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BERNSTEIN-TYPE INEQUALITIES FOR THE DERIVATIVES OF CONSTRAINED POLYNOMIALS

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ABSTRACT. Generalizing a number of earlier results, P. Borwein established a sharp Markov-type inequality on $[-1, 1]$ for the derivatives of polynomials $p \in \pi_n$ having at most k $(0 \leq k \leq n)$ zeros in the complex unit disk. Using Lorentz representation and a Markov-type inequality for the derivative of Müntz polynomials due to D. Newman, we give a surprisingly short proof of Borwein's Theorem. The new result of this paper is to obtain a sharp Bernstein-type analogue of Borwein's Theorem. By the same method we prove a sharp Bernstein-type inequality for another wide family of classes of constrained polynomials.

1. INTRODUCTION, NOTATIONS

Markov's inequality, which plays a significant role in approximation theory and related areas, states that

$$\max_{-1\leq x\leq 1} |p'(x)| \leq n^2 \max_{-1\leq x\leq 1} |p(x)| \tag{1}$$

for every polynomial $p \in \pi_n$, where π_n denotes the set of all real algebraic polynomials of degree at most n. The pointwise algebraic Bernstein-type analogue asserts that

$$|p'(y)| \leq \frac{n}{\sqrt{1-y^2}} \max_{-1\leq x\leq 1} |p(x)| \quad (-1 < y < 1) \tag{2}$$

for every polynomial $p \in \pi_n$. On every fixed subinterval $[-a, a]$ $(0 < a < 1)$, (2) gives a much better upper bound than (1). Let $S_n^k(z, r)$ be the family of polynomials from π_n which have at most k zeros in the open disk of the complex plane with center z and radius r. A number of papers were written on Markov- and Bernstein-type inequalities for the derivatives of polynomials from $S_n^k(0, 1)$ in certain special cases. When $k = 0$, see [5], [6], and [9]; when k is small compared with n, [7] and [11] give reasonable results. Finally, proving J. Szabados's conjecture, P. Borwein [1] verified the following sharp inequality:

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Theorem 1. *We have*

$$\max_{0\le x\le 1}|p'(x)|\le cn(k+1)\max_{0\le x\le 1}|p(x)|$$

for every $p\in S_n^k(1/2,\,1/2)$ *with some absolute constant* $c\le 18$.

Using a Lorentz representation of a polynomial from $S_n^0(1/2,\,1/2)$ and a Markov-type inequality for the derivative of Müntz polynomials, we will present a very short proof of Theorem 1. A sharp Markov-type inequality was established in [2] for another family of classes of constrained polynomials.

Theorem 2. *We have*

$$\max_{-1\le x\le 1}|p'(x)|\le \min\left\{n^2,\,\frac{cn}{\sqrt{r}}\right\}\max_{-1\le x\le 1}|p(x)|$$

for every polynomial from π_n *having no zeros in the open disks with diameters* $[-1,\,-1+2r]$ *and* $[1-2r,\,1]$, *respectively, where* $0<r\le 1$ *and* $c\le 20$ *is an absolute constant.*

For $0<r\le 1$, we define

$$K(r)=\bigcup_{a\in[-1+r,\,1-r]}\{z\in\mathbb{C}:\,|z-a|<r\}$$

and denote by $W_n^0(r)$ the set of those polynomials from π_n which have no zeros in $K(r)$. The main goal of this paper is to obtain sharp Bernstein-type inequalities for the derivatives of polynomials from $S_n^k(0,\,1)$ and $W_n^0(r)$, respectively.

2. New results

We will prove the following Bernstein-type inequalities:

Theorem 3. *We have*

$$|p'(y)|\le c\frac{\sqrt{n(k+1)}}{1-y^2}\max_{-1\le x\le 1}|p(x)|\qquad(-1<y<1)$$

for every $p\in S_n^k(0,\,1)$, *where* c *is an absolute constant.*

Theorem 4. *We have*

$$\text{(i)}\qquad |p'(y)|\le c\sqrt{\frac{n}{r(1-y^2)}}\max_{-1\le x\le 1}|p(x)|\qquad(-1+r<y<1-r)$$

and

$$\text{(ii)}\qquad |p'(y)|\le c\sqrt{\frac{n}{r}}\frac{1}{1-y^2}\max_{-1\le x\le 1}|p(x)|\qquad(-1<y<1)$$

for every polynomial $p\in W_n^0(r)$ $(0<r\le 1)$ *with certain absolute constants* c.

