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THE STABLE RANK OF CROSSED PRODUCTS OF SECTIONAL C*-ALGEBRAS BY COMPACT LIE GROUPS

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ABSTRACT. Let X be a second countable, locally compact Hausdorff space, and let a compact Lie group D act on a C*-bundle (E, X), the fibres of which are of bounded, finite dimension. Denote by A the C*-algebra of sections vanishing at infinity, and by $\hat{\alpha}$ the induced action of D on A. The stable ranks of both, the fixed point algebra $A^{\hat{\alpha}}$ and the crossed product algebra $A \times_{\hat{\alpha}} D$ are determined.

1. INTRODUCTION

In this paper, we study the stable rank of fixed point algebras and C*-crossed products arising from actions of compact Lie groups on commutative or "nearly" commutative C*-algebras. In particular, let (E, X) be a C*-bundle over the second countable, locally compact Hausdorff space X, and assume that all fibres E_x are finite dimensional, $\dim(E_x) \le M$ for some fixed M and all $x \in X$. Let D be a compact group acting on (E, X), that is, there exist actions α_d and φ_d of D on E and X respectively, such that the following diagram

$$\begin{array}{ccc} E & \xrightarrow{\alpha_d} & E \\ \downarrow & & \downarrow \\ X & \xrightarrow{\varphi_d} & X \end{array}$$

commutes, and α_d is a C*-isomorphism on fibres; call such a bundle a $D-C^*$-bundle. Set $\Gamma_o(E) = \Gamma_o(E, X)$, the C*-algebra of continuous sections vanishing at infinity, and let $\Gamma_o^D(E) = \{e \in \Gamma_o(E) : e(\varphi_d(x)) = \alpha_d(e(x)) \text{ for all } d \in D, x \in X\}$, the algebra of equivariant sections. There is an induced action $\hat{\alpha}$ of D on $\Gamma_o(E)$ given by $[\hat{\alpha}_d(e)](x) = \alpha_d(e(\varphi_{d^{-1}}(x)))$, and $\Gamma_o^D(E)$ is the fixed point algebra of $\Gamma_o(E)$ under this action, which by Evans' tilde construction [Ev] can be represented as the algebra $\Gamma_o(\widetilde{E})$ of sections of a new C*-bundle $(\widetilde{E}, \widetilde{X})$.

Nistor [Ni] has given an explicit formula for the computation of the stable rank of certain separable type I C*-algebras: If the C*-algebra A has a finite

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composition series $\{0\} = I_0 \subset I_1 \subset \cdots \subset I_{n+1} = A$ such that each subquotient is of the form $I_{k+1}/I_k \cong c_o - \sum_{m\in\mathbf{N}} \Gamma_o(E_m^k, X_m^k)$, where each (E_m^k, X_m^k) is a locally trivial homogeneous C*-bundle with fibres isomorphic to $M_m(\mathbf{C})$, then

$$\operatorname{sr}(I_{k+1}/I_k) = \sup_m \left\lceil \frac{\dim(X_m^k) - 1}{2\,m} \right\rceil + 1 \quad \text{and} \quad \operatorname{sr}(A) = \max_{0\le k\le n} \operatorname{sr}(I_{k+1}/I_k).$$

Here $\dim(X)$ denotes the covering dimension of the space X, and $\lceil x\rceil$ denotes the least integer $\geq x$. ($\dim(X) = \infty$ is permitted in an obvious way.)

This, in combination with Evans' results, allows us to determine $\operatorname{sr}(\Gamma_o^D(E))$ for an action of a compact Lie group D on a C*-bundle (E, X) as above, and to obtain that $\operatorname{sr}\Gamma_o(E) \times_{\hat\alpha} D \leq \operatorname{sr}\Gamma_o(E)$, provided that X has locally finite orbit structure.

2. Some facts and lemmas

It will be useful to review Evans' tilde construction and state some lemmas needed in the main proofs. Given an action of the compact group D on the C*-bundle (E, X) by pairs (α_d, φ_d), let D_x denote the stabilizer of $x \in X$, and let F_x be the fixed point algebra of the fibre E_x under the restriction of α to D_x. Put $F = \bigcup_{x\in X} F_x$ and give it the subspace topology of E. Define $\widetilde{X} = X/D$ and $\widetilde{E} = F/D$ and give them the quotient topologies. Then $(\widetilde{E}, \widetilde{X})$ is a C*-bundle, the map $F \to \widetilde{E}$ is an isomorphism when restricted to a fibre F_x, and the quotient map $\pi : X \to \widetilde{X}$ is open and closed. If, as will be assumed throughout this paper, X is locally compact, second countable and Hausdorff, then $\widetilde{X}$ inherits these properties, and both spaces are σ-compact, separable and metrizable. If Y is a locally compact D-invariant subset of X, then $\widetilde{E}_{|\widetilde{Y}} \cong \widetilde{E_{|Y}}$.

The following lemma clarifies part (b) of Nistor's Theorem 7 in our C*-bundle case.

