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ON THE PRIMARINESS OF THE BANACH SPACE l_∞/C_0

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ABSTRACT. In a response to a question asked by Leonard and Whitfield (1983) we show that, under the Continuum Hypothesis, the Banach space l_∞/C_0 is primary.

1. INTRODUCTION

A Banach space E is said to be primary if for every direct sum decomposition $E = X \oplus Y$ of E, at least one of the summands is isomorphic to E. The question of whether or not the Banach space $C = l_\infty/C_0$ is primary was raised by Leonard and Whitfield [6], and was motivated by the following two natural decompositions of C: $C \cong C \oplus C$ and $C \approx l_\infty \oplus C$. J. Lindenstrauss has shown that l_∞ is prime [7]. The primary question for Banach spaces has been studied extensively. [2] is good reference for this subject. The space C has been used before by Y. Benyamini to construct an example of an M-space which is not isomorphic to a $C(K)$ space [1].

In this paper we show that, under the continuum hypothesis (henceforth abbreviated CH), the space C is primary. We accomplish this in two steps, as in Pelczynski's classical proof that the spaces l_p are prime (see [8]): In §2, as a corollary to our result on operators on C, we obtain the following.

(i) If $C = X \oplus Y$, then X or Y contains a complemented copy of C. In §3, we prove that

(ii) $l_\infty(C)$, the l_∞-sum of countably many copies of C, is isomorphic to a complemented subspace of C; in consequence, $l_\infty(C) \approx C$.

From (i) and (ii), using Pelczynski's decomposition method, it follows easily that C is primary.

As is well known (see e.g. [6]), C can be identified with $C(\omega^*)$, the space of continuous scalar-valued functions on $\omega^* = \beta\omega \sim \omega$, the growth of the Stone–Cech compactification of the discrete space $\omega = \{1, 2, \dots\}$. In §§2 and 3 we work exclusively with C thus represented. We do not need CH in (i); however,

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our proof of (ii) relies heavily on the fact [10] that open F_σ sets in ω^* are retracts of ω^* which was established assuming CH, and is now known to be false without CH [4; pp. 205–206]. Of course, it might still be true that the primariness of C can be established within ZFC though our personal feeling is that this is not the case.

In §4, which concludes the paper, we collect some observations on the space $l_\infty(C)$. In particular, we show that an isomorphic embedding of this space into C cannot be obtained as a map induced by an operator from l_∞ into itself.

Our Banach space terminology and notation are standard, as in [8]; the symbols $\approx$ and $\cong$ are used to denote isomorphisms and isometric isomorphisms between Banach spaces. For the information on the Stone–Cech compactifications the reader is referred to [14] and [9]; basic facts about $\beta\omega$ and ω^* are gathered together on p. 532 of [6].

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2. Operators on $C = l_\infty/C_0 = C(\omega^*)$

We start by introducing some notation. If $A \subset \omega^*$, then 1_A denotes the characteristic function of A relative to ω^*, $\mathscr{A}(A)$ stands for the algebra of clopen subsets in the subspace A, and $\mathscr{A}_0(A) = \mathscr{A}(A) \sim \{\varnothing\}$; we write simply $\mathscr{A}$ and $\mathscr{A}_0$ when $A = \omega^*$. Recall that if $A \in \mathscr{A}_0$, then A is homeomorphic to ω^* and hence $C(A) \cong C$; in what follows we often identify $C(A)$ with the subspace $\{x\colon 1_A x = x\}$ of C. We recall also that the algebra $\mathscr{A}$ has the following property (called Cantor separability in [14]): For every decreasing sequence (A_n) in $\mathscr{A}_0$ there is an $A \in \mathscr{A}_0$ which is contained in all A_n. Finally, if $T\colon C \to C$ is a (bounded linear) operator and $A \in \mathscr{A}$, then the operator $T_A\colon C(A) \to C(A)$ is defined by $T_A(x) = T(x)1_A$.

Our main result in this section is the following.

