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SOME TRACE FORMULAS FOR ALMOST UNPERTURBED SCHRÖDINGER PAIRS OF OPERATORS

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ABSTRACT. Some trace formulas are given for an almost unperturbed Schrödinger pair of operators $\{U, V\}$ and a self-adjoint operator V_0, where V_0 satisfies the condition that $\{U, V_0\}$ is a Schrödinger pair of operators and $\{V, V_0\}$ is almost commuting.

1. INTRODUCTION

This paper is a continuation of previous works [5] and [6]. Let $\mathscr{H}$ be a Hilbert space and $\{U, V\}$ be a pair of self-adjoint operators on $\mathscr{H}$. This pair $\{U, V\}$ is said to be a Schrödinger pair of operators if

$$e^{isU}e^{iVt}e^{-iUs}e^{-iVt} = e^{ist}I, \qquad -\infty < s, t < +\infty. \tag{1}$$

A condition equivalent to (1) is

$$i[(U-\lambda I)^{-1}, (V-\mu I)^{-1}] = (U-\lambda I)^{-1}(V-\mu I)^{-2}(U-\lambda I)^{-1} \tag{2}$$

for some $\lambda, \mu \in \mathbb{C} \setminus \mathbb{R}$. A pair of self-adjoint operators is said to be an almost unperturbed Schrödinger pair of operators if there is a trace class operator D such that

(3)
$$i[(U-\lambda I)^{-1}, (V-\mu I)^{-1}] = (U-\lambda I)^{-1}(V-\mu I)^{-1}(I+D)(V-\mu I)^{-1}(U-\lambda I)^{-1}.$$

(cf. [5, p. 243] with $a = 1$). For this pair $\{U, V\}$, a cyclic cocycle is given by

$$\begin{aligned} &\operatorname{tr}([e^{is_1U}e^{it_1V}, e^{is_2U}e^{it_2V}] - e^{i(s_1+s_2)U}e^{i(t_1+t_2)V}(e^{-is_2t_1} - e^{-is_1, t_2})) \\ &\quad = \tau(s_1+s_2, t_1+t_2)(e^{-is_2t_1} - e^{-it_1s_2}), \end{aligned} \tag{4}$$

where $[\cdot, \cdot]$ is the commutator, and the function τ may be written

$$\tau(s, t) = \operatorname{tr}(e^{isU}\int_0^t e^{i\tau U}De^{i(t-\tau)V}\,d\tau)/t. \tag{5}$$

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Under certain conditions, the operator V in the almost unperturbed Schrödinger operator $\{U, V\}$ is a perturbation of the operator V_0 in the Schrödinger pair $\{U, V_0\}$ (see [5, Theorem 2]). Therefore, in this paper we consider a triple $\{U, V, V_0\}$, where $\{U, V\}$ is an almost unperturbed Schrödinger pair of operators, $\{U, V_0\}$ is a Schrödinger pair of operators, and the pair $\{V, V_0\}$ satisfies the following condition (C) that this pair is almost commuting in the following sense:

$$\|[e^{isV_0}, e^{itV}]\|_1 \le k\min(|s|, |t|), \qquad -\infty < s, t < \infty, \tag{6}$$

where $\|\cdot\|_1$ is the trace norm and k is a constant. This triple $\{U, V, V_0\}$ is said to be an *almost unperturbed Schrödinger triple satisfying the condition* (C). From [5] it is easy to see that if there is a trace class operator D_1 such that

$$[(V-\lambda I)^{-1}, (V_0-\mu I)^{-1}] = (V-\lambda I)^{-1}(V_0-\mu I)^{-1}D_1(V_0-\mu I)^{-1}(V-\lambda I)^{-1} \tag{7}$$

for some $\lambda, \mu \in \mathbb{C}\setminus\mathbb{R}$, then (6) holds well.

