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COMMON FIXED POINTS IN HYPERBOLIC RIEMANN SURFACES AND CONVEX DOMAINS

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ABSTRACT. In this paper we prove that a commuting family of continuous self-maps of a bounded convex domain in $\mathbb{C}^n$ which are holomorphic in the interior has a common fixed point. The proof makes use of three basic ingredients: iteration theory of holomorphic maps, a precise description of the structure of the boundary of a convex domain, and a similar result for commuting families of self-maps of a hyperbolic domain of a compact Riemann surface.

0. INTRODUCTION

Let $D \Subset \mathbb{C}^n$ be a bounded domain. In this paper we shall study families of continuous maps of $\overline{D}$ into itself that are holomorphic in D and commute with each other under composition. In particular, we shall be interested in the existence of points which are fixed by any element of the family.

For the sake of brevity, let us make some definitions. Let X be a topological space, and $\mathscr{F}$ a family of continuous maps of X into itself (shortly, of *self-maps* of X). We shall say that $\mathscr{F}$ is a *commuting family* if $f \circ g = g \circ f$ for every $f, g \in \mathscr{F}$. A point $x \in X$ is a *fixed point* of the family $\mathscr{F}$ if $f(x) = x$ for all $f \in \mathscr{F}$. Then the starting point of our paper is the following theorem, proved by Shields in 1964:

Theorem 0.1 (Shields [S]). *Let Δ be the unit disk in $\mathbb{C}$, and $\mathscr{F}$ a commuting family of continuous self-maps of $\overline{\Delta}$ holomorphic in Δ. Then $\mathscr{F}$ has a fixed point in $\overline{\Delta}$.*

We point out that, although every continuous self-map of $\overline{\Delta}$ has a fixed point, this result is a feature of the holomorphic structure, and not just some sort of consequence of Brouwer's theorem: indeed, there are examples of commuting continuous functions mapping the closed unit interval $[-1, 1] \subset \mathbb{R}$ into itself without common fixed points (cf. Boyce [B] and Huneke [H]). See also [Co] for a detailed study of commuting holomorphic functions in Δ.

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The natural generalization of Shields' theorem to several complex variables would be an analogous statement for bounded convex domains of $\mathbb{C}^n$. The first step in this direction was made by Eustice [E], who in 1972 extended Shields' theorem to $\Delta^2 = \Delta \times \Delta \subset \mathbb{C}^2$. Later, Suffridge [Su] found a proof valid for the euclidean unit ball B^n of $\mathbb{C}^n$ (a slightly simpler argument is presented in [A4]), and Heath and Suffridge [HS] generalized the theorem to the unit polydisk Δ^n.

Shields', Eustice's, and Suffridge's proofs were all based on the iteration theory of holomorphic maps. Recently, the iteration theory in convex domains has been thoroughly studied (see [A1, 2, 4]); this allowed the first author to generalize Shields' theorem to strongly convex domains (in [A3]). His proof was then considerably simplified by the second author and, independently, by Kuczumow and Stachura [KS], who also extended the theorem to product domains.

In this paper we shall be able to prove the complete generalization of Shields' theorem to convex domains; namely,

Theorem 0.2. *Let D be a bounded convex domain in $\mathbb{C}^n$, and $\mathscr{F}$ a commuting family of continuous self-maps of $\overline{D}$ holomorphic in D. Then $\mathscr{F}$ has a fixed point in $\overline{D}$.*

The proof is contained in the second section of this paper; in the first section we shall extend Shields' theorem to hyperbolic domains of compact Riemann surfaces, introducing a very mild restriction on the families under consideration.

1. Hyperbolic Riemann surfaces

We begin recalling some notations and definitions. Let X and Y be two complex manifolds; we shall denote by $\mathrm{Hol}(X, Y)$ the space of holomorphic maps from X into Y, endowed with the compact-open topology. $\mathrm{Aut}(X) \subset \mathrm{Hol}(X, X)$ will denote the automorphism group of X. A sequence $\{f_k\} \subset \mathrm{Hol}(X, Y)$ is said to be *compactly divergent* if for every pair of compact sets $H \subset X$ and $K \subset Y$ we have $f_k(H) \cap K = \varnothing$ for k sufficiently large. A family $\mathscr{F} \subset \mathrm{Hol}(X, Y)$ is said to be *normal* if every sequence in $\mathscr{F}$ admits a subsequence which is either compactly divergent or uniformly convergent on compact subsets.