Lemma 2.1. *Let $\{X_j\}_{j=1}^t$ be a partition of X such that $\bigcup_{l\geq j} X_l$ is closed in X for all j. Then,*

$$\operatorname{sr}(\Gamma_o(E)) = \max_j \operatorname{sr}(\Gamma_o(E_{|X_j})).$$

Proof. Proceeding by induction, we may assume that $t = 2$. Pick a sequence $\{F_n\}$ of compact subsets of X_1 such that $F_n \subset \overset{\circ}{F}_{n+1}$ and $X_1 = \bigcup_{n=1}^\infty F_n$. Then by [She, Proposition 3.15],

$$\begin{aligned}\operatorname{sr}(\Gamma_o(E)) &\leq \sup_n\{\operatorname{sr}(\Gamma_o(E_{|X_2})), \operatorname{sr}(\Gamma_o(E_{|F_n}))\} \\ &\leq \max\{\operatorname{sr}(\Gamma_o(E_{|X_2})), \operatorname{sr}(\Gamma_o(E_{|X_1|}))\} \leq \operatorname{sr}(\Gamma_o(E)).\end{aligned}$$

Recall that as a consequence of [Fe, Theorem 3.1], homogeneous C*-bundles with finite dimensional fibres are locally trivial. For convenience of notation, let us denote by M_∞ the algebra $\mathscr{K}(\mathscr{H})$ of compact operators on the infinite dimensional, separable Hilbert space $\mathscr{H}$.

Lemma 2.2. *Let (E, X) be a locally trivial $D - C^*$-bundle with $E_x \subset M_\infty$ for all $x \in X$. Let $\{[A_i]\}_{i=1}^\infty$ denote the collection of isomorphism classes of fibres of E and set $X_i = \{x \in X : E_x \in [A_i]\}$. Then,*

(i) $\{X_i\}_{i=1}^\infty$ *is a collection of mutually disjoint clopen subsets covering X.*
(ii) *The bundle $E_{|X_i}$ obtained by restriction is D-invariant and*

$$\Gamma_o^D(E) \cong c_o - \sum_{i=1}^\infty \Gamma_o^D(E_{|X_i}).$$

Proof. (i) is obvious by local triviality.

(ii) By (i), E decomposes into the disjoint union of clopen sets, $E = \bigcup_{i=1}^\infty E_{|X_i}$. Because D acts as an isomorphism on fibres, each $E_{|X_i}$ is D-invariant. One checks easily that the map

$$\phi : \begin{cases} \Gamma_o^D(E) \to c_o - \sum_{i=1}^\infty \Gamma_o^D(E_{|X_i}) \\ e \to \sum_{i=1}^\infty e_{|X_i}, \quad e \in \Gamma_o^D(E) \end{cases}$$

is a C*-isomorphism.

Now let A be a fixed C*-algebra of the form $A = c_o - \sum_{k \in \mathbf{N}} n_k \circ M_k$, where $0 \le n_k \le \aleph_0$ and $n_k \circ M_k = c_o - \sum_{i=1}^{n_k} M_k$.

Proposition 2.3. *Let (E, X) be a locally trivial, homogeneous $D - C^*$-bundle whose fibres are isomorphic to A. Then there exist (locally compact, second countable Hausdorff) spaces Y^k and $D - C^*$-bundles (F^k, Y^k) such that*

(i) *each F^k is homogeneous, locally trivial with fibres M_k,*
(ii) $\Gamma_o(E) = c_o - \sum_k \Gamma_o(F^k)$ *and* $\Gamma_o^D(E) = c_o - \sum_k \Gamma_o^D(F^k)$,
(iii) $\dim(Y^k) = \dim(X)$ *for all k.*

Proof. Let $\bar{F}_x^k$ be the subalgebra of the fibre E_x isomorphic to $n_k \circ M_k$, and set $\bar{F}^k = \bigcup_{x \in X} \bar{F}_x^k$. Then $(\bar{F}^k, X)$ is a locally trivial subbundle of (E, X), and it is D-invariant as D acts by isomorphism on fibres.

Let e_x^k denote the projection of E_x onto $\bar{F}_x^k$. A partition of unity argument and [Le, Lemma 2] show that $e^k : x \to e_x^k$ is a section in the multiplier algebra $\mathscr{M}(\Gamma_o(E))$ which also is D-invariant under the extension of $\hat{\alpha}$ to $\mathscr{M}(\Gamma_o(E))$. Then, $\Gamma_o(\bar{F}^k) = e^k \Gamma_o(E)$, $\Gamma_o^D(\bar{F}^k) = e^k \Gamma_o^D(E)$, and since $\{e^k\}_{k=1}^\infty$ is a decomposition of the identity of $\mathscr{M}(\Gamma_o(E))$ into mutually orthogonal central projections, one obtains that $\Gamma_o(E) = c_o - \sum_k \Gamma_o(\bar{F}^k)$ and $\Gamma_o^D(E) = c_o - \sum_k \Gamma_o^D(\bar{F}^k)$.

Now set $Y^k = \Gamma_o(\bar{F}^k)^\wedge$. By [Fe, Theorem 1.1], Y^k can be identified as a set with $X \times (1, \dots, n_k)$ for all k. Pick an open cover $\{U_i\}_{i=1}^\infty$ of X such that $\bar{U}_i$ is compact and $E_{|U_i}$ is trivial. Then $\Gamma_o(\bar{F}_{|U_i}^k)^\wedge$ is homeomorphic to the product space $U_i \times (1, \dots, n_k)$, and because $\Gamma_o(\bar{F}^k_{|U_i})$ is a closed ideal in $\Gamma_o(\bar{F}^k)$, $U_i \times (1, \dots, n_k)$ is open in Y^k. Thus, $\{U_i \times \{s\}\}_{i=1,s=1}^{\infty \ \ n_k}$ is an open cover of Y^k, in particular Y^k is locally compact, second countable Hausdorff. By the sum theorem, $\dim(Y^k) = \sup_{i,s} \dim(\bar{U}_i \times \{s\}) = \sup_i \dim(\bar{U}_i) = \dim(X)$. Now it is known ([Fe, Theorem 3.2]) that the algebra of sections $\Gamma_o(\bar{F}^k)$ of the homogeneous bundle $(\bar{F}^k, X)$ is isomorphic to the algebra $\Gamma_o(F^k)$ of sections of a bundle (F^k, Y^k) with fibres M_k. Since this isomorphism maps $\Gamma_o(\bar{F}^k_{|U_i}) \cong C_o(U_i) \otimes (c_o - \sum_{i=1}^{n_k} M_k) \cong C_o(U_i \times (1, \dots, n_k)) \otimes M_k$ onto $\Gamma_o(F^k_{|U_i \times (1, \dots, n_k)})$, the bundle (F^k, Y^k) is locally trivial. Here $C_o(X)$ denotes the set of continuous functions on a locally compact space X that vanish at infinity.