There are several examples of this kind of triple. Let $\mathscr{H} = L^2(\mathbb{R})$,

$$(Qf)(x) = xf(x) \qquad f \in \mathscr{D}(Q), \tag{8}$$

where $\mathscr{D}(Q) = \{f \in L^2(\mathbb{R}), (\cdot)f(\cdot) \in L^2(\mathbb{R})\}$, and

$$(Pf)(x) = i\left(\frac{d}{dx}f\right)(x), \qquad f \in \mathscr{D}(P), \tag{9}$$

where $\mathscr{D}(P) = \{f \in L^2(\mathbb{R}),\ f$ is absolutely continuous, and $(\cdot)f(\cdot) \in L^2(\mathbb{R})\}$. Moreover, let

$$(Vf)(x) = i\frac{d}{dx}f(x) + \frac{\alpha(x)}{2\pi i}\int\frac{\beta(s)f(s)\,ds}{x-(s+io)} + \frac{\overline{\beta(x)}}{2\pi i}\int\frac{\overline{\alpha(s)}f(s)\,ds}{x-(s+io)}$$

for $f \in \mathscr{D}(P)$, where $\alpha(\cdot)$ and $\beta(\cdot)$ satisfy certain conditions. For example, if $\alpha(x) = c(x-\mu)(x-\lambda)^{-1}$ and $\beta(x) = b\alpha(x)^{-1}$ where $\alpha, \mu \in \mathbb{C}\setminus\mathbb{R}$ and $b, c \in \mathbb{C}$, then $\{Q, P, V\}$ is an almost unperturbed Schrödinger triple of operators satisfying the condition (C). Another example is the triple $\{Q, P, \Omega P\Omega^{-1}\}$, where Ω is unitary and $\Omega - I \in \mathscr{L}^1$.

In this paper, we study trace formulas for the almost unperturbed Schrödinger triple satisfying the condition (C). A cyclic cocycle is given by the trace formula

$$\begin{aligned}&\operatorname{tr}([e^{ir_0U}e^{is_0V_0}e^{it_0V}, e^{ir_1U}e^{is_1V_0}e^{it_1V}]\\ &\qquad - (e^{-ir_1(s_0+t_0)} - e^{-ir_0(s_1+t_1)})e^{i(r_0+r_1)U}e^{(s_0+s_1)V_0}e^{i(t_0+t_1)V})\\ &\quad = \operatorname{tr}(r_0+r_1, s_0+s_1, t_0+t_1)(e^{-ir_1(s_0+t_0)U} - e^{ir_0(s_1+t_1)})\end{aligned} \tag{10}$$

for $(r_0+r_1)^2 + (s_0+s_1+t_0+t_1)^2 \ne 0$, where $\tau(\cdot, \cdot, \cdot)$ is a continuous function (see Theorem 2). Formula (10) is a refinement of (4). We also prove that there is a function $c(t)$ such that

$$e^{itV} = V_t + c(t)e^{itV_0},$$

with

$$(V_t f)(x) = -\frac{1}{\sqrt{2\pi}} \int \hat{\tau}(y, y-x, t) f(y)\, dy,$$

where $\hat{\tau}(\cdot, s, t)$ is the inverse Fourier transform of $\tau(\cdot, s, t)$ for fixed s and t (see Theorem 3). Under certain conditions, this function $c(t) \equiv 1$ (see Theorem 4).

2. Cyclic Cocycle

Let G be a Lie group with elements $e^{ik} g(p)$, where $k \in \mathbb{R}$ and $p = (r, s, t) \in \mathbb{R}^3$. Suppose the elements in G are subject to the multiplication formula

$$e^{ik_1} g(p_1) e^{ik_2} g(p_2) = e^{i(k_1 + k_2 + \langle p_1, p_2 \rangle)} g(p_1 + p_2),$$

where $\langle p_1, p_2 \rangle = (r_1(s_2 + t_2) - r_2(s_1 + t_1))/2$ for $p_j = (r_j, s_j, t_j)$, $j = 1, 2$.

For an almost unperturbed Schrödinger triple $\{U, V_0, V\}$ satisfying the condition (C) and the group G, define

$$\rho(e^{ik} g(r, s, t)) = e^{i(k - r(s+t)/2)} e^{irU} e^{isV_0} e^{itV}.$$

Then $\{\rho(e^{ik} g(p)) : k \in \mathbb{R}, p \in \mathbb{R}^3\}$ is an "almost" Lie group of unitary operators in the sense of [4], and

$$\operatorname{tr}([\rho(h_1), \rho(h_2)] - (\rho(h_1 h_2) - \rho(h_2 h_1))), \qquad h_1, h_2 \in G,$$

is a cyclic one cocycle on G. Let

$$\phi(p_1, p_2) = \operatorname{tr}([\phi(g(p_1)), \phi(g(p_2))] - \phi(g(p_1) g(p_2)) + \phi(g(p_2) g(p_1)))$$

for $p_j \in \mathbb{R}^3$. Then $\phi(\cdot, \cdot)$ is a locally bounded Baire function satisfying the following conditions:

$$\phi(0, p) = 0, \tag{11}$$

$$\phi(p_0, p_1) = -\phi(p_1, p_0), \tag{12}$$

and

$$\phi(p_1 + p_2, p_0) e^{i\langle p_1, p_2 \rangle} + \phi(p_2 + p_0, p_1) e^{i\langle p_2, p_0 \rangle} + \phi(p_0 + p_1, p_2) e^{i\langle p_0, p_1 \rangle} = 0. \tag{13}$$