3. The sharpness of our theorems

The sharpness of Theorem 1 was proved by J. Szabados [10]. It was shown in [3] that

$$\sup_{p\in S_n^k(0,1)} \frac{|p'(0)|}{\max_{-1\le x\le 1}|p(x)|} \ge c\sqrt{n(k+1)}$$

with some absolute constant $c > 0$.

Conjecture 1. The pointwise factor $(1-y^2)^{-1}$ in Theorem 3 can be replaced by $(1-y^2)^{-1/2}$.

Conjecture 2. We have

$$\sup_{p\in W_n^0(r)} \frac{|p'(0)|}{\max_{-1\le x\le 1}|p(x)|} \ge c\sqrt{\frac{n}{r}} \quad \left(\frac{1}{n}\le r\le 1\right)$$

with some absolute constant $c > 0$.

Remark 1. With J. Szabados [4], we proved that

$$|p'(y)| \le c\frac{\sqrt{n}(k+1)^2}{\sqrt{1-y^2}} \max_{-1\le x\le 1}|p(x)| \quad (-1<y<1)$$

for every $p\in S_n^k(0,1)$, where c is an absolute constant.

4. A new proof of Borwein's theorem

In this section we give a short proof of Theorem 1. Let $\Lambda = \{\lambda_j\}_{j=1}^N$ be an increasing set of positive numbers. Denote by $\pi(\Lambda)$ the collection of Λ polynomials of the form

$$p(x) = a_0 + \sum_{j=1}^N a_j x^{\lambda_j} \qquad (0\le x<\infty) \tag{3}$$

with real coefficients a_j. We will use the following Markov-type theorem for the derivative of Λ polynomials.

Theorem 5 (D. Newman). *For every Λ polynomial p of type* (3), *we have*

$$\frac{2}{3}\sum_{j=1}^N \lambda_j \le \sup_{\pi(\Lambda)} \frac{\max_{0\le x\le 1}|p'(x)x|}{\max_{0\le x\le 1}|p(x)|} \le 11\sum_{j=1}^N \lambda_j .$$

The proof of Theorem 5 may be found in [8]. As a straightforward consequence of Theorem 5, we obtain the following:

Proposition. *Let $p(x) = x^{n-k}Q_k(x)$, where $Q_k\in\pi_k$. Then*

$$|p'(1)| \le 11n(k+1)\max_{0\le x\le 1}|p(x)|.$$

Proof of the proposition. By Theorem 5 we have

$$|p'(1)| \le 11 \left(\sum_{j=n-k}^{n} j \right) \max_{0\le x\le 1} |p(x)| \le 11n(k+1) \max_{0\le x\le 1} |p(x)| . \quad \square$$

Now let $p \in S_n^k(1/2, 1/2)$. By an observation of G. G. Lorentz [9], we have $p = wQ_k$, where $Q_k \in \pi_k$ and

$$w(x) = \sum_{j=0}^{n-k} a_j (1-x)^j x^{n-k-j}, \quad \text{with all } a_j \ge 0 .$$

We may assume that $n - k \ge 1$; otherwise, (1) gives Theorem 1. Using the Proposition and $a_j \ge 0$ $(0 \le j \le n-k)$, we obtain

$$\begin{aligned}
|p'(1)| &= |(a_0 x^{n-k} Q_k(x))'(1) + (a_1(1-x)x^{n-k-1} Q_k(x))'(1)| \\
&= |(x^{n-k-1}(a_0 x Q_k(x) + a_1(1-x)Q_k(x)))'(1)| \\
&\le 11n(k+2) \max_{0\le x\le 1} \left| \sum_{j=0}^{1} a_j (1-x)^j x^{n-k-j} Q_k(x) \right| \\
&\le 11n(k+2) \max_{0\le x\le 1} \left| \sum_{j=0}^{n-k} a_j (1-x)^j x^{n-k-j} Q_k(x) \right| \\
&= 11n(k+2) \max_{0\le x\le 1} |p(x)| .
\end{aligned} \tag{4}$$