Suppose now, the bundle automorphism (α_d, φ_d) maps the element e_i in the ith component of F_x^k to the element f_j in the jth component of $F^k_{\varphi_d(x)}$. Define

$$\begin{cases} \gamma_d : e_i \in M_k \to f_j \in M_k \\ \psi_d : (x, i) \in Y^k \to (\varphi_d(x), j) \in Y^k. \end{cases}$$

One easily checks that (γ_d, ψ_d) defines an action of D on (F^k, Y^k) and that the above isomorphism maps $\Gamma_o^D(\bar{F}^k)$ onto $\Gamma_o^D(F^k)$.

Let D be a second countable compact group, L a closed subgroup, and denote by m_D and m_L the normalized Haar measures on D and L, respectively. If $\pi : D \to D/L$ denotes the quotient map, then we have an invariant regular Borel measure μ on D/L defined by $\mu(Q) = m_D(\pi^{-1}(Q))$ on Borel sets $Q \subset D/L$. Fix a Borel cross-section $\gamma : D/L \to D$. Then by the uniqueness of the Haar measure, $\int_D f(d)\, dm_D(d) = \int_{D/L} \int_L f(\gamma(q)l)\, dm_L(l)\, d\mu(q)$ for all $f \in C(D)$, the set of continuous functions on D. (See [Lo], §33.)

Lemma 2.4. *The* C^**-dynamical systems*

$\left(\mathscr{K}(L^2(D)), \operatorname{Ad}\rho(D)_{|L}, L\right)$ *and* $\left(\mathscr{K}(L^2(L)) \otimes \mathscr{K}(L^2(D/L)), \operatorname{Ad}\tilde{\rho}(L) \otimes id, L\right)$

are covariantly isomorphic. Here ρ *and* $\tilde{\rho}$ *denote the right regular representations of* D *and* L*, respectively.*

Proof. First identify $L^2(D/L) \otimes L^2(L)$ with $L^2(D/L, L^2(L))$. On elementary tensors $p \otimes f \in L^2(D/L) \otimes L^2(L)$, the identification is given by $[(p \otimes f)(q)](l) = p(q)f(l)$ for $q \in D/L$, $l \in L$. Now define

$$U : L^2(D) \to L^2(D/L, L^2(L))$$

by

$$[(Uf)(q)](l) = f(\gamma(q)l)$$

for $f \in L^2(D)$. As

$$||Uf||^2 = \int_{D/L} \int_L |f(\gamma(q)l)|^2\, dm_L(l)\, d\mu(q) = \int_D |f(d)|^2\, dm_D(d) = ||f||^2,$$

it follows that U is an isomorphism of the two Hilbert spaces.

Note that for $q \in D/L$ and $d, l \in L$,

$$[(U\rho_d U^*)(p \otimes f)](q)(l) = (p \otimes f)(q)(ld) = p(q)f(ld) = (p \otimes \widetilde{\rho}_d f)(q)(l),$$

so that by linearity and continuity,

$$U\rho_d U^* = id \otimes \widetilde{\rho}_d$$

and thus,

$$\mathrm{Ad}\, U \ \mathrm{Ad}\, \rho_d \ \mathrm{Ad}\, U^* = id \otimes \mathrm{Ad}\, \widetilde{\rho}_d.$$

Two actions α and β of a locally compact group D on a C*-algebra A are *exterior equivalent*, if there exists a strictly continuous map $d \to u_d$ of D into the unitary group $\mathscr{U}(M(A))$ such that

$$\alpha_d(a) = u_d \beta_d(a) u_d^* \qquad d \in D, \ a \in A$$

and

$$u_{cd} = u_c \beta_c(u_d) \qquad c, d \in D,$$

where β is extended to the multiplier algebra $\mathscr{M}(A)$ in a unique way. Exterior equivalent actions give rise to isomorphic crossed products.

Given a separable Hilbert space $\mathscr{H}$, identify $C_o(X, \mathscr{K}(\mathscr{H}))$ with the algebra $\Gamma_o(\mathscr{K}(\mathscr{H}) \times X)$ of sections of the trivial bundle. Its multiplier algebra is isomorphic to the set $C^b(X, \mathscr{B}(\mathscr{H}))$ of bounded, strictly (and hence strong-$*$) continuous functions from X to $\mathscr{B}(\mathscr{H})$. There is a one-to-one correspondence between bundle actions of the form (α, id) and center-fixing actions $\hat{\alpha}$ on $C_o(X, \mathscr{K}(\mathscr{H}))$. Denote by α^x the restriction of α to the fibre $\mathscr{K}(\mathscr{H}) \times \{x\}$.

The following result is probably well known, although I have found no reference.