Now we have to determine the form of $\phi(\cdot, \cdot)$. For fixed $\xi \in \mathbb{R}^3$, denote

$$\eta(p) = \eta(p; \xi) = \phi(p, \xi - p).$$

Then (12) and (13) imply that

$$\eta(p_1 + p_2) = \eta(p_1) e^{i\langle p_2, \xi \rangle} + \eta(p_2) e^{i\langle p_1, \xi \rangle}. \tag{14}$$

Exchanging p_1 and p_2 in (14), we get another equation, and if we eliminate $\eta(p_1 + p_2)$ from it and (14), we obtain

$$\eta(p_1)(e^{i\langle p_2, \xi \rangle} - e^{-i\langle p_2, \xi \rangle}) = \eta(p_2)(e^{i\langle p_1, \xi \rangle} - e^{-i\langle p_1, \xi \rangle}). \tag{15}$$

Let $M(\xi) = \{p \in \mathbb{R}^3 : \langle p, \xi\rangle = 0\}$. If $M(\xi) \neq \mathbb{R}^3$, then choose $p_2 \in \mathbb{R}^3 \setminus M(\xi)$ in (15) such that $e^{2i\langle p_2, \xi\rangle} \neq 1$. Hence, by (15), $\eta(p) = 0$ if $e^{2i\langle p, \xi\rangle} = 1$. But if $e^{2i\langle p, \xi\rangle} \neq 1$, then (15) shows that

$$\eta(p; \xi)(e^{i\langle p, \xi\rangle} - e^{-\langle p, \xi\rangle})^{-1}$$

is independent of p. Therefore, if $M(\xi) \neq \mathbb{R}^3$, there is a number $q(\xi)$ such that

$$\eta(p; \xi) = q(\xi)(e^{i\langle p, \xi\rangle} - e^{-i\langle p, \xi\rangle}), \qquad p \in \mathbb{R}^3.$$

The condition $M(r, s, t) = \mathbb{R}^3$ is equivalent to $r^2 + (s+t)^2 = 0$. Thus, if $(r_1 + r_2)^2 + (s_1 + s_2 + t_1 + t_2)^2 \neq 0$, then

$$\phi(p_1, p_2) = q(p_1 + p_2)(e^{i\langle p_1, p_2\rangle} - e^{i\langle p_2, p_1\rangle}).$$

Denote $\tau(r, s, t) = q(r, s, t)e^{ir(s+t)/2}$.

Lemma 1. *Let $\{U, V_0, V\}$ be an almost unperturbed Schrödinger triple satisfying the condition* (C). *Then there is a locally bounded Baire function $\tau(r, s, t)$ defined on $\mathcal{M} = \{(r, s, t) : r^2 + (s+t)^2 \neq 0\}$ which is continuous in each variable separately such that*

$$\begin{aligned} &\operatorname{tr}([e^{ir_0U}e^{is_0V_0}e^{it_0V}, e^{ir_1U}e^{is_1V_0}e^{it_1V}] \\ &\qquad - (e^{-ir_1(s_0+t_0)} - e^{-ir_0(s_1+t_1)})e^{i(r_0+r_1)U}e^{i(s_0+s_1)V_0}e^{i(t_0+t_1)V}) \qquad (16) \\ &\quad = \tau(r_0 + r_1, s_0 + s_1, t_0 + t_1)(e^{-ir_1(s_0+t_0)} - e^{-ir_0(s_1+t_1)}). \end{aligned}$$