2.1. **Theorem.** *If $T\colon C \to C$ is an operator, then for every $A \in \mathscr{A}_0$ there exists a $B \in \mathscr{A}_0(A)$ and a scalar γ such that*

$$T_B(x) = \gamma x \quad \textit{for all } x \in C(B).$$

This will follow directly from Lemma 2.3 below applied to the representing measure μ of T: $\mu(E) = T(1_E)$ for $E \in \mathscr{A}$.

We first prove the following (comp. [11; Lemma 1]):

2.2. **Lemma.** *Let $\lambda\colon \mathscr{A}_0 \to R$ be a nondecreasing set function. Then for every $A \in \mathscr{A}_0$ there exist $B \in \mathscr{A}_0(A)$ and $\beta \in R$ such that*

$$\lambda(E) = \beta \quad \textit{for all } E \in \mathscr{A}_0(B).$$

Proof. Define $\tilde{\lambda}\colon \mathscr{A}_0 \to R \cup \{-\infty\}$ by

$$\tilde{\lambda}(E) = \inf\{\lambda(F)\colon F \in \mathscr{A}_0(E)\}.$$

Now, given $A \in \mathscr{A}_0$, chose by induction a sequence $A = A_1 \supset A_2 \supset A_3 \supset \cdots$ in $\mathscr{A}_0$ so that, for each n,

$$\lambda(A_n) \le \tilde{\lambda}(A_{n-1}) + \tfrac{1}{n} \quad \text{if } \tilde{\lambda}(A_{n-1}) \ne -\infty,$$

or

$$\lambda(A_n) \le -n \quad \text{if } \tilde{\lambda}(A_{n-1}) = -\infty.$$

Since the sequence $\tilde{\lambda}(A_n)$ is nondecreasing, the limit

$$\beta = \lim \tilde{\lambda}(A_n) = \lim \lambda(A_n) \text{ exists in } R \cup \{-\infty\}.$$

By the Cantor separability of $\mathscr{A}$, there is a $B \in \mathscr{A}_0$ such that $B \subset A_n$ for all n. Now, if $E \in \mathscr{A}_0(B)$, then

$$\tilde{\lambda}(A_n) \le \lambda(E) \le \lambda(A_n) \quad \text{for all } n;$$

hence $\lambda(E) = \beta$ (and $\beta \in R$).

2.3. **Lemma.** *Let $\mu: \mathscr{A} \to C = C(\omega^*)$ be a bounded finitely additive vector measure. Then for every $A \in \mathscr{A}_0$ there exists a $B \in \mathscr{A}_0(A)$ and a scalar γ such that*

$$1_B \mu(E) = \gamma 1_E \quad \textit{for all } E \in \mathscr{A}(B).$$

Proof. Clearly, it suffices to prove this for the real space C. Define a nondecreasing set function $\lambda_\mu: \mathscr{A} \to R$ by

$$\lambda_\mu(E) = \sup_{F \in \mathscr{A}(E)} \max_E \mu(F),$$

and define $\lambda_{-\mu}$ likewise.

Let $A \in \mathscr{A}_0$. Applying Lemma 2.2 first to $\lambda = \lambda_\mu$, and next to $\lambda = \lambda_{-\mu}$, we find $B \in \mathscr{A}_0(A)$ and scalars α, β such that

$$\lambda_\mu(E) = \beta$$

and

$$(*) \qquad -\lambda_{-\mu}(E) = \inf_{F \in \mathscr{A}(E)} \min_E \mu(F) = \alpha \quad \text{for all } E \in \mathscr{A}_0(B).$$

Clearly, $\alpha \le 0 \le \beta$. If $\alpha = \beta = 0$, then $1_B \mu(E) = 0 \cdot 1_E$ for all $E \in \mathscr{A}(B)$. Now let us consider the case when $\alpha < 0$ or $\beta > 0$; by switching from μ to $-\mu$ if necessary, we may assume $\beta > 0$.

Claim $\alpha = 0$.