Proposition 2.5. *Let $\hat{\alpha}$ and $\hat{\beta}$ be center-fixing actions of a second countable compact group D on $C_o(X, \mathscr{K}(\mathscr{H}))$, and assume that for some $x_o \in X$, the actions α^{x_o} and β^{x_o} are exterior equivalent. Then there exists a neighborhood V of x_o such that the restrictions of $\hat{\alpha}$ and $\hat{\beta}$ to $C_o(V, \mathscr{K}(\mathscr{H}))$ are exterior equivalent.*

Proof. As in the proof of [PhR, Proposition 1.3], there exist a compact neighborhood V of x_o and Borel maps $v, w : D \to C^b(V, \mathscr{B}(\mathscr{H}))$ such that

$$\hat{\alpha}_d(a) = v_d a v_d^* \quad \text{and} \quad \hat{\beta}_d(a) = w_d a w_d^*$$

for $a \in C(V, \mathscr{K}(\mathscr{H}))$ and $d \in D$. Let $u : D \to \mathscr{U}(\mathscr{H})$ be the unitary 1-cocycle implementing the exterior equivalence of α^{x_o} and β^{x_o}. Since $u_d w_d(x_o)$ and $v_d(x_o)$ differ only by a scalar, we may assume w.l.o.g. that $v_d(x_o) = u_d w_d(x_o)$. Then

$$\rho(c, d) = v_c v_d v_{cd}^* \quad \text{and} \quad \psi(c, d) = w_c w_d w_{cd}^*$$

define Borel cocycles $D \times D \to C(V, \mathbf{T})$ such that

$$\begin{aligned}\rho(c, d)(x_o) &= u_c w_c(x_o) u_d w_d(x_o) w_{cd}^*(x_o) u_{cd}^* \\ &= u_{cd} w_c(x_o) w_d(x_o) w_{cd}^*(x_o) u_{cd}^* = \psi(c, d)(x_o)\end{aligned}$$

so that the 2-cocycle $\sigma = \rho\overline{\psi}$ satisfies $\sigma(c, d)(x_o) = 1$ for all $c, d \in D$. The map $y \in V \to \sigma(\cdot, \cdot)(y)$ gives a family of cocycles $D \times D \to \mathbf{T}$ varying continuously in y, and as $H^2(D, \mathbf{T})$ is discrete by compactness of D, $\sigma(\cdot, \cdot)(y)$ is a coboundary for y close to x_o. Cutting down on V if necessary, we may by [Ro, Theorem 2.1] assume that σ is a coboundary, that is, there exists a Borel map $\lambda : D \to C(V, \mathbf{T})$ such that $\sigma(c, d) = \lambda(c)\lambda(d)\overline{\lambda(cd)}$ for almost all $c, d \in D$. Set $z_d = \overline{\lambda(d)}v_d w_d^*$. Then

$$z_d \hat{\beta}_d(a) z_d^* = v_d w_d^* \hat{\beta}_d(a) w_d v_d^* = v_d a v_d^* = \hat{\alpha}_d(a) \tag{1}$$

and

$$\begin{aligned} z_{cd} &= \overline{\lambda(cd)} v_{cd} w_{cd}^* = \sigma(c, d)\overline{\lambda(c)\lambda(d)\rho(c, d)} v_c v_d \psi(c, d) w_d^* w_c^* \\ &= z_c w_c \overline{\lambda(d)} v_d w_d^* w_c^* = z_c \hat{\beta}_c(z_d) \end{aligned} \tag{2}$$

for almost all $c, d \in D$. Finally, by [Mo, Theorem 3], we may assume that $d \to z_d$ is strictly continuous, so that (1) and (2) hold for all $c, d \in D$.

Rieffel [Ri] has shown that given a compact C*-dynamical system (A, D, α), the crossed product $A \times_\alpha D$ is isomorphic to the fixed point algebra $[A \otimes \mathscr{K}(L^2(D))]^{\alpha \otimes \mathrm{Ad}\,\rho}$, and he has given an identification of $\mathscr{K}(L^2(D))$ with $C(D) \times_\lambda D$ under which the D-action $\mathrm{Ad}\,\rho$ on $\mathscr{K}(L^2(D))$ corresponds to the action γ defined on the dense subset $C(D \times D) \cong C(D, C(D))$ of $C(D) \times_\lambda D$ by $[\gamma_s(f)](c, d) = f(c, ds)$ for $f \in C(D \times D)$, $s, c, d \in D$. Here λ and ρ denote the left and right regular representations of D on $L^2(D)$, and $\mathrm{Ad}\,\rho_d(T) = \rho_d T \rho_{d^{-1}}$ for all $T \in \mathscr{K}(L^2(D))$ and $d \in D$.

Given an action (α_d, φ_d) of the compact group D on the C*-bundle (E, X), let $(\bar{E}, X)$ denote the tensor product bundle, that is the bundle with fibres $\bar{E}_x = E_x \otimes \mathscr{K}(L^2(D))$, and give it an action $(\bar{\alpha}_d, \varphi_d)$ by

$$\bar{\alpha}_d : a \otimes T \in E_x \otimes \mathscr{K}(L^2(D)) \to [\alpha_d \otimes \mathrm{Ad}\,\rho_d](a \otimes T) \in E_{\varphi_d(x)} \otimes \mathscr{K}(L^2(D)). \tag{3}$$

Then $\Gamma_o(E) \otimes \mathscr{K}(L^2(D)) \cong \Gamma_o(\bar{E})$ and $\Gamma_o(E) \times_{\hat{\alpha}} D \cong [\Gamma_o(E) \otimes \mathscr{K}(L^2(D))]^{\hat{\alpha} \otimes \mathrm{Ad}\,\rho} \cong \Gamma_o^D(\bar{E})$.

Corollary 2.6. *Let the second countable compact group D act on the trivial bundle $(E, X) = \mathscr{K}(\mathscr{H}) \times X$ by actions (α_d, id). Then, given $x \in X$, there exists a neighborhood V of x such that $\widetilde{\bar{E}_{|V}}$ is trivial. In particular, $\Gamma_o(E_{|V}) \times_{\hat{\alpha}} D \cong C_o(V) \otimes (\mathscr{K}(\mathscr{H}) \times_{\alpha^x} D)$.*

Proof. Define an action (β, id) of D on (E, X) by $\beta_d^y = \alpha_d^x$ for $y \in X$ and pick V as in the last proposition with $x_o = x$.