Proof. We have to prove only the separate continuity of τ. Suppose $r \neq 0$. In order to prove that $\tau(r, s, t)$ is a continuous function of r, we have to prove only that the left-hand side of (16) is a continuous function of $r_0 (= r)$ for fixed r_1, s_0, t_0 and t_1, satisfying $r_1 = 0$, and $e^{-ir(s_1+t_1)} \neq 1$. It is easy to see that (16) is a sum of the function $\operatorname{tr}(e^{irU}A)$ and

$$q(r) = \operatorname{tr}((e^{irU}e^{itV} - e^{irt}e^{itV}e^{irU})B),$$

where $t = t_0 + t_1$, $A \in \mathcal{L}^1$, and $B \in \mathcal{L}$ are independent of r. It is obvious that $\operatorname{tr}(e^{irU}A)$ is a continuous function of r. Besides, we have

$$\begin{aligned} q(r+\delta) - q(r) = {} & \operatorname{tr}((e^{i\delta U}e^{itV} - e^{i\delta t}e^{itV}e^{i\delta U})Be^{irU}) \\ & + \operatorname{tr}((e^{irU}e^{itV} - e^{irt}e^{itV}e^{irU})(e^{i\delta U}e^{i\delta t} - I)B). \end{aligned}$$

However, from [5, (2)] we have

$$\|e^{irU}e^{itV} - e^{irt}e^{itV}e^{irU}\|_1 \leq |r|\|D\|_1.$$

Therefore, $\lim_{\delta\to 0} q(r+\delta) = q(r)$. Thus $\tau(r, s, t)$ is a continuous function of $r \neq 0$ for fixed s and t. Similarly, we may prove the continuity of τ in the other cases.

In order to analyze the function τ, define functions

$$\phi(r, s, t) = \operatorname{tr}(e^{irU}[e^{isV_0}, e^{itV}])$$

and

$$\psi(r, s, t) = \operatorname{tr}((e^{irU}e^{itV} - e^{irt}e^{itV}e^{irU})e^{isV_0})$$

on $\mathbb{R}^3$. These functions are continuous in each variable separately. In fact, it is obvious that $\phi(r, s, t)$ is a continuous function of r for fixed s and t, since $[e^{isV_0}, e^{itV}] \in \mathscr{L}^1$. To show that ϕ is a continuous function of s for fixed r and t, we only have to use the formulas

$$\begin{aligned}\operatorname{tr}(e^{irU}[e^{i(s+\delta)V_0}, e^{itV}]) = {} & \operatorname{tr}(e^{irU}e^{isV_0}[e^{i\delta V_0}, e^{itV}]) \\ & + \operatorname{tr}(e^{irU}[e^{isV_0}, e^{itV}]e^{i\delta V_0})\end{aligned}$$

and (6). We omit the details of the proof in the other cases.

Putting $r_0 = r$, $s_0 = s$, $t_0 = r_1 = s_1 = 0$, $t_1 = t$ into (16), we get

$$\tau(r, s, t)(e^{irt} - 1) = \phi(r, s, t) + \psi(r, s, t) \tag{17}$$

for $(r, s, t) \in \mathscr{M}$. If $r = 0$ and $s + t \neq 0$, then (17) implies

$$\operatorname{tr}([e^{isV_0}, e^{itV}]) = 0 \tag{18}$$

for $s + t \neq 0$, since $\psi(0, s, t) = 0$. By continuity, (18) holds for $(s, t) \in \mathbb{R}^2$. Therefore we get the following theorem as a byproduct of the foregoing analysis:

Theorem 1. *Let $\{U, V_0, V\}$ be an almost unperturbed Schrödinger triple satisfying the condition that there is an operator $D_0 \in \mathscr{L}^1$ such that*

(19)
$$[(V-\lambda_0 I)^{-1}, (V_0-\mu_0 I)^{-1}] = (V_0-\mu_0 I)^{-1}(V_0-\lambda_0 I)^{-1}D_0(V-\lambda_0 I)^{-1}(V_0-\mu_0 I)^{-1}$$

for some $\lambda_0, \mu_0 \in \mathbb{C} \setminus \mathbb{R}$. Then the Pincus principal function for the almost commuting pair of self-adjoint (unbounded) operators $\{V_0, V\}$ equals zero.

Notice that (19) implies (6), so Theorem 1 is a consequence of (18).