Suppose $\alpha < 0$, and let $0 < \varepsilon < \frac{1}{2}\min(-\alpha, \beta)$. Since $\lambda_\mu(B) = \beta$, there exists $E \in \mathscr{A}(B)$ such that $\max_B \mu(E) > \beta - \varepsilon$, and hence there exists an $H \in \mathscr{A}(B)$ with

$$\mu(E) > \beta - \varepsilon \quad \text{on } H.$$

Observe that $H \subset E$: In fact, if $H \sim E \ne \varnothing$, then $\lambda_\mu(H \sim E) = \beta$ and we could find $F \in \mathscr{A}(H \sim E)$ so that $\max_{H \sim E} \mu(F) > \beta - \varepsilon$. But in that case $\max_H \mu(E \cup F) > 2(\beta - \varepsilon) > \beta$, which is impossible. Next, using $(*)$,

choose $F \in \mathscr{A}(H)$ with $\min_H \mu(F) < \alpha/2$. Then $\max_H \mu(E \sim F) = \max_H[\mu(E) - \mu(F)] > \beta + \alpha/2 - \alpha/2$ and so $\lambda_\mu(E) > \beta$, which is impossible.

From the above it now follows that the vector measure $\mu_B : \mathscr{A}(B) \to C$ defined by $\mu_B(E) = 1_B\mu(E)$ is nonnegative and hence monotone:

(1) If E, $F \in \mathscr{A}(B)$ and $F \subset E$, then $0 \le \mu_B(F) \le \mu_B(E)$.

From this and the definition of λ_μ it follows immediately that

$$\max 1_B\mu(E) = \max 1_E\mu(E) = \beta \quad \text{for all } E \in \mathscr{A}_0(B). \tag{2}$$

Now we show that

$$\mu_B(E) \le \beta 1_E \quad \text{for all } E \in \mathscr{A}(B). \tag{3}$$

Suppose it is not so for some $E \in \mathscr{A}_0(B)$; thus there is an $H \in \mathscr{A}_0(B \sim E)$ and an $\varepsilon > 0$ such that $\mu_B(E) \ge \varepsilon 1_H$. Then, as $\max 1_H\mu(H) = \beta$, we have

$$\max 1_H\mu(E \cup H) = \max[1_H\mu(E) + 1_H\mu(H)] \ge \max[\varepsilon + 1_H\mu(H)] = \varepsilon + \beta;$$

a contradiction.

From (1), (2), and (3) it now follows that $\mu_B(E) = \beta 1_E$ for all $E \in \mathscr{A}(B)$, which concludes the proof.

2.4. **Corollary.** *Let I be the identity operator on C, and let (T_n) be a (finite or infinite) sequence of operators on C such that*

$$I = T_1 + T_2 + \cdots \quad \textit{pointwise on } C.$$

Then for every $A \in \mathscr{A}_0$ there exists a $B \in \mathscr{A}_0(A)$ and an m such that T_m maps $C(B)$ isomorphically onto a complemented subspace of C.

Proof. By Theorem 2.1 we can find a decreasing sequence (B_n) in $\mathscr{A}_0$, with $B_1 \subset A$, and a sequence of scalars (γ_n) so that for each n,

$$T_n(x)1_{B_n} = \gamma_n x \quad \text{for all } x \in C(B_n).$$

By the Cantor separability of $\mathscr{A}$ there exists $B \in \mathscr{A}_0$ which is contained in all B_n's. Then

$$T_n(x)1_B = \gamma_n x \quad \text{for all } x \in C(B) \text{ and all } n.$$

Moreover, by assumption we have $x = I(x)1_B = \gamma_1 x + \gamma_2 x + \cdots$ for all $x \in C(B)$; therefore, $\gamma_m \ne 0$ for some m. Since

$$|\gamma_m|\,\|x\| \le \|T_m(x)\| \le \|T_m\|\,\|x\| \quad \text{for } x \in C(B), \quad T_{m|C(B)}$$

is an isomorphism. Finally, the map $P : C \to C$ defined by $P(x) = T_m(x1_B)$ is a projection onto $T_m[C(B)]$.