Define a bundle isomorphism $(\phi, id) : \bar{E}_{|V} \to \bar{E}_{|V}$ by

$$\phi(a)(c, d) = z_{d^{-1}}(y) a(c, d) z_{d^{-1}c}(y)^*$$

for a an element of the dense subset $C(D \times D, E_y)$ of the fibre $\overline{E}_y$ and $c, d \in D$. Then,

$$\begin{aligned}[(\alpha_s \otimes \gamma_s)\phi(a)](c, d) &= \alpha_s[(\phi(a))(c, ds)] = \alpha_s(z_{(ds)^{-1}}(y)a(c, ds)z_{(ds)^{-1}c}(y)^*) \\ &= z_s(y)\beta_s(z_{(ds)^{-1}}(y)a(c, ds)z_{(ds)^{-1}c}(y)^*)z_s(y)^* \\ &= z_{d^{-1}}(y)\beta_s(a(c, ds))z_{d^{-1}c}(y)^* \\ &= z_{d^{-1}}(y)[((\beta_s \otimes \gamma_s)(a))(c, d)]z_{d^{-1}c}(y)^* \\ &= \varphi((\beta_s \otimes \gamma_s)(a))(c, d).\end{aligned}$$

So ϕ intertwines the two bundle actions, and thus restricts to an isomorphism of the tilde bundles. As the tilde bundle associated with $(\beta_s \otimes \gamma_s)$ is trivial, the statement follows.

3. Compact group actions

Throughout this section, (E, X) will denote a C*-bundle whose fibres are of bounded finite dimension, that is, there exists an integer M such that $\dim(E_x) \leq M$ for all $x \in X$.

Let us first determine the stable rank of $\Gamma_o(E)$: Denote by $[A_1], \dots, [A_r]$ the collection of isomorphism classes of fibres of E, ordered such that $i < k \Rightarrow \dim(A_i) \geq \dim(A_k)$ $(1 \leq i, k \leq r)$. Each A_k is a sum of matrix algebras,

$$A_k = \sum_{q=N_k}^{M_k} n_k^q \circ M_q \qquad (0 \leq n_k^q < \infty), \tag{4}$$

where M_{N_k} denotes the smallest matrix component of A_k. Set $X_i = \{x \in X : E_x \in [A_i]\}$. Because of [Fe, Theorem 3.1], $\bigcup_{i \geq k} X_i$ is closed in X for all k. Setting $I_k = \{f \in \Gamma_o(E) : f \text{ vanishes on } \bigcup_{i \geq k} X_i\} \cong \Gamma_o(E_{|X - \cup_{i \geq k} X_i})$, we obtain a decomposition series of closed ideals

$$\{0\} = I_1 \subset I_2 \subset \cdots \subset I_r \subset I_{r+1} = \Gamma_o(E) \tag{5}$$

with subquotients

$$I_{k+1}/I_k \cong \Gamma_o(E_{|X_k}). \tag{6}$$

By Proposition 2.3, each of I_{k+1}/I_k satisfies condition (A) of Nistor's Theorem 7, so that

$$sr(\Gamma_o(E)) = \max_{1 \leq k \leq r} \left\lceil \frac{\dim(X_k) - 1}{2\, N_k} \right\rceil + 1. \tag{7}$$

Note also that

$$sr(C_o(X)) = \max_{1 \leq k \leq r} \left\lceil \frac{\dim(X_k) - 1}{2} \right\rceil + 1 \tag{8}$$

by the same theorem.

Now let D be a compact group acting on the C*-bundle (E, X). Let $\pi : X \to \widetilde{X} = X/D$ denote the quotient map. We say that π has *local cross-sections* if given $\tilde{x} \in \widetilde{X}$ there exists a neighborhood U of $\tilde{x}$ such that $\pi_{|\pi^{-1}(U)}$ has a cross-section. Note that this implies that $\dim(\widetilde{X}) \leq \dim(X)$ by the sum theorem. If D is compact Lie, then always $\dim(\widetilde{X}) \leq \dim(X)$ (see [Pa], Theorem 3.16). In both cases, $\dim(X) - t \leq \dim(\widetilde{X})$ by a theorem on dimension lowering mappings ([En],1.12.4), where t denotes the largest of the dimensions of the orbits.

Proposition 3.1. *If there exist local cross-sections or if D is a Lie group, then*

$$\sup_{1\leq k\leq r} \{\mathrm{sr}(C_o(\widetilde{X_k}) \otimes M_{N_k})\} \leq \mathrm{sr}(\Gamma_o^D(E)) \leq \mathrm{sr}(C_o(\widetilde{X})) \leq \mathrm{sr}(C_o(X)).$$