Now let $y \in [0, 1]$ be arbitrary. To estimate $|p'(y)|$, we may assume that $1/2 \le y \le 1$; otherwise, $P(x) = p(1-x) \in S_n^k(1/2, 1/2)$ can be studied. If $p \in S_n^k(1/2, 1/2)$, then $p \in S_n^k(y/2, y/2)$; hence, by a linear transformation, (4) yields

$$|p'(y)| \le \frac{11}{y} n(k+2) \max_{0\le x\le y} |p(x)| \le 22n(k+2) \max_{0\le x\le y} |p(x)| ,$$

which finishes the proof of Borwein's Theorem. $\square$

Remark 2. Observe that $p \in S_n^k(1/2, 1/2)$ does not necessarily imply $p' \in S_n^k(1/2, 1/2)$; therefore, the generalization of Borwein's inequality for higher derivatives does not follow immediately from the case of the first derivative. Nevertheless, the inequality

$$\max_{0\le x\le 1} |p^{(m)}(x)| \le c(m)(n(k+1))^m \max_{0\le x\le 1} |p(x)|$$

can be proved about as briefly, relying only on a Lorentz representation of $p \in S_n^k(1/2, 1/2)$ and Newman's inequality.

5. Lemmas for Theorem 3

It is sufficient to prove Theorem 3 when $y = 0$, since from this we will obtain the statement in the general case by a linear transformation. By our first lemma, we introduce an extremal polynomial $Q \in S_n^k(0, 1)$.

Lemma 1. *For every* n *and* k $(0 \le k \le n)$ *natural numbers, there exists a polynomial* $Q \in S_n^k(0, 1)$ *such that*

$$\frac{|Q'(0)|}{\max_{-1\le x\le 1} |Q(x)|} = \sup_{p\in S_n^k(0,1)} \frac{|p'(0)|}{\max_{-1\le x\le 1} |p(x)|}.$$

The next lemma gives some information on the zeros of the extremal polynomial Q introduced by Lemma 1.

Lemma 2. *Let* Q *be defined by Lemma* 1. *Then* Q *has only real zeros, and at most* $k+1$ *of them are different from* ± 1 *(counting multiplicities).*

From Theorem 1 we will easily deduce

Lemma 3. *Let* $\delta = (36n(k+1))^{-1}$. *Then*

$$\max_{-\delta\le x\le 1} |q(x)| < 2 \max_{0\le x\le 1} |q(x)|$$

for every $q \in S_n^k(1/2, 1/2)$ *having all its zeros in* $[0, \infty)$.

From Lemma 3 we will easily obtain

Lemma 4. *Let* $z_0 = i(36n(k+1))^{-1/2}$ *where* i *is the imaginary unit. Then*

$$|p(z_0)| < \sqrt{2} \max_{-1\le x\le 1} |p(x)|$$

for every $p \in S_n^k(0, 1)$ *having only real zeros.*

Remark 3. The inequalities of Lemmas 3 and 4 can be proved for all $p \in S_n^k(1/2, 1/2)$ and $p \in S_n^k(0, 1)$, respectively, but under our additional assumptions their proofs are much shorter.

We will prove Theorem 3 by Cauchy's integral formula and these lemmas.

6. Lemmas for Theorem 4

The way to prove Theorem 4 is very similar to the previous section. We introduce an extremal polynomial by

Lemma 5. *For every* n *natural and* $0 < r \le 1$ *real numbers, there exists a polynomial* $Q \in S_n^0(0, r)$ $(0 < r \le 1)$ *such that*

$$\frac{|Q'(0)|}{\max_{-1\le x\le 1} |Q(x)|} = \sup_{p\in S_n^0(0,r)} \frac{|p'(0)|}{\max_{-1\le x\le 1} |p(x)|}.$$

By a variational method we will obtain

Lemma 6. *Let* Q *be defined by Lemma* 5. *Then* Q *has only real zeros.*

From Theorem 2 we will easily prove

Lemma 7. *Let* $0 < R \le 1$ *and* $\delta = \sqrt{R}/(8n)$. *Then*

$$\max_{-\delta\le x\le 1} |q(x)| < 2 \max_{0\le x\le 1} |q(x)|$$

for every $q \in S_n^0(0, R)$ *having all its zeros in* $[R, \infty)$.