2.5. **Corollary.** *For every (finite or infinite) Schauder decomposition $C = X_1 + X_2 + \cdots$ of C at least one of the summands X_n contains a subspace isomorphic to C and complemented in C.*

2.6. *Remark.* It is also known ([6; 5.1], [13; Prop. 2]) that every infinite-dimensional complemented subspace of C contains an isomorphic copy of l_∞.

3. C IS PRIMARY

Throughout this section we assume the continuum hypothesis (CH). The following result is due to Negrepontis [10; Corollary 3.2], and it cannot be proved without CH [4].

3.1. **Theorem** (CH). *If A is an open F_σ subset of ω^*, then $\overline{A}$ is a retract of ω^*.*

(Equivalently, there is a Boolean isomorphism σ from the algebra $\mathscr{A}(A)$ into the algebra $\mathscr{A}(\omega^*)$ such that $\sigma(B) \cap A = B$ for all $B \in \mathscr{A}(A)$.)

3.2. **Proposition** (CH). *$l_\infty(C)$ is isometric to a complemented subspace of C. As a consequence $l_\infty(C) \approx C$.*

Proof. Let (A_n) be an infinite sequence of disjoint nonempty clopen subsets of ω^* and let A be its union. Then $l_\infty(C)$ is isomorphically isometric to the Banach space $C_b(A)$ of bounded continuous functions on A. Moreover, as is easily seen $\overline{A} = \beta A$, and therefore $C_b(\overline{A}) \cong C(\overline{A})$. Hence $l_\infty(C) \cong C(\overline{A})$. Now, by Theorem 3.1, there is a retraction r of ω^* onto $\overline{A}$. Then the corresponding composition operator $R\colon C(\overline{A}) \to C$; $x \to x \circ r$ is an isometry, and the operator P defined by $P(x) = R(x_{|\overline{A}})$ is a projection from C onto $R[C(\overline{A})]$.

The second assertion follows easily from the first by using Pelczynski's decomposition method: Let $C \approx l_\infty(C) \oplus Z$; then

$$l_\infty(C) \approx l_\infty(C) \oplus C \approx l_\infty(C) \oplus l_\infty(C) \oplus Z \approx l_\infty(C) \oplus Z \approx C.$$

3.3. **Theorem** (CH). *Let $C = X_1 \oplus X_2 \oplus \cdots$ be a (finite or infinite) Schauder decomposition of C. Then there is an m such that $X_m \approx C$. In particular, C is primary.*

Proof. By Corollary 2.5 there is an m such that X_m contains a subspace V which is isomorphic to C and complemented in C. Assuming, as we may, that $m = 1$, and denoting $X = X_1$, $Y = X_2 \oplus X_3 \oplus \cdots$, we thus have

$$C = X \oplus Y, \qquad X = U \oplus V, \quad \text{and} \quad V \approx C.$$

(for some subspace U of X). Moreover, by Proposition 3.2, $C \approx l_\infty(C)$. Now, applying Pelczynski's decomposition method,

$$\begin{aligned} X &\approx U \oplus l_\infty(C) \approx U \oplus C \oplus l_\infty(C) \approx X \oplus l_\infty(X \oplus Y) \\ &\approx X \oplus l_\infty(X) \oplus l_\infty(Y) \approx l_\infty(X) \oplus l_\infty(Y) \approx l_\infty(X \oplus Y) \approx l_\infty(C) \approx C. \end{aligned}$$