Now let X be a compact Riemann surface. A *hyperbolic domain* of X is an open connected subset D of X that is hyperbolic as Riemann surface. It is known (Montel's theorem for hyperbolic Riemann surfaces; see [A4, Theorem 1.1.43, Proposition 1.1.25]) that if X is any Riemann surface and Y is a hyperbolic Riemann surface then the family $\mathrm{Hol}(X, Y)$ is normal. For hyperbolic domains something more is true:

Proposition 1.1. *Let D be a hyperbolic domain of a compact Riemann surface X, and Y another Riemann surface. Then* $\mathrm{Hol}(Y, D)$ *is relatively compact in* $\mathrm{Hol}(Y, X)$.

Proof. Let $D_1 \subset X$ be a hyperbolic domain containing D and such that $X \backslash D_1$ is finite (containing three points if X is the Riemann sphere, one point if X

is a torus, even empty if X is hyperbolic); it suffices to show that $\mathrm{Hol}(Y, D_1)$ is relatively compact in $\mathrm{Hol}(Y, X)$.

Let $\{f_k\}$ be a sequence in $\mathrm{Hol}(Y, D_1)$; we must find a subsequence converging in $\mathrm{Hol}(Y, X)$. By Montel's theorem, up to a subsequence we can assume that $\{f_k\}$ is compactly divergent in $\mathrm{Hol}(Y, D_1)$; otherwise we can extract a subsequence already converging in $\mathrm{Hol}(Y, D_1)$. Note that if $D_1 = X$ the proof is finished.

Write $X\backslash D_1 = \{x_1, \dots, x_p\}$, and fix a compact connected subset K_0 of Y. We claim that there exists $x_{j_0} \in X\backslash D_1$, a subsequence $\{f_{k_\nu}\}$ of $\{f_k\}$ and a connected neighborhood U_0 of x_{j_0} in X such that for every compact connected subset K of Y containing K_0 and every neighborhood U of x_{j_0} in X contained in U_0 we have $f_{k_\nu}(K) \subset U$ for all ν large enough. Since Y is the increasing union of compact connected subsets containing K_0, this will imply that $f_{k_\nu} \to x_{j_0}$ in $\mathrm{Hol}(Y, X)$ as $\nu \to +\infty$, and the assertion will follow.

For $j = 1, \dots, p$ choose disjoint connected neighborhoods U_j of x_j in X. Since $\{f_k\}$ is compactly divergent, we have $f_k(K_0) \subset \bigcup_j U_j$ for k sufficiently large. But K_0 is connected; hence there is a subsequence $\{f_{k_\nu}\}$ and an index j_0 such that $f_{k_\nu}(K_0) \subset U_{j_0} = U_0$ for all $\nu \in \mathbb{N}$.

Now let K be any compact connected subset of Y containing K_0, and U any neighborhood of x_{j_0} in X contained in U_0. Since $\{f_{k_\nu}\}$ is still compactly divergent, we have

$$f_{k_\nu}(K) \subset U \cup \bigcup_{j \neq j_0} U_j$$

for ν large enough. But $K \supset K_0$ is connected, and $f_{k_\nu}(K) \cap U_0 \neq \emptyset$ for every $\nu \in \mathbb{N}$; therefore, $f_{k_\nu}(K) \subset U$ for ν sufficiently large, and we are done. Q.E.D.

This proposition can be expressed by saying that D is *hyperbolically imbedded* in X. It is worth remarking that Kiernan [K] has shown that this is equivalent to the following geometrical fact: if $\{z_\nu\}$ and $\{w_\nu\}$ are sequences in D such that $z_\nu \to z \in \overline{D}$, $w_\nu \to w \in \overline{D}$, and $k_D(z_\nu, w_\nu) \to 0$ as $\nu \to +\infty$, where k_D is the Poincaré distance on D, then $z = w$.