Our last lemma will be a straightforward consequence of Lemma 7.

Lemma 8. *Let* $0 < r \le 1$ *and* $z_0 = i\sqrt{r/(8n)}$, *where* i *is the imaginary unit. Then*

$$|p(z_0)| < \sqrt{2} \max_{-1\le x\le 1} |p(x)|$$

for every $p \in S_n^0(0, r)$ *having only real zeros.*

Remark 4. The inequalities of Lemmas 7 and 8 can be verified for all $p \in S_n^0(0, R)$ $(0 < R \le 1)$ and $p \in S_n^0(r)$ $(0 < r \le 1)$, respectively, but under our additional assumptions their proofs are simpler.

We will prove Theorem 4 by Cauchy's integral formula, similarly to the proof of Theorem 3.

7. A Bernstein-Walsh type problem for polynomials from $S_n^k(0, 1)$

We would like some information on the magnitude of $|p(z)|$ $(z \in \mathbb{C})$ when $p \in S_n^k(0, 1)$ and $\max_{-1\le x\le 1} |p(x)| = 1$.

Conjecture 3. Let D be the ellipse of the complex plane with axes $[-a, a]$ and $[-b, b]$, where $a = 1+(n(k+1))^{-1}$ and $b = i(n(k+1))^{-1/2}$ (i is the imaginary unit). Then there is an absolute constant c such that

$$|p(z)| \le c \max_{-1\le x\le 1} |p(x)| \quad (z \in D)$$

for every $p \in S_n^k(0, 1)$ $(0 \le k \le n)$.

Conjecture 1 could be obtained immediately from Conjecture 3 and Cauchy's integral formula.

8. Proofs of the lemmas for Theorem 3

The proof of Lemma 1 is a straightforward application of Hurwitz's Theorem.

Proof of Lemma 2. Assume indirectly that there are at least two zeros of Q outside the open unit disk and different from ± 1. If z_1 is a nonreal zero of Q, then the polynomial

$$Q_\varepsilon(x) = Q(x) - \varepsilon \frac{Q(x)x^2}{(x-z_1)(x-\overline{z}_1)} \in S_n^k(0, 1)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of Q. If a and b are real zeros of Q such that $|a| \ge |b| > 1$, then the polynomial

$$Q_\varepsilon(x) = Q(x) - \varepsilon\, \mathrm{sgn}(ab) \frac{Q(x)x^2}{(x-a)(x-b)} \in S_n^k(0, 1)$$

with a sufficiently small $\varepsilon > 0$ contradicts the maximality of Q. Thus Lemma 2 is proved. □

Lemma 3 follows immediately from Theorem 1 and the Mean Value Theorem.

Proof of Lemma 3. Assume indirectly that there is a $-\delta \leq y < 0$ such that

$$|q(y)| = 2 \max_{0 \leq x \leq 1} |q(x)| \tag{5}$$

for a polynomial $q \in S_n^k(1/2, 1/2)$ having all its zeros in $[0, \infty)$. Then using (5), $-\delta \leq y < 0$, $\delta = (36n(k+1))^{-1}$, the Mean Value Theorem, the monotonicity of $|q'|$ on $(-\infty, 0]$, and Theorem 1 transformed linearly to the interval $[y, 1]$, we can find a $\xi \in (y, 0)$ such that

$$\begin{aligned} 36n(k+1) \max_{0 \leq x \leq 1} |q(x)| &\leq \left| \frac{q(y) - q(0)}{y} \right| = |q'(\xi)| \leq |q'(y)| \\ &\leq \frac{18}{1-y} n(k+1) \max_{y \leq x \leq 1} |q(x)| \\ &< 18n(k+1) \max_{y \leq x \leq 1} |q(x)| \\ &= 36n(k+1) \max_{0 \leq x \leq 1} |q(x)|, \end{aligned}$$

a contradiction. □

Proof of Lemma 4. We may assume that $p \in S_n^k(0, 1)$ is monic; thus, let $p(x) = \prod_{j=1}^s (x - u_j)$ with some $s \leq n$. Applying Lemma 3 to the polynomial