3.4. *Remarks.* (1) In view of the natural decompositions $C \approx C \oplus C$ and $C \approx l_\infty(C)$, one might conjecture that an infinite-dimensional complemented

subspace of C must be isomorphic either to l_∞ or to C. It is however easy to see that it is not so if we assume CH: Let $L = L_\infty([0, 1]^c)$, where $C = 2^{\chi_0} = \chi_1$ and $[0, 1]^c$ is considered with its product Lebesgue measure. Then the measure algebra of $[0, 1]^c$ has cardinality $c = \chi_1$; moreover, by Parovicenko's theorem [14; p. 81], every Boolean algebra of cardinality $\leq \chi_1$ can be embedded via a Boolean isomorphism into $\mathscr{A}(\omega^*)$. From this it follows that L is isometric to a closed subspace X of C. Since L is an injective Banach space, X is complemented in C. Moreover, $L \not\approx l_\infty$ because no sequence of continuous linear functionals separates points in L, and $L \approx C$ because C is not injective (see [12]). (By Theorem 3.3, every complement of X in C is isomorphic to C.)

(2) The proof of the decomposition $C \approx l_\infty \oplus C$ given in [6; 5.4] uses implicitly CH, which would be easily avoided: $C \approx (l_\infty \oplus l_\infty)/(\{0\} \oplus C_0) \approx l_\infty \oplus C$, where the first isomorphism is justified by the separable homogeneity of l_∞ [8; 2.f.12(i) and p. 112].

4. Some observations on the space $l_\infty(C)$

In the proof of Proposition 3.2 we represented $l_\infty(C)$ isometrically in the form $C_b(A_1 \cup A_2 \cup \cdots)$, where (A_k) was any disjoint sequence of nonempty clopen sets in ω^*. The corresponding "quotient" representations of $l_\infty(C)$ are of course of the form

$$l_\infty \Big/ \left(\sum_k C_0[N_k]\right)_{l_\infty},$$

where (N_k) is a partition of ω into a sequence of infinite subsets. Here for $M \subset \omega$ we let $l_\infty[M] = \{x = (x_n) \in l_\infty : x_n = 0 \text{ for } n \notin M\}$, $C_0[M] = C_0 \cap l_\infty[M]$, and we identify $(\sum l_\infty[N_k])_{l_\infty}$ in a natural manner with l_∞, and $(\sum C_0[N_k])_{l_\infty}$ with subspace of l_∞. Clearly, the latter space is isometrically isomorphic to $l_\infty(C_0)$. In Corollary 4.2 below we show that whenever X is a subspace of l_∞ isomorphic to $l_\infty(C_0)$, then $l_\infty/X \approx l_\infty(C)$. We digress momentarily to recall some results concerning l_∞.

(A) If X and Y are isomorphic closed subspaces of l_∞ such that both l_∞/X and l_∞/Y are nonreflexive, then every isomorphism between X and Y extends to an automorphism of l_∞; in particular, $l_\infty/X \approx l_\infty/Y$ [8; 2.f.12(i)].

(B) If X is a closed subspace of l_∞, then l_∞/X is nonreflexive if and only if l_∞/X contains an (isomorphic) copy of l_∞ [8; the proof of 2.f.12(i)].

(C) If a Banach space E has a subspace isomorphic to l_∞, and X is a closed subspace of E, then X or E/X must contain a copy of l_∞ [5].

4.1. **Proposition.** *Let X be a closed subspace of l_∞. Then l_∞/X is nonreflexive in each of the following cases*:

(a) $X \not\cong l_\infty$ (*i.e. X contains no copy of l_∞*),

(b) $X = U \oplus V$, *where* $U \approx l_\infty$, $\dim V = \infty$, *and* $V \not\cong l_\infty$,

(c) $X = U \oplus V$, *where V is nonreflexive and* $V \not\cong l_\infty$.

Proof. (a) See [8; 2.f.13].

Let $Q: l_\infty \to l_\infty/U$ be the quotient map, and note that

$$l_\infty/X \approx (l_\infty/U)/Q(V).$$

(b) Since $\dim(l_\infty/U) = \infty$ and $U \approx l_\infty$, $l_\infty/U \approx l_\infty$ ([8; 2.a.7]); moreover, $Q(V) \approx V \not\cong l_\infty$, hence l_∞/X is nonreflexive by (a) (or (C)).