Now, we have to obtain an expression for the function τ. Using the Weyl commutation relation (1) of e^{irU} and e^{isV_0}, we may calculate that

$$(1 - e^{irs})\psi(r, s, t) + (1 - e^{ir(s+t)})\phi(r, s, t) = 0. \tag{20}$$

Multiplying both sides of (17) by $(1 - e^{irs})$ and using (20) to eliminate ψ, we obtain

$$\tau(r, s, t)(e^{-irs} - 1) = \phi(r, s, t) \quad \text{for } (r, s, t) \in \mathscr{M}, \tag{21}$$

if $e^{irt} \neq 1$. But both sides of (21) are continuous functions; therefore, (21) is still true even if $e^{irt} = 1$. Thus

$$\tau(r, s, t) = \operatorname{tr}(e^{irU}[e^{isV_0}, e^{itV}])(e^{-irs} - 1)^{-1} \quad \text{for } (r, s, t) \in \mathscr{M}, e^{irs} \neq 1. \tag{22}$$

If $e^{irs} = 1$ then $\phi(r, s, t) = 0$. By the continuity of τ, we have

$$\tau(r, s, t) = is^{-1}\frac{\partial}{\partial r}\phi(r, s, t), \quad \text{for } (r, s, t) \in \mathscr{M}, s \neq 0, e^{irs} = 1. \tag{23}$$

On the other hand, by [5, (53)] we also have

$$\tau(r, 0, t) = ir^{-1}\frac{\partial}{\partial s}\phi(r, 0, t) \quad \text{for } r \neq 0, \tag{24}$$

and

$$\tau(0, 0, t) = \operatorname{tr}(De^{itV}). \tag{25}$$

The next step is to examine the case of $M(\xi) = \mathbb{R}^3$; i.e., $\xi = (0, s, -s)$. Then (14) becomes

$$\eta(p_1 + p_2) = \eta(p_1) + \eta(p_2).$$

So, there are functions $K(s)$ and $K_j(s)$, $j = 1, 2$, such that

$$\eta(r_0, s_0, t_0) = r_0K(s) + s_0K_1(s) + t_0K_2(s),$$

since $\eta(p)$ is a locally bounded Baire function.

Thus $\tau(r_0, s_0, t_0; -r_0, s_1, t_1)$ equals

$$\begin{aligned} &\operatorname{tr}[e^{ir_0U}e^{is_0V_0}e^{it_0V}, e^{-ir_0U}e^{is_1V_0}e^{it_1V}] \\ &\quad = r_0K(s_0 + s_1) + s_0K_1(s_0 + s_1) + t_0K_2(s_0 + s_1), \end{aligned} \tag{26}$$

provided that $s_0 + s_1 + t_0 + t_1 = 0$. If $r_0 = 0$, then, from Theorem 1, we have $s_0K_1(s_0 + s_1) + t_0K_2(s_0 + s_1) = 0$. Hence

$$rK(s) = \operatorname{tr}([e^{irU}e^{isV_0}e^{-isV}, e^{-irU}]).$$

Therefore, by [5, (58)],

$$K(s) = i\operatorname{tr}([(U, e^{-isV}] + se^{-isV})e^{-isV_0}). \tag{27}$$

Theorem 2. *Let $\{U, V_0, V\}$ be an almost unperturbed Schrödinger triple satisfying the condition* (C). *If $(r_0+r_1)^2+(s_0+s_1+t_0+t_1)^2 \neq 0$, then* (16) *holds, where the function $\tau(r, s, t)$ is defined by* (22)–(25). *If $(r_0+r_1)^2+(s_0+s_1+t_0+t_1)^2 = 0$, then the left-hand side of* (16) *equals $r_0K(s_0 + s_1)$, where the function $K(\cdot)$ is defined by* (27).

By continuity, (20) and (21) imply that

$$\psi(r, s, t) = (e^{irt} - e^{-irs})\tau(r, s, t), \quad \text{for } r^2 + (s + t)^2 \neq 0. \tag{28}$$

Lemma 2. *Let $\{U, V_0, V\}$ be an almost unperturbed Schrödinger triple of operators satisfying the condition* (C). *If $(r_1 + r_2)^2 + (s_1 + s_2 + t)^2 \neq 0$, then*

$$\begin{aligned} &\operatorname{tr}(e^{ir_1U}e^{is_1V_0}(e^{ir_2U}e^{is_2V_0}e^{itV} - e^{ir_2t}e^{itV}e^{ir_2U}e^{is_2V_0})) \\ &\quad = (e^{i(r_2t - r_1s_2)} - e^{-is_1r_2})\tau(r_1 + r_2, s_1 + s_2, t). \end{aligned} \tag{29}$$

Proof. Denote the left-hand side of (29) by $\psi(r_1, r_2, s_1, s_2, t)$. It is easy to see that