As already anticipated in the introduction, theorems on common fixed points are often proved using iteration theory. In our case, we shall need only a few easy results, that we are now going to describe.

Let X be a complex manifold; we shall denote by id_X the identity map of X. Take a map $f \in \mathrm{Hol}(X, X)$; the kth *iterate* f^k of f is inductively defined by $f^0 = \mathrm{id}_X$ and $f^k = f \circ f^{k-1}$. A *limit point* of the sequence $\{f^k\}$ is the limit of a subsequence $\{f^{k_\nu}\}$ such that $k_\nu \to +\infty$ as $\nu \to +\infty$.

Our first goal is a precise description of the limit points of a sequence of iterates. To obtain it, we need two preliminary lemmas. The first one is of topological character:

Lemma 1.2. *Let X, Y be two locally compact locally connected Hausdorff spaces, and $\{\varphi_\nu\}$ a sequence of continuous open maps of X into Y, converging for the compact-open topology to a continuous map $\varphi\colon X \to Y$. Suppose that $z_0 \in X$ is an isolated point of $\varphi^{-1}(\varphi(z_0))$. Then for any neighborhood U of z_0 there is a $\nu_0 \in \mathbb{N}$ such that*

$$\varphi(z_0) \in \varphi_\nu(U) \qquad \forall \nu \geq \nu_0 .$$

Proof. See [A4, Lemma 2.1.19] or [N, Proposition 5, p. 79]. Q.E.D.

The second lemma holds in a general setting too:

Lemma 1.3. *Let X be a complex manifold, and $f \in \mathrm{Hol}(X, X)$. Then an automorphism of X can be a limit of $\{f^k\}$ only if f itself is an automorphism of X.*

Proof. Assume $g \in \mathrm{Aut}(X)$ be a limit point of $\{f^k\}$. Clearly, f is one-to-one; in particular, it is an open map. Let $\{f^{k_\nu}\}$ be a subsequence converging to g, and take $z_0 \in X$. Then Lemma 1.2 applied to the sequence $\varphi_\nu = f^{k_\nu}$ and to the point $g^{-1}(z_0)$ shows that $z_0 = g(g^{-1}(z_0)) \in f(X)$. Being z_0 arbitrary, this implies that f is onto, and we are done. Q.E.D.

It should be remarked that if X actually is a Riemann surface, then it is possible to prove this lemma using the classical Hurwitz theorem for Riemann surfaces [A4, Corollary 1.1.36] instead of Lemma 1.2.

And now we can completely describe the limit points of a sequence of iterates:

Theorem 1.4. *Let D be a hyperbolic domain of a compact Riemann surface X, and $f \in \mathrm{Hol}(D, D)$. Let $h\colon D \to X$ be a limit point in $\mathrm{Hol}(D, X)$ of the sequence $\{f^k\}$. Then either*

(i) *h is a constant $z_0 \in \overline{D}$, or*
(ii) *h is an automorphism of D. In this case, f is an automorphism too.*

Proof. Clearly, $h(D) \subset \overline{D}$. If $h(D) \cap \partial D \neq \varnothing$, by the open mapping theorem h is constant, and we are in case (i); therefore, we can assume $h \in \mathrm{Hol}(D, D)$.

Choose a subsequence $\{f^{k_\nu}\}$ converging to h; we can assume that $m_\nu = k_{\nu+1} - k_\nu$ tends to infinity as $\nu \to +\infty$. By Montel's theorem, up to a subsequence we can also suppose that $\{f^{m_\nu}\}$ either converges to a holomorphic map g or is compactly divergent.

Suppose h is not constant; we must prove that both f and h are automorphisms of D. For any $z \in D$ we have

$$\lim_{\nu\to\infty} f^{m_\nu}(f^{k_\nu}(z)) = \lim_{\nu\to\infty} f^{k_{\nu+1}}(z) = h(z);$$

therefore, $\{f^{m_\nu}\}$ cannot be compactly divergent, and g is the identity on the open subset $h(D)$ of D. Hence $g = \mathrm{id}_D$; by Lemma 1.3, f is an automorphism.