$$q(x) = \prod_{j=1}^s (x - u_j^2) \in S_n^k \left(\frac{1}{2}, \frac{1}{2} \right),$$

we easily deduce that

$$\begin{aligned} |q(-(36n(k+1))^{-1})| &< 2 \max_{0 \leq x \leq 1} |q(x)| = 2 \max_{0 \leq x \leq 1} |q(x^2)| \\ &= 2 \max_{-1 \leq x \leq 1} |p(x)p(-x)| \leq 2 \left(\max_{-1 \leq x \leq 1} |p(x)| \right)^2 . \end{aligned} \tag{6}$$

Observe that

$$|p(z_0)|^2 = \prod_{j=1}^s (u_j^2 + (36n(k+1))^{-1}) = |q(-(36n(k+1))^{-1})|, \tag{7}$$

and together with (6) this yields

$$|p(z_0)| < \sqrt{2} \max_{-1 \leq x \leq 1} |p(x)|.$$

Thus Lemma 4 is proved. □

9. Proofs of lemmas for Theorem 4

The proofs of Lemmas 5–8 are very similar to the corresponding ones from §8. The proofs of Lemmas 5 and 6 are exactly the same as those of Lemmas 1 and 2; therefore, we do not give any details.

Proof of Lemma 7. We have

$$|q'(0)| \le \frac{4n}{\sqrt{R}} \max_{0\le x\le 1} |q(x)| \tag{8}$$

for every polynomial $q \in \pi_n$ having all its zeros in $[R, \infty)$. This inequality follows from the proof of Theorem 1 in [2] with a certain multiplicative constant c instead of 4. The fact that $c = 4$ can be chosen was pointed out by M. v. Golitschek and G. G. Lorentz. Now assume indirectly that there exists a $-\delta \le y < 0$ such that

$$|q(y)| = 2 \max_{0\le x\le 1} |q(x)| \tag{9}$$

for a polynomial $q \in \pi_n$ having all its zeros in $[R, \infty)$. Then (9), $-\delta \le y < 0$, $\delta = \sqrt{R}/(8n)$, the Mean Value Theorem, the monotonicity of $|q'|$ on $(-\infty, 0]$, and (8) transformed linearly to the interval $[y, 1]$ imply that there exists a $\xi \in (y, 0)$ such that

$$\begin{aligned} \frac{8n}{\sqrt{R}} \max_{0\le x\le 1} |q(x)| &\le \left|\frac{q(y)-q(0)}{y}\right| = |q'(\xi)| \le |q'(y)| \\ &\le \frac{4n}{(1-y)\sqrt{\frac{R-y}{1-y}}} \max_{y\le x\le 1} |q(x)| \\ &< \frac{4n}{\sqrt{R}} \max_{y\le x\le 1} |q(x)| = \frac{8n}{\sqrt{R}} \max_{0\le x\le 1} |q(x)|, \end{aligned}$$

a contradiction. □

Proof of Lemma 8. We may assume that $p \in S_n^0(0, r)$ is monic; thus, let $p(x) = \prod_{j=1}^s (x - u_j)$ with some $s \le n$. Applying Lemma 7 to the polynomial

$$q(x) = \prod_{j=1}^{s} (x - u_j^2) \in S_n^0(0, r^2),$$

we easily deduce that

$$\begin{aligned} \left| q\left(-\frac{\sqrt{r^2}}{8n}\right)\right| &< 2 \max_{0\le x\le 1} |q(x)| = 2 \max_{0\le x\le 1} |q(x^2)| \\ &= 2 \max_{-1\le x\le 1} |p(x)p(-x)| \le 2 \left(\max_{-1\le x\le 1} |p(x)|\right)^2 . \end{aligned} \tag{10}$$

Further,

$$|p(z_0)|^2 = \prod_{j=1}^{s} \left(u_j^2 + \frac{r}{8n}\right) = \left|q\left(-\frac{r}{8n}\right)\right|, \tag{11}$$

and together with (10) this yields

$$|p(z_0)| < \sqrt{2} \max_{-1\le x\le 1} |p(x)|.$$

Thus Lemma 8 is proved. □

10. Proofs of Theorems 3 and 4

Proof of Theorem 3. Let

$$D = \left\{ z \in \mathbb{C} : |\operatorname{Re} z| \le \frac{1}{2}, |\operatorname{Im} z| \le \frac{1}{12}(n(k+1))^{-1/2} \right\}. \tag{12}$$

