that $f(\kappa)$ is increasing, constant or decreasing according as $\nu > 1/2$, $\nu = 1/2$ or $0 \leq \nu < 1/2$. Also $f(\kappa) \to 0$, as $\kappa \to \infty$ by (5.3), so the stated results follow.

We establish now some consequences of Theorem 5.1. Integration of (5.1) between $\kappa_0$ and $\kappa$, $\kappa > \kappa_0 > 0$, gives

\begin{equation}
(5.4) \quad j_{\nu\kappa} - j_{\nu\kappa_0} > \pi(\kappa - \kappa_0), \quad \nu > 1/2
\end{equation}
where the inequality must be reversed when $0 \leq \nu < 1/2$ and becomes equality when $\nu = 1/2$. If $j_{\nu \kappa_0}$ and $j_{\nu \kappa}$ are two consecutive zeros of the same cylinder function, we find the known result that the distance between two consecutive zeros is larger than, equal to or less than $\pi$ according as $\nu > 1/2$, $\nu = 1/2$ or $\nu < 1/2$.

An interesting particular case of (5.4) is when $\kappa_0 = n - 1/2$ and $\kappa = n$. In this case we find that

$$j_{\nu, n} > y_{\nu, n} + \pi/2, \quad \nu > 1/2, \quad n = 1, 2, \ldots$$

Clearly other inequalities of this kind can be found taking into consideration, in (5.4), different particular values of $\kappa_0$ and $\kappa$.

6. INEQUALITIES ARISING FROM CONVEXITY OR CONCAVITY WITH RESPECT TO $\kappa$

The function $j_{\nu \kappa}$ is known to be concave (convex) with respect to $\kappa$, $0 < \kappa < \infty$, for $\nu > 1/2$ ($0 < \nu < 1/2$) [3, p. 1485]. By considering the chord joining two points on the graph of this function, we thus obtain:

\begin{equation}
(6.1) \quad j_{\nu \kappa} > \frac{\kappa - k}{K - k} (j_{\nu K} - j_{\nu k}) + j_{\nu k}, \quad k < \kappa < K, \quad \nu > 1/2,
\end{equation}

and

\begin{equation}
(6.2) \quad j_{\nu \kappa} < \frac{\kappa - k}{K - k} (j_{\nu k} - j_{\nu K}) + j_{\nu k}, \quad k < \kappa < K, \quad 0 \leq \nu < 1/2.
\end{equation}

These become equalities when $\nu = 1/2$.

If we put $k = n$, $K = n + 1$, we get

\begin{equation}
(6.3) \quad j_{\nu \kappa} > (\kappa - n)(j_{\nu, n+1} - j_{\nu n}) + j_{\nu n}, \quad \nu > 1/2,
\end{equation}

\begin{equation}
(6.4) \quad j_{\nu \kappa} < (\kappa - n)(j_{\nu, n+1} - j_{\nu n}) + j_{\nu n}, \quad 0 \leq \nu < 1/2.
\end{equation}

In the case where $\kappa = n + 1/2$, we get

\begin{equation}
(6.5) \quad y_{\nu, n+1} > (j_{\nu, n+1} + j_{\nu n})/2, \quad \nu > 1/2,
\end{equation}

\begin{equation}
(6.6) \quad y_{\nu, n+1} < (j_{\nu, n+1} + j_{\nu n})/2, \quad 0 \leq \nu < 1/2.
\end{equation}

For example, taking $\nu = 0$, $n = 39$ in (6.6) gives $y_{0, 40} < 123.3085255$, where the values of $j_{0, 39}$, $j_{0, 40}$ are taken from [11, p. 748]. The “exact” value of $y_{0, 40}$, from the same source, is 123.3085253.

On the other hand, if in (6.2) we take $k = n - 1/2$, $\kappa = n$, $K = n + 1/2$, we get

\begin{equation}
(6.7) \quad j_{\nu, n} < (y_{\nu, n+1} + y_{\nu n})/2, \quad \nu < 1/2.
\end{equation}

In particular, with $n = 39$, $\nu = 0$, we get $y_{0, 40} > 2j_{0, 39} - y_{0, 39} = 123.308525$. 

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The upper and lower bounds
\[123.308525 < y_{0.40} < 123.3085255\]
show the sharpness of these inequalities.

We have found the inequalities to be similarly sharp in the case \( \nu = 1 \) and even more so in the case \( \nu = 1/3 \), as may be expected from the fact that they become equalities when \( \nu = 1/2 \).

Inequalities of a similar kind may be obtained by using, for example, the increase of \( j_{\nu} / k \) with respect to \( k \) (0 < \( k \) < \( \infty \)) for 0 \( \leq \nu < 1/2 \) (Theorem 4.1). Thus, for example, with \( \nu = 1/3 \), we get
\[
\frac{y_{1/3,40}}{39.5} = \frac{j_{1/3,39.5}}{39.5} > \frac{j_{1/3,39}}{39}
\]
or
\[y_{1/3,40} > \left(\frac{79}{78}\right)j_{1/3,39} = 123.8283293;\]
the "actual" value of \( y_{1/3,40} \) is 123.8316712. The numerical values are from [11, p. 75].

An advantage of the results enunciated here is that the knowledge of the zeros of one cylinder function (\( J_\nu(x) \), for example) enables us to establish approximations for the zeros of any other cylinder function, those of \( Y_\nu(x) \) being but one example. Also, some of the inequalities found in this section are more informative than some of those which can be obtained by a direct application of the Sturm comparison theorem. For example, when \( \nu > 1/2 \), one can use the Sturm theorem to show that \( y_{\nu,n+1} \) belongs to the interval \((j_{\nu,n}, j_{\nu,n+1})\), but formula (6.5) asserts that it belongs to the second half of this interval.

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