It remains to show that h itself is an automorphism. Set $f_\nu = f^{k_\nu}$ and $g_\nu = f_\nu^{-1}$. By Montel's theorem, up to a subsequence we can assume that either $g_\nu \to g \in \mathrm{Hol}(D, D)$ or $\{g_\nu\}$ is compactly divergent. Since for any $z \in D$ we have $f_\nu(z) \to h(z) \in D$ and $g_\nu(f_\nu(z)) = z$, the sequence $\{g_\nu\}$ cannot be compactly divergent. Then

$$g(h(z)) = \lim_{\nu\to\infty} g_\nu(f_\nu(z)) = z \qquad \forall z \in D,$$

and, since $g(D) \subset D$,

$$h(g(z)) = \lim_{\nu\to\infty} f_\nu(g_\nu(z)) = z \qquad \forall z \in D;$$

therefore $g = h^{-1}$, and h is an automorphism. Q.E.D.

We are now able to prove the announced generalization of Shields' theorem to hyperbolic domains:

Theorem 1.5. *Let D be a hyperbolic domain of a compact Riemann surface X, and $\mathscr{F}$ a commuting family of continuous self-maps of $\overline{D}$ holomorphic in D. Assume either:*

(i) *D is simply connected, or*
(ii) *D is multiply connected and $\mathscr{F}$ is not contained in* $\mathrm{Aut}(D)$.

Then $\mathscr{F}$ has a fixed point.

Proof. We begin remarking that if $\mathscr{F}$ contains a constant map $z_0 \in \overline{D}$, then z_0 is clearly a fixed point of $\mathscr{F}$; hence we shall assume that $\mathscr{F}$ does not contain constant maps. By the open map theorem, then, every element of $\mathscr{F}$ sends D into itself.

Suppose, for the moment, that $\mathscr{F}$ is not contained in $\mathrm{Aut}(D)$, and take $f_0 \in \mathscr{F} \setminus \mathrm{Aut}(D)$. Let h be a limit point in $\mathrm{Hol}(D, X)$ of the sequence of iterates of f_0; h exists by Proposition 1.1, and it is a constant $z_0 \in \overline{D}$ by Theorem 1.4. Choose a subsequence $\{f_0^{k_\nu}\}$ converging to h. Then for every $g \in \mathscr{F}$ we have

$$g(z_0) = \lim_{\nu\to\infty} g(f_0^{k_\nu}(z)) = \lim_{\nu\to\infty} f_0^{k_\nu}(g(z)) = z_0,$$

where z is any point of D; therefore, z_0 is a fixed point of $\mathscr{F}$.

It remains to study the case $\mathscr{F} \subset \mathrm{Aut}(D)$ and D simply connected; in particular, D is biholomorphic to the upper half-plane H^+ of $\mathbb{C}$. Take $f_0 \in \mathscr{F}$. If f_0 has no fixed points in D, using the explicit expression of automorphisms of H^+ it is easy to see that the sequence $\{f_0^k\}$ is compactly divergent. Then (by Proposition 1.1 and Theorem 1.4) there is a subsequence $\{f_0^{k_\nu}\}$ converging toward a constant $z_0 \in \partial D$, and we find as before that z_0 is a fixed point of $\mathscr{F}$.

Finally, assume every $f \in \mathscr{F}$ has a fixed point in D. Take $f_0 \in \mathscr{F}$. Clearly, we can assume $f_0 \neq \mathrm{id}_D$; so f_0 has a unique fixed point $z_0 \in D$. Now, for every $f \in \mathscr{F}$ we have

$$f_0(f(z_0)) = f(f_0(z_0)) = f(z_0);$$

therefore, $f(z_0) = z_0$, and we are done. Q.E.D.

Two remarks are in order. First of all, in multiply-connected domains there actually exist commuting families of automorphisms without fixed points; indeed, there are automorphisms (continuous up to the boundary and) without fixed points. Take for instance

$$D = \{z \in \mathbb{C} | 1 < |z| < 2\}$$

and $f(z) = -z$. Second, Theorem 1.5 is more general then [A4, Theorem 1.3.22], and its proof requires considerably less technical tools; in particular, it is not necessary to study the boundary behavior of the universal covering map.