Since $Q \in S_n^k(0, 1)$ defined by Lemma 1 has only real zeros (see Lemma 2), from Lemma 4, by a linear transformation, we obtain

$$|Q(z)| \le \sqrt{2} \max_{a \le x \le b} |Q(x)| \le \sqrt{2} \max_{-1 \le x \le 1} |Q(x)| \quad (z \in D), \tag{13}$$

where, with the notation $y = \operatorname{Re} z$, $[a, b] = [-1, 1+2y]$ if $-\frac{1}{2} \le y \le 0$, and $[a, b] = [2y-1, 1]$ if $0 \le y \le \frac{1}{2}$. Now let S be the circle with center 0 and radius $(n(k+1))^{-1/2}/12$. By Cauchy's integral formula, (12), and (13), we obtain

$$\begin{aligned} |Q'(0)| &= \frac{1}{2}\left| \int_S \frac{Q(\xi)}{\xi^2}\, d\xi \right| \le \frac{1}{2} \int_S \left| \frac{Q(\xi)}{\xi^2} \right| |d\xi| \\ &\le \pi \frac{1}{12}(n(k+1))^{-1/2} (12(n(k+1))^{1/2})^2 \sqrt{2} \max_{-1 \le x \le 1} |Q(x)| \\ &\le c\sqrt{n(k+1)} \max_{-1 \le x \le 1} |Q(x)|. \end{aligned}$$

From this and the maximality of Q, we deduce

$$|p'(0)| \le c\sqrt{n(k+1)} \max_{-1 \le x \le 1} |p(x)| \quad (p \in S_n^k(0, 1)),$$

and from here we obtain the desired result in the general case $-1 < y < 1$ by a linear transformation. Thus Theorem 3 is proved. □

Proof of Theorem 4. We proceed in a very similar way. Let

$$D = \left\{ z \in \mathbb{C} : |\operatorname{Re} z| \le \frac{r}{2}, |\operatorname{Im} z| \le \sqrt{\frac{r}{32n}} \right\}. \tag{14}$$

Since $Q \in S_n^0(0, r)$ defined by Lemma 5 has only real zeros (see Lemma 6), from Lemma 8, by a linear transformation, we deduce

$$|Q(z)| \le \sqrt{2} \max_{-1 \le x \le 1} |Q(x)| \quad (z \in D). \tag{15}$$

Let S be the circle with center 0 and radius $\sqrt{r/(32n)}$. Since $r \ge 1/n$, we have $\sqrt{r/(32n)} \le r/2$; hence, by Cauchy's integral formula, (14), and (15), we get

$$\begin{aligned} |Q'(0)| &= \frac{1}{2}\left| \int_S \frac{Q(\xi)}{\xi^2}\, d\xi \right| \le \frac{1}{2} \int_S \left| \frac{Q(\xi)}{\xi^2} \right| |d\xi| \\ &\le \pi \sqrt{\frac{r}{32n}} \frac{32n}{r} \sqrt{2} \max_{-1 \le x \le 1} |Q(x)| \le c\sqrt{\frac{n}{r}} \max_{-1 \le x \le 1} |Q(x)|. \end{aligned} \tag{16}$$

Therefore the maximality of Q yields

$$|p'(0)| \le c\sqrt{\frac{n}{r}} \max_{-1\le x\le 1} |p(x)| \quad (p \in S_n^0(0, r)). \tag{17}$$

Now observe that $p \in W_n^0(r)$ implies $p \in S_n^0(y, r)$ if $y \in [-r, r]$, and $p \in S_n^0(y, 1-y)$ if $r \le |y| < 1$. Applying (17) transformed linearly to the interval $[2y-1, 1]$ when $0 \le y \le 1$ and $[-1, 2y+1]$ when $-1 \le y < 0$, we obtain (i) and (ii) immediately. Thus Theorem 4 is proved. □

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