Proof. Since the ideals I_k in (5) are D-invariant, $\Gamma_o^D(E) \cong \Gamma_o(\widetilde{E})$ has a composition series whose subquotients are isomorphic to $\Gamma_o^D(E_{|X_k}) \cong \Gamma_o(\widetilde{E}_{|\widetilde{X}_k})$. An application of the above process shows that $\Gamma_o(\widetilde{E}_{|\widetilde{X}_k})$ satisfies Nistor's condition A) and allows us to compute $\mathrm{sr}(\Gamma_o(\widetilde{E}_{|\widetilde{X}_k}))$. Since the fibres of $\widetilde{E}_{|\widetilde{X}_k}$ are subalgebras of A_k $(1 \leq k \leq r)$, we obtain the following estimate:

$$\left\lceil \frac{\dim(\widetilde{X_k}) - 1}{2\, N_k} \right\rceil + 1 \leq \mathrm{sr}(\Gamma_o(\widetilde{E}_{|\widetilde{X}_k})) \leq \left\lceil \frac{\dim(\widetilde{X_k}) - 1}{2} \right\rceil + 1$$

and therefore,

$$\begin{aligned} &\sup_{1\leq k\leq r} \left\lceil \frac{\dim(\widetilde{X_k}) - 1}{2\, N_k} \right\rceil + 1 \leq \mathrm{sr}(\Gamma_o(\widetilde{E})) \\ &\leq \sup_{1\leq k\leq r} \left\lceil \frac{\dim(\widetilde{X_k}) - 1}{2} \right\rceil + 1 = \mathrm{sr}(C_o(\widetilde{X})). \end{aligned} \tag{9}$$

Remark. Note that $\mathrm{sr}(\Gamma_o^D(E))$ may be larger or smaller than $\mathrm{sr}(\Gamma_o(E))$.

(1) If the r-torus $\mathbf{T}^r$ $(r \geq 4)$ acts on the trivial bundle $E = \mathbf{C} \times \mathbf{T}^r$ by $t : (a, x) \to (a, tx)$ $(a \in \mathbf{C},\ t, x \in \mathbf{T}^r)$, then $\Gamma_o(\widetilde{E}) \cong \mathbf{C}$ and $\Gamma_o(E) \cong C(\mathbf{T}^r)$, so that $1 = \mathrm{sr}(\Gamma_o^D(E)) < \mathrm{sr}(\Gamma_o(E)) = \left\lceil \frac{r-1}{2} \right\rceil + 1$.

(2) If the r-torus acts on $E = M_r \times \mathbf{T}^r$ by

$$t = (t_1, \ldots, t_r) : (a, x) \to \left(\left(\sum t_i e_{ii}\right) a \left(\sum \bar{t}_j e_{jj}\right), x\right),$$

then $\Gamma_o(E) \cong C(\mathbf{T}^r, M_r)$ while $\Gamma_o(\widetilde{E}) \cong C(\mathbf{T}^r, \mathbf{C}^r)$, so that $\left\lceil \frac{r-1}{2} \right\rceil + 1 = \mathrm{sr}(\Gamma_o^D(E)) > \mathrm{sr}(\Gamma_o(E)) = 2$.

However, we have the following, which always holds for finite groups:

Corollary 3.2. *If all orbits are finite, then* $\mathrm{sr}(\Gamma_o(E)) \leq \mathrm{sr}(\Gamma_o^D(E)) \leq \mathrm{sr}(C_o(X))$.

Proof. By [En], Theorem 1.12.8, $\dim(\widetilde{X_k}) = \dim(X_k)$ for all k. The assertion then follows from (7) and (9).

Corollary 3.3. *Let G be a second countable, locally compact group and A an abelian normal subgroup of G of finite index, say $D = G/A$ has order n. Then,*

$$\mathrm{sr}(C^*(A) \otimes M_n) \leq \mathrm{sr}(C^*(G)) \leq \mathrm{sr}(C^*(A)).$$

Proof. Taylor [Ta] has shown that $C^*(G) \cong \Gamma_o^D(E)$, where D acts on the trivial bundle $E = M_n \times \hat{A}$. Since $C^*(A) \cong C_o(\hat{A})$, the statement follows immediately from Proposition 3.1.

We say that X has *locally finite orbit structure*, if given $x \in X$ there exists an open neighborhood U of x such that the types of orbits touching U are finite in number. By [Bre], IV.1.2, this always happens if D is compact Lie and X is a topological manifold.

Theorem 3.4. *Let the compact Lie group D act on (E, X) by actions (α_d, φ_d), and assume that X has locally finite orbit structure. Then there exist numbers $l, s \in \mathbf{N} \cup \{\infty\}$ such that*

$$\mathrm{sr}(C_o(\widetilde{X}) \otimes M_l) \leq \mathrm{sr}(\Gamma_o(E) \times_{\hat{\alpha}} D) \leq \mathrm{sr}(\Gamma_o(E) \otimes M_s).$$

Proof. Let $(\bar{E}, X)$ denote the tensor product bundle with fibres $E_x \otimes \mathscr{K}(L^2(D))$ and D-action $(\bar{\alpha}_d, \varphi_d)$ as in (3), so that $\Gamma_o(E) \times_{\hat{\alpha}} D \cong \Gamma_o^D(\bar{E})$.

Assume first that X is compact, so that it has finite orbit structure globally, and that E is homogeneous with fibres isomorphic to the matrix algebra M_n. Let $[D_1], \dots, [D_t]$ denote the conjugacy classes of stabilizer subgroups ordered such that $\exists G \in [D_j]$ with $G \subset D_l \Rightarrow j < l$. Set $X_j = \{x \in X : D_x \in [D_j]\}$. Then X_j is D-invariant, and $\bigcup_{l \geq j} X_l$ is closed in X for $1 \leq j \leq t$. One obtains a composition series of $\Gamma_0(E)$ by D-invariant ideals $I_j = \{f \in \Gamma_o(E) : f \text{ vanishes on } \bigcup_{l \geq j} X_l\}$, with subquotients isomorphic to $\Gamma_o(E_{|X_j})$, giving rise to a composition series $I_j \times_{\hat{\alpha}} D$ of the crossed product $\Gamma_o(E) \times_{\hat{\alpha}} D$ with subquotients isomorphic to $\Gamma_o(E_{|X_j}) \times_{\hat{\alpha}} D$.