2. Convex domains

This section is devoted to the proof of Theorem 0.2. The main tool (besides some results on iteration theory and fixed point sets in convex domains to be recalled later) is a description of the structure of the boundary of a convex domain. We begin with a (very easy) sort of maximum principle:

Lemma 2.1. *Let D be a bounded convex domain in $\mathbb{C}^n$, X a (connected) complex manifold, and $f \in \mathrm{Hol}(X, \mathbb{C}^n)$ such that $f(X) \subset \overline{D}$. Then either $f(X) \subset D$ or $f(X) \subset \partial D$.*

Proof. Assume there exists $w_0 \in X$ such that $x_0 = f(w_0) \in \partial D$; we must show that $f(X) \subset \partial D$. By the Hahn–Banach theorem, we can find a $\mathbb{C}$-linear map $\sigma: \mathbb{C}^n \to \mathbb{C}$ such that

$$\operatorname{Re} \sigma(z) < \operatorname{Re} \sigma(x_0) \qquad \forall z \in D.$$

Therefore $\operatorname{Re} \sigma \circ f \leq \operatorname{Re} \sigma \circ f(w_0)$ on X; the maximum principle then implies $\sigma \circ f \equiv \sigma \circ f(w_0)$, and in particular $f(X) \subset \partial D$. Q.E.D.

Using a more refined version of the previous argument, we can associate with every point x_0 in the boundary of a bounded convex domain $D \Subset \mathbb{C}^n$, a convex domain of smaller dimension which is, in a certain sense, the natural range of every holomorphic map $f: X \to \overline{D}$ such that $x_0 \in f(X)$ More precisely, we have

Proposition 2.2. *Let D be a bounded convex domain in $\mathbb{C}^n$, and take $x_0 \in \partial D$. Then there exists a complex affine subspace L of $\mathbb{C}^n$ satisfying the following conditions:*

(i) $D \cap L = \varnothing$;
(ii) $\overline{D} \cap L$ *is the closure of a bounded convex domain* D_0 *of* L;
(iii) $x_0 \in D_0$;

(iv) *for every connected complex manifold* X *and holomorphic map* $f: X \to \mathbb{C}^n$ *such that* $f(X) \subset \overline{D}$ *and* $f(X) \cap \overline{D}_0 \neq \emptyset$ *we have* $f(X) \subset \overline{D_0} = \overline{D} \cap L$;

(v) *for every connected complex manifold* X *and holomorphic map* $f: X \to \mathbb{C}^n$ *such that* $f(X) \subset \overline{D}$ *and* $x_0 \in f(X)$ *we have* $f(X) \subset D_0$.

Proof. We shall prove by induction on n the existence of L and D_0 satisfying (i)–(iv); (v) will then follow from (iii), (iv), and Lemma 2.1.

For $n = 1$, by the open mapping theorem we can take $L = D_0 = \{x_0\}$; so assume $n > 1$. Without loss of generality, we can suppose that x_0 is the origin. By the Hahn-Banach theorem, there exists a $\mathbb{C}$-linear map $\sigma_1: \mathbb{C}^n \to \mathbb{C}$ such that $\operatorname{Re}\sigma_1 < 0$ on D; set $V_1 = \ker\sigma_1$. Let W_1 be the real subspace of V_1 generated by $\overline{D} \cap V_1$, and set $L_1 = W_1 \cap iW_1$. Since $\overline{D} \cap V_1$ is convex and generates W_1, there is a convex open subset D_1' of W_1 such that $\overline{D} \cap V_1 = \overline{D_1'}$; set $D_1 = D_1' \cap L_1$. Clearly, $x_0 = 0 \in \overline{D_1} = \overline{D} \cap L_1$. Note that the complex dimension of L_1 is strictly less than n.

Let $f \in \operatorname{Hol}(X, \mathbb{C}^n)$ be such that $f(X) \subset \overline{D}$ and $f(X) \cap \overline{D_1} \neq \emptyset$; fix $w_0 \in f^{-1}(\overline{D_1})$. Since $\operatorname{Re}\sigma_1 \circ f \le 0$ on X and $\sigma_1 \circ f(w_0) = 0$, the maximum principle implies $\sigma_1 \circ f \equiv 0$, i.e., $f(X) \subset \overline{D} \cap V_1$. In particular, f is a holomorphic map from X to V_1.