(30)

$$\psi(r_1, r_2, s_1, s_2, t) = e^{-is_1r_2}\phi(r_1 + r_2, s_1, s_2, t) + e^{-ir_1s_2}\psi(r_1, r_2, s_1 + s_2, t),$$

where

$$\phi(r, s_1, s_2, t) = \operatorname{tr}(e^{irU} e^{is_1 V_0}[e^{is_2 V_0}, e^{itV}])$$

and

$$\psi(r_1, r_2, s, t) = \operatorname{tr}(e^{ir_1 U} e^{isV_0}(e^{ir_2 U} e^{itV} - e^{ir_2 t} e^{itV} e^{ir_2 U})).$$

We may calculate that

$$\psi(r_1, r_2, s, t) + e^{-i(r_2 s + r_1 t)} \psi(r_2, r_1, s, t) = e^{-ir_1 t} \psi(r_1 + r_2, s, t). \tag{31}$$

Eliminating ψ from (31) and using another equation obtained from (31) by exchanging r_1 and r_2, we get

$$\psi(r_1, r_2, s, t) = (e^{ir_2 t} - e^{-ir_2 s}) \tau(r_1 + r_2, s, t) \tag{32}$$

if $e^{i(r_1+r_2)(s+t)} \neq 1$. However, both sides of (32) are continuous functions. Therefore (32) holds even if $e^{i(r_1+r_2)(s+t)} = 1$. Similarly, we have

$$\phi(r, s_1, s_2, t) = (e^{-irs_2} - 1)\tau(r, s_1 + s_2, t). \tag{33}$$

Equation (29) now follows from (30), (32), and (33).

3. A LEMMA

For $\xi \in L^2(\mathbb{R})$, let $\hat{\xi}$ be the inverse Fourier transform of ξ.

Lemma 3. *Let $\{Q, P\}$ be the Schrödinger pair of operators on $L^2(\mathbb{R})$ defined by* (8) *and* (9), *let $A \in \mathscr{L}^1$, and let $\xi, \eta \in L^2(\mathbb{R}) \cap L^1(\mathbb{R})$. Then*

$$\frac{1}{2\pi} \int \left(\iint \operatorname{tr}(e^{i(r-p)Q} e^{isP} A) \hat{\xi}(r) \overline{\hat{\eta}(p)} e^{ips} \, dr \, dp \right) ds = (A\xi, \eta). \tag{34}$$

Proof. For $f \in L^2(\mathbb{R})$, the vector-valued integral

$$L(s; f) = \frac{1}{2\pi} \iint e^{i(r-p)Q} e^{isP} f \hat{\xi}(r) \overline{\hat{\eta}(p)} e^{ips} \, dr \, dp$$

converges in $L^2(\mathbb{R})$. As a function of x, the vector $L(s; f)$ is

$$\frac{1}{2\pi} \iint e^{ixr} f(x-s) \hat{\xi}(r) \hat{\eta}(p) e^{i(x-s)p} \, dr \, dp = f(x-s)\xi(x)\overline{\eta(x-s)},$$

for a.e. x.

Therefore, for every $h \in L^2(\mathbb{R})$, we have

$$(L(s; f), h) = \int f(x-s)\xi(x)\overline{\eta(x-s)h(x)} \, dx.$$

Hence, the weak integral $\int L(s; f) \, ds$ exists and

$$\left(\int L(s; f) \, ds, h \right) = (f, \eta)(\xi, h).$$

Thus

$$\int L(s; f) \, ds = (f, \eta)\xi. \tag{35}$$

Without loss of generality, we may assume $A \geq 0$. Let $\{e_k\}$ be an orthonormal basis satisfying $Ae_k = \lambda_k e_k$, $\lambda_k \geq 0$ and $\sum \lambda_k < +\infty$. It is obvious that

$$\frac{1}{2\pi}\iint \operatorname{tr}(e^{i(r-p)Q}e^{isP}A)\hat{\xi}(r)\overline{\hat{\eta}(p)}e^{irs}\,dr\,dp = \sum \lambda_k(L(s\,;e_k), e_k). \tag{36}$$

From (35) and (36), (34) follows.

For fixed s_2 and t, $\phi(r, s_1, s_2, t)$ is a continuous function of (r, s_1). Therefore, by (33), it is easy to see that $\tau(r, s, t)$ is a continuous function of (r, s) for fixed t.