Set $Y_j = \{x \in X_j : D_x = D_j\}$, so that $X_j = \bigcup_{d \in D} \varphi_d(Y_j)$. Given $\tilde{x} \in \widetilde{X_j}$, pick a point x in Y_j whose image is $\tilde{x}$. By [Ev, Theorem 3.2], there exists a slice at x, i.e. a subset S of Y_j containing x satisfying

(i) S is closed in $DS = \bigcup_{d \in D} \varphi_d(S)$,
(ii) DS is an open neighborhood of the orbit $Dx = \bigcup_{d \in D} \varphi_d(x)$, so that $\widetilde{S}$ is an open neighborhood of $\tilde{x}$,
(iii) D_j is the stabilizer subgroup of $y \in S$,
(iv) $d \notin D_j \Rightarrow \varphi_d(S) \cap S = \varnothing$,

and $\widetilde{\bar{E}_{|S}} \cong \widetilde{\bar{E}_{|\widetilde{DS}}}$, where $\bar{E}_{|S}$ is viewed as a D_j-bundle. Applying the tilde construction and Corollary 2.6 to $\bar{E}_{|S}$ shows that $\widetilde{\bar{E}_{|S}}$ and hence $\widetilde{\bar{E}_{|\widetilde{X_j}}}$ are locally trivial, and that the fibre $\widetilde{\bar{E}}_{\tilde{y}}$ (for $y \in S$) is isomorphic to

$$[M_n \otimes \mathscr{K}(L^2(D))]^{\alpha^y \otimes \mathrm{Ad}\, \rho(D)|D_j},$$

which by Lemma 2.4 is again isomorphic to

$$[M_n \otimes \mathscr{K}(L^2(D_j))]^{\alpha^y \otimes \mathrm{Ad}\, \rho(D_j)} \otimes M_{s_j} \cong (M_n \times_{\alpha^y} D_j) \otimes M_{s_j},$$

$s_j = |D/D_j|$ $(1 \leq s_j \leq \infty)$.

By Lemma 2.2, $\widetilde{\bar{E}_{|\widetilde{X}_j}}$ decomposes into a countable, disjoint union of locally trivial, homogeneous bundles $(E_{j,p}, Z_{j,p})$, where $Z_{j,p}$ is a partition of $\widetilde{X}_j$ into clopen sets, and by the sum theorem $\dim(\widetilde{X}_j) = \sup_p \dim(Z_{j,p})$. If $y \in Z_{j,p}$, then by [Gr, Theorem 18], the fibre $M_n \times_{\alpha^y} D_j$ over y decomposes into a c_0-sum of matrix algebras with components $M_{n\cdot r}$ contained in $M_n \otimes \mathscr{K}(L^2(D_j))$, where $1 \le r \le \sqrt{|D_j|} < \infty$; say the smallest of the components is of the form $M_n \otimes M_{r_{j,p}}$. An application of Proposition 2.3 and Nistor's Theorem 7 shows that

$$\mathrm{sr}(\Gamma_o(E_{j,p})) = \left\lceil \frac{\dim(Z_{j,p}) - 1}{2nr_{j,p}s_j} \right\rceil + 1.$$

Set $l_j = \sup_p r_{j,p}$ and $v_j = \inf_p r_{j,p}$. Then,

$$\begin{aligned}\left\lceil \frac{\dim(\widetilde{X}_j) - 1}{2nl_js_j} \right\rceil + 1 &\le \mathrm{sr}(\Gamma_o(\widetilde{\bar{E}}_{|\widetilde{X}_j}))\\ &= \sup_p \left\lceil \frac{\dim(Z_{j,p}) - 1}{2nr_{j,p}s_j} \right\rceil + 1 \le \left\lceil \frac{\dim(\widetilde{X}_j) - 1}{2nv_js_j} \right\rceil + 1.\end{aligned}$$

By Lemma 2.1, we obtain the following estimate for $\mathrm{sr}(\Gamma_o(E) \times_{\hat{\alpha}} D)$:

$$\begin{aligned}\max_j \left\lceil \frac{\dim(\widetilde{X}_j) - 1}{2nl} \right\rceil + 1 &\le \max_j \mathrm{sr}(\Gamma_o(\widetilde{\bar{E}}_{|\widetilde{X}_j}))\\ &= \mathrm{sr}(\Gamma_o^D(\bar{E})) \le \max_j \left\lceil \frac{\dim(\widetilde{X}_j) - 1}{2ns} \right\rceil + 1\end{aligned}$$

where $l = \max_j l_j \cdot |D/D_j|$ and $s = \min_j v_j \cdot |D/D_j|$. Thus,

$$\mathrm{sr}\big(C_o(\widetilde{X}) \otimes M_{n\cdot l}\big) \le \mathrm{sr}(\Gamma_o^D(\bar{E})) \le \mathrm{sr}(\Gamma_o(E) \otimes M_s). \tag{10}$$

Now if E is an arbitrary bundle, then by (5), (6) and Proposition 2.3, we can find a finite composition series of $\Gamma_o(E)$ by D-invariant ideals, with subquotients of the form $\Gamma_o(F_k^q, Y_k^q)$ for homogeneous D-bundles whose fibres are the matrix components M_q of A_k as in (4), and we obtain a corresponding decomposition series of $\Gamma_o^D(\bar{E})$ with subquotients $\Gamma_o^D(\bar{F}_k^q) \cong \Gamma_o(\widetilde{\bar{F}_k^q})$. Note that $\dim(X_k) = \dim(Y_k^q)$ and consequently, $\dim(\widetilde{X}_k) = \dim(\widetilde{Y_k^q})$ for all q. If l_k^q and s_k^q are chosen for the bundle F_k^q as in (10), and if $l = \max_{k,q} q \cdot l_k^q$ and $s = \min_{k,q} s_k^q$, then by (10) and Nistor's Theorem 7,

$$\mathrm{sr}\big(C_o(\tilde{X}) \otimes M_l\big) \le \max_{k,q} \mathrm{sr}\big(\Gamma_o^D(\bar{F}_k^q)\big) = \mathrm{sr}\big(\Gamma_o^D(\bar{E})\big) \le \mathrm{sr}\big(\Gamma_o(E) \otimes M_s\big).$$