Now, there exists a (possibly empty) set $\{\tau_1, \dots, \tau_r\}$ of $\mathbb{C}$-linear forms on V_1 such that

$$W_1 = \bigcap_{j=1}^{r} \ker \operatorname{Re}\tau_j \quad \text{and} \quad L_1 = \bigcap_{j=1}^{r} \ker\tau_j .$$

Since $\overline{D} \cap V_1$ is contained in W_1, we have $\operatorname{Re}\tau_j \circ f \equiv 0$ on X. Being $\tau_j \circ f(w_0) = 0$, this yields $\tau_j \circ f \equiv 0$ on X for every $j = 1, \dots, r$, that is $f(X) \subset \overline{D} \cap L_1 = \overline{D_1}$.

In other words we have constructed a complex subspace L_1 and a bounded convex domain D_1 of L_1 satisfying (i), (ii) and (iv). If $x_0 \in D_1$ we are done; otherwise, $x_0 \in \partial D_1$ and we can apply the induction hypothesis. Q.E.D.

We shall use this proposition to build up an induction argument; but before the proof of Theorem 0.2 we recall another few facts on convex domains. First of all, we shall make use of the following result of iteration theory:

Theorem 2.3 [A2; A4, Theorems 2.4.3, 2.4.20; KS]. *Let* D *be a bounded convex domain of* $\mathbb{C}^n$, *and* $f \in \operatorname{Hol}(D, D)$. *Then the following facts are equivalent*:

(i) f *has a fixed point in* D;

(ii) *the sequence of iterates of* f *is not compactly divergent*;

(iii) *the sequence of iterates of* f *is relatively compact in* $\operatorname{Hol}(D, D)$;

(iv) *for every point* $z \in D$ *the sequence* $\{f^k(z)\}$ *is relatively compact in* D;

(v) *for one point* $z_0 \in D$ *the sequence* $\{f^k(z_0)\}$ *is relatively compact in* D.

It should be remarked that the convexity of D is required only to prove that (v) $\Rightarrow$ (i); the equivalence of (ii), (iii), (iv), and (v) holds for any taut manifold, and (i) $\Rightarrow$ (ii) is obvious.

To state the second result we recall a couple of definitions. Let X be a complex manifold; a *holomorphic retraction* of X is a map $\rho \in \mathrm{Hol}(X, X)$ such that $\rho^2 = \rho$; its image is a *holomorphic retract* of X. It should be noticed that a holomorphic retract is always a smooth submanifold [R, C]. In convex domains, fixed point sets and holomorphic retracts are one and the same thing:

Theorem 2.4 [V1, 2]. *Let D be a bounded convex domain of $\mathbb{C}^n$, and $f \in \mathrm{Hol}(D, D)$. Let F be the set of fixed points of f in D, and assume $F \neq \emptyset$. Then there exists a holomorphic retraction $\rho_f \colon D \to D$ such that $\rho_f(D) = F$. In particular, F is a closed connected submanifold of D.*

We are almost ready to prove Theorem 0.2. We only need another observation:

Lemma 2.5. *Let D be a bounded convex domain of $\mathbb{C}^n$, and let L be a complex affine subspace of $\mathbb{C}^n$ such that $L \cap D = \emptyset$ and $L \cap \partial D \neq \emptyset$. Let D_1 be the interior of $L \cap \partial D$ in L. Then for every continuous function $f \colon \overline{D} \to \mathbb{C}^m$ which is holomorphic in D the restriction of f to D_1 is still holomorphic.*

Proof. Without loss of generality we can assume $0 \in D$. For every $k \in \mathbb{N}^*$ define $f_k \colon D_1 \to \mathbb{C}^m$ by

$$f_k(z) = f((1 - 1/k)z).$$

It is clear that every f_k is holomorphic; since $f_k \to f|_{D_1}$ uniformly, the assertion follows. Q.E.D.