4. Form of e^{itV}

In this section, an expression for e^{itV} is given by using $\tau(x, s, t)$. Let K be the space of all indefinitely differentiable functions with compact supports endowed with the topology defined in [2] and [3] and let K' be the space of all linear continuous functionals (distributions) on K. Let Z be the space of Fourier transforms of the functions in K, and Z' be the space of the Fourier transforms of the distributions in K'. It is obvious that the function $\tau(\cdot, s, t) \in K$ for fixed s and t and the function $\psi(\cdot, r_2, s_1, s_2, t) \in K$ for fixed r_2, s_1, s_2, and t. Let $\hat{\tau}(\cdot, s, t) \in Z'$ and $\hat{\psi}(\cdot, r_2, s_2, t) \in Z'$ be the inverse Fourier transforms of $\tau(\cdot, s, t)$ and $\psi(\cdot, r_2, s_1, s_2, t)$, respectively. Then, by (29),

$$\hat{\psi}(x, r_2, s_1, s_2, t) = \hat{\tau}(x+s_2, s_1+s_2, t)e^{ir_2(x+s_2+t)} - \hat{\tau}(x, s_1+s_2, t)e^{ir_2(x-s_1)}.$$

Let

$$A(r, s, t) = e^{irt}e^{itV}e^{irU}e^{isV_0} - e^{irU}e^{isV_0}e^{itV}.$$

Without loss of generality, we may assume that $\mathscr{H} = L^2(\mathbb{R}, \mathscr{D})$, $U = Q$, and $V_0 = P$. For $\xi, \eta \in K$, we obtain by (34)

$$\begin{aligned}(A(r, s, t)\xi, \eta) = \frac{1}{(2\pi)^{3/2}}\int\Big(\iiint & e^{i(r_1-p)x}(\hat{\tau}(x, s_1+s, t)e^{ir(x-s_1)} \\ & - \hat{\tau}(x+s, s_1+s, t)e^{ir(x+s+t)})\hat{\xi}(r_1)\overline{\hat{\eta}(p)}e^{ips_1}\,dr_1\,dp\,dx\Big)\,ds_1.\end{aligned}$$

Therefore

$$\begin{aligned}(A(r, s, t)\xi)(x) = {} & \frac{1}{\sqrt{2\pi}}\int \hat{\tau}(y, y-x+s, t)e^{irx}\xi(y)\,dy \\ & - \frac{1}{\sqrt{2\pi}}\int \hat{\tau}(y+s, y-x+s, t)e^{ir(y+s+t)}\xi(y)\,dy.\end{aligned} \tag{37}$$

Define an operator V_t:

$$(V_t\xi)(x) = -\frac{1}{\sqrt{2\pi}}\int \hat{\tau}(y, y-x, t)\xi(y)\,dy, \qquad \xi \in K. \tag{38}$$

Then (37) shows that

$$(e^{irU}e^{isV_0}e^{-itV_0}(V_t - e^{itV}) - e^{-itV_0}(V_t - e^{itV})e^{irU}e^{isV_0})\xi = 0$$

for all $r, s, t \in \mathbb{R}$ and $\xi \in K$. Therefore the operator $e^{-itV_0}(V_t - e^{itV})$ commutes $e^{irU}e^{isV_0}$ for $r, s \in \mathbb{R}$. Thus, there is a function $c(t)$ such that

$$e^{itV} = V_t + c(t)e^{itV_0}. \tag{39}$$

Theorem 3. *Let $\{Q, P, V\}$ be an almost unperturbed Schrödinger triple of operators on $L^2(\mathbb{R})$ satisfying* (C), *and let V_t be the operator defined by* (38). *Then there is a function $c(t)$ such that* (39) *holds. If V is not of the form $V = P + \alpha I$, $\alpha \in \mathbb{C}$, then the function $c(t)$, satisfying the condition that $\{V_t + c(t)e^{itP} : t \in \mathbb{R}\}$ is a one-parameter group of operators, is unique.*

Thus the operator V is determined by the function $\tau(r, s, t)$ if the almost unperturbed Schrödinger triple $\{U, V_0, V\}$ satisfies the condition that U is simple.