For noncompact X, pick a sequence $\{T_m\}_{m=1}^\infty$ of D-invariant compact sets such that $T_m \subset \mathring{T}_{m+1}$ and $X = \bigcup_{m=1}^\infty T_m$. By the above, there exist l_m and

s_m so that $\mathrm{sr}\big(C_o(\widetilde{T_m}) \otimes M_{l_m}\big) \leq \mathrm{sr}\big(\Gamma_o(E_{|T_m}) \times_{\hat{\alpha}} D\big) \leq \mathrm{sr}\big(\Gamma_o(E_{|T_m}) \otimes M_{s_m}\big)$. Set $J_m = \Gamma_o(E_{|\mathring{T}_m}) \times_{\hat{\alpha}} D$ and $K_m = \Gamma_o(E_{|X-T_m}) \times_{\hat{\alpha}} D$; then $\Gamma_o(E) \times_{\hat{\alpha}} D = \overline{\bigcup_{m=1}^{\infty} J_m}$ and $\Gamma_o(E) \times_{\hat{\alpha}} D/K_m \cong \Gamma_o(E_{|T_m}) \times_{\hat{\alpha}} D$, so that by [She, Proposition 3.15],

$$\mathrm{sr}\big(\Gamma_o(E) \times_{\hat{\alpha}} D\big) = \sup_m \mathrm{sr}\big(\Gamma_o(E_{|T_m}) \times_{\hat{\alpha}} D\big).$$

Since a similar calculation works for $\mathrm{sr}\big(\Gamma_o(E)\big)$ and $\mathrm{sr}\big(C_o(\widetilde{X})\big)$, the assertion holds with $l = \sup_m l_m$ and $s = \inf_m s_m$.

Remark. The proof of the theorem is valid for an arbitrary second countable compact group D provided that slices exist on all X_j. This is true for example if all orbits are finite, in which case $\dim(\widetilde{X}) = \dim(X)$. Inspection of the above process shows that then even $\mathrm{sr}\big(\Gamma_o(E) \otimes \mathscr{K}(L^2(D))\big) \leq \mathrm{sr}\big(\Gamma_o(E) \otimes M_l\big) \leq \mathrm{sr}\big(\Gamma_o(E) \times_{\hat{\alpha}} D\big) \leq \mathrm{sr}\big(\Gamma_o(E) \otimes M_s\big)$ for some $1 \leq s, l \leq \infty$.

Remark. If (E, X) is a locally trivial, homogeneous C*-bundle with fibres M_∞, then $\mathrm{sr}(\Gamma_o(E) \times_{\hat{\alpha}} D) \leq \mathrm{sr}(\Gamma_o(E))$ by a similar proof.

Remark. We can weaken the assumption that the bundle (E, X) have fibres of bounded dimension. In fact, set $X_n = \{x \in X : \dim(E_x) \leq n\}$. Then, X_n is D-invariant, closed, and $X = \bigcup_{n=1}^{\infty} X_n$. Now set $J_n = \{f \in \Gamma_o^D(E) : f(X - \mathring{X}_n) = \{0\}\}$, $K_n = \{f \in \Gamma_o^D(E) : f(X_n) = \{0\}\}$, and $J = \overline{\bigcup_{n=1}^{\infty} J_n}$. Then, J_n, K_n and J are closed ideals in $\Gamma_o^D(E)$. If $\Gamma_o^D(E)^+$ denotes the C*-algebra with adjoined unit, then by Fell's Stone-Weierstrass theorem, $\Gamma_o^D(E)^+/K_n \cong \Gamma_o^D(E_{|X_n})^+$ and $\Gamma_o^D(E)^+/J \cong \Gamma_o^D(E_{|Z^0})^+$ where $Z^0 = X - \bigcup_{n=1}^{\infty} \mathring{X}_n$. One obtains by [She, Proposition 3.15] that

$$\mathrm{sr}(\Gamma_o^D(E)) = \sup_n \Big\{\mathrm{sr}\big(\Gamma_o^D(E_{|Z^0})\big), \mathrm{sr}\big(\Gamma_o^D(E_{|X_n})\big)\Big\}.$$

The same computation can be done for $\mathrm{sr}\big(\Gamma_o^D(\bar{E})\big)$ and $\mathrm{sr}\big(\Gamma_o(E)\big)$.

(a) If the bundle has fibres of finite dimension locally, i.e. if given $x \in X$, there exists an open neighborhood U of x and an M such that $\dim(E_y) \leq M$ for all $y \in U$, then $x \in \mathring{X}_M$. Thus $Z^0 = \varnothing$, so that the results of this section, in particular (7), (9), and (10) hold.

(b) Set $X^k = Z^{k-1}$ and repeat the above construction inductively with X^k instead of X and Z^k instead of Z^0. In most nonpathological situations, $Z^k = \varnothing$ after a finite number of steps. One can show that $\mathrm{sr}\big(\Gamma_o^D(E_{|Z^k})\big) \leq \sup_n \mathrm{sr}\big(\Gamma_o^D(E_{|X_n})\big)$ for all k, and that similar inequalities apply to $\mathrm{sr}\big(\Gamma_o^D(\bar{E})\big)$ and $\mathrm{sr}\big(\Gamma_o(E)\big)$, so again the results of this section hold.

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