And finally:

Theorem 0.2. *Let D be a bounded convex domain in $\mathbb{C}^n$, and $\mathscr{F}$ a commuting family of continuous self-maps of $\overline{D}$ holomorphic in D. Then $\mathscr{F}$ has a fixed point in $\overline{D}$.*

Proof. By induction on n. For $n = 1$ it is just Theorem 1.5; so assume $n > 1$. We must examine three mutually exclusive (by Lemma 2.1) cases:

(a) *Every $f \in \mathscr{F}$ has a fixed point in D*. For every $f \in \mathrm{Hol}(D, D)$ let $\mathrm{Fix}(f)$ denote the fixed point set of f in D. Take $f, g \in \mathscr{F}$; since f and g commute, we have $g(\mathrm{Fix}(f)) \subset \mathrm{Fix}(f)$. Let $\rho_f \colon D \to \mathrm{Fix}(f)$ be the holomorphic retraction given by Theorem 2.4, and set $\tilde{g} = \rho_f \circ g \in \mathrm{Hol}(D, D)$. For every $z \in \mathrm{Fix}(f)$ and $k \in \mathbb{N}$ we have $\tilde{g}^k(z) = g^k(z)$; hence, by Theorem 2.3, $\mathrm{Fix}(\tilde{g}) \neq \emptyset$. Since $\mathrm{Fix}(\tilde{g}) = \mathrm{Fix}(f) \cap \mathrm{Fix}(g)$, it follows from Theorem 2.4 that $\mathrm{Fix}(f) \cap \mathrm{Fix}(g)$ is a holomorphic retract of D.

Using an induction argument it is now easy to show that $\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)$ is a holomorphic retract (and so a nonempty closed smooth submanifold) of D for every $r \in \mathbb{N}$ and $f_1, \ldots, f_r \in \mathscr{F}$. Set

$$d = \min\{\dim[\mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_r)] \mid r \in \mathbb{N}, f_1, \ldots, f_r \in \mathscr{F}\} \geq 0,$$

and pick $f_1, \dots, f_{r_0} \in \mathscr{F}$ so that the dimension of $F = \mathrm{Fix}(f_1) \cap \cdots \cap \mathrm{Fix}(f_{r_0})$ is exactly d. This implies that $F \cap \mathrm{Fix}(f)$ is (a closed connected submanifold of equal dimension and hence) equal to F for every $f \in \mathscr{F}$, and so every point in F is fixed by $\mathscr{F}$.

(b) *There exists $f_0 \in \mathscr{F}$ such that $f_0(D) \subset \partial D$*; in particular, f_0 has no fixed points in D. Fix $z_0 \in D$ and let $x_0 = f(z_0) \in \partial D$. Using Proposition 2.2 we associate to x_0 a complex affine subspace L of $\mathbb{C}^n$ and a bounded convex domain D_0 of L whose closure is $\partial D \cap L$ such that $x_0 \in f_0(D) \subset D_0$; clearly, $f_0(\overline{D}) \subset \overline{D_0}$.

Take $g \in \mathscr{F}$. By Lemma 2.5, $g|_{D_0} : D_0 \to \mathbb{C}^n$ is holomorphic. Now $g(x_0) = f(g(z_0)) \in \overline{D_0}$; hence, by Proposition 2.2 (iv), $g(\overline{D_0}) \subset \overline{D_0}$. Therefore

$$\mathscr{F}|_{\overline{D_0}} = \{g|_{\overline{D_0}} | g \in \mathscr{F}\}$$

is a commuting family of continuous self-maps of $\overline{D_0}$ holomorphic in D_0, and we can conclude by applying the induction hypothesis.

(c) *We have $f(D) \subset D$ for every $f \in \mathscr{F}$ and there is $f_0 \in \mathscr{F}$ without fixed points in D.* By Theorem 2.3, the sequence of iterates of f_0 is compactly divergent; therefore, if $h \in \mathrm{Hol}(D, \mathbb{C}^n)$ is a limit point of $\{f_0^k\}$ (such an h exists because D is bounded) we have $h(D) \subset \partial D$. Furthermore,

$$f(h(z)) = h(f(z)) \qquad \forall f \in \mathscr{F}, \ \forall z \in D,$$

because every $f \in \mathscr{F}$ commutes with f_0 and sends D into itself. Therefore we can repeat the argument used in case (b), replacing f_0 by h, and we are done. Q.E.D.

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