Proof. We have to prove only the uniqueness of $c(t)$. From the equation

$$(V_{t_1} + c(t_1)e^{it_1P})(V_{t_2} + c(t_2)e^{it_2P}) = V_{t_1+t_2} + c(t_1 + t_2)e^{i(t_1+t_2)P},$$

it is easy to see that

(40)
$$\begin{aligned}
&\frac{1}{2\pi}\int \hat{\tau}(v - u + t_2, z - u - t_1, t_1)\hat{\tau}(v, u - t_2, t_2)\,du \\
&\qquad - \frac{1}{\sqrt{2\pi}}c(t_1)\hat{\tau}(v, z - t_2, t_2) - \frac{1}{\sqrt{2\pi}}c(t_2)\hat{\tau}(v + t_2, z - t_1, t_1) + c(t_1)c(t_2)\delta(z) \\
&\quad = -\frac{1}{\sqrt{2\pi}}\hat{\tau}(v, z - t_1 - t_2, t_1 + t_2) + c(t_1 + t_2)\delta(z).
\end{aligned}$$

Suppose $b(\cdot)$ is another function satisfying the condition that $\{V_t + b(t)e^{itP} : t \in \mathbb{R}\}$ is also a one-parameter group. Then (40) implies that

(41)
$$\begin{aligned}
&\frac{1}{\sqrt{2\pi}}(b(t_1) - c(t_1))\hat{\tau}(v, z - t_2, t_2) + \frac{1}{\sqrt{2\pi}}(b(t_2) - c(t_2))\hat{\tau}(v + t_2, z - t_1, t_1) \\
&\quad = (c(t_1 + t_2) - b(t_1 + t_2) - c(t_1)c(t_2) + b(t_1)b(t_2))\delta(z).
\end{aligned}$$

Exchanging the roles of t_1 and t_2 in (41), we obtain the result that, if $b(t_j) - c(t_j) \neq 0$ $j = 1, 2$, then

$$\tau_1(v, t_2) + \tau_1(v + t_2, t_1) = \tau_1(v, t_1) + \tau_1(v + t_1, t_2), \tag{42}$$

where

$$\tau_1(v, t) = \hat{\tau}(v, z - t, t)(b(t) - c(t))^{-1}$$

for t satisfying $b(t) \neq c(t)$ and fixed z. Taking the Fourier transform of both sides of (42), we may conclude that there is a function $\tau(x, z)$ such that

$$\tau(x, s, t) = (1 - e^{-ixt})(b(t) - c(t))\tau(x, s + t). \tag{43}$$

It is easy to see that if $b(\cdot) \neq c(\cdot)$ then there exists a sequence $\{t_n\}$ satisfying $b(t_n) \neq c(t_n)$. From (41) and (43), there is a number c_n such that

$\hat{\tau}(y, z-2t_n, 2t_n) = c_n\delta(z)$. Thus $V_{2t_n} = -\frac{1}{\sqrt{2\pi}}c_n e^{i2t_nP}$. Hence $V-P$ is a multiple of identity. This proves the theorem.

Theorem 4. *Let $\{Q, P, V\}$ be an almost unperturbed Schrödinger triple of operators on $L^2(\mathbb{R})$ satisfying the condition* (C). *If $e^{itV} - e^{itP} \in \mathscr{L}^1$ for $t \in \mathbb{R}$, then the function $c(t)$ in* (39) *is the constant* 1.

Proof. Let

$$\tau_0(r, s, t) = \operatorname{tr}(e^{irQ}e^{isP}(e^{itP} - e^{itV})). \tag{44}$$

Then it is easy to calculate that

$$\psi(r_1, r_2, s_1, s_2, t) = (e^{i(r_2t - r_1s_2)} - e^{-is_1r_2})\tau_0(r_1+r_2, s_1+s_2, t). \tag{45}$$

Comparing equations (29) and (41), we have

$$\tau = \tau_0. \tag{46}$$

Since $e^{itV} - e^{itV_0} \in \mathscr{L}^1$ is a product of two Hilbert-Schmidt operators, it is easy to see that there is a kernel $k(x, y, t)$ such that

$$((e^{itV} - e^{itP})f)(x) = \frac{1}{\sqrt{2\pi}}\int k(y, x, t)f(y)\,dy \tag{47}$$

and

$$\operatorname{tr}(e^{irQ}e^{isP}(e^{itV} - e^{itP})) = \frac{1}{\sqrt{2\pi}}\int e^{irx}k(x, x-s, t)\,dx.$$

Therefore, $\hat{\tau}_0(x, s, t) = -k(x, x-s, t)$ or $k(y, x, t) = -\hat{\tau}_0(y, y-x, t)$. From (46) and (47) it follows that $e^{itV} = V_t + e^{itP}$, and the theorem is proved.

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