(c) Since $l_\infty/U \supset Q(V) \approx V$ and V is nonreflexive, neither is l_∞/U. Hence, by (B), $l_\infty/U \not\cong l_\infty$. But $Q(V) \not\cong l_\infty$, hence l_∞/X must contain a copy of l_∞, by (C).

4.2. **Corollary.** *Let X be a subspace of l_∞ isomorphic to $l_\infty(C_0)$. Then*

$$l_\infty/X \approx l_\infty(C).$$

Proof. If $Y = (\sum C_0[N_k])_{l_\infty} \subset l_\infty$, as at the beginning of this section, then $Y \approx l_\infty(C_0)$ and $l_\infty/Y \approx l_\infty(C)$. Since $l_\infty(C_0) \cong l_\infty(C_0) \oplus C_0$, it is enough to apply (A) and Proposition 4.1(c).

From Proposition 3.2 we know that, under CH, $l_\infty(C)$ is isomorphic to a subspace of C. We are going to show now that such an isomorphic embedding cannot be induced by an operator of l_∞ into itself.

4.3. **Proposition.** *Let X and Y be subspaces of l_∞, with $X \approx l_\infty(C_0)$ and $Y \approx C_0$. There exists no operator $T: l_\infty \to l_\infty$ mapping X into Y and such that the induced operator $\widetilde{T}: l_\infty/X \to l_\infty/Y$ is an isomorphic embedding.*

Proof. Recall that if $Q: l_\infty \to l_\infty/X$ and $q: l_\infty \to l_\infty/Y$ are the quotient maps then $\widetilde{T}$ is the unique operator for which $\widetilde{T}Q = qT$.

In view of Corollary 4.2, Proposition 4.1(a), and (A) (see [8; p. 211]), we may assume that $X = (\sum C_0[N_k])_{l_\infty}$ (as at the beginning of this section) and $Y = C_0$. In what follows we denote X thus represented simply by $l_\infty(C_0)$.

Now, let $T: l_\infty \to l_\infty$ be an operator such that $T[l_\infty(C_0)] \subset C_0$, and let $f_j \in (l_\infty)^*$ $(j = 1, 2, \dots)$ be the coordinate maps of T: $T(x) = (f_j(x))$ for every $x \in l_\infty$. For each j let μ_j be the bounded finitely additive measure on the power set $P(\omega)$ of ω representing f_j. Note that $\sup_j |\mu_j(A)| = \|T(1_A)\|$ for all $A \subset \omega$.

$(*)$ *Claim.* For every $\varepsilon > 0$, there exists a k such that whenever A is a finite subset of $N_k \cup N_{k+1} \cup \cdots$, then $\|T(1_A)\| < \varepsilon$.

Suppose it is not so. Then we can find an $\varepsilon > 0$, a strictly increasing sequence (k_n) of indices, and a sequence (A_n) of finite sets such that, for each n, $A_n \subset \cup[N_k : k_n \le k < k_{n+1}\}$ and $\|T(1_{A_n})\| \ge \varepsilon$.

Since $l_\infty \cong (\sum l_\infty[A_n]_{l_\infty} \subset l_\infty(C_0)$, a result of H. P. Rosenthal [9] applied to the operator $T|(\sum l_\infty[A_n])_{l_\infty} : (\sum l_\infty[A_n])_{l_\infty} \to C_0$ shows that this operator must be an isomorphism on a subspace isomorphic to l_∞; a contradiction.

Now, for each k, pick an infinite subset M_k of N_k so that $\mu_j|(M_k)$ is σ-additive for all j (see [1; p.38]). Then from $(*)$ it follows that $(**)$ for every

$\varepsilon > 0$, there exists a k such that if $p \geq K$ and A is any subset of $M_k \cup \cdots \cup M_p$, then $\|T(1_A)\| < \varepsilon$.

Suppose $\widetilde{T}$ is an isomorphic embedding. Then

$$\eta = \inf\{\|\widetilde{T}(Qx)\| : \|Qx\| = 1\} > 0;$$

in particular, since $\widetilde{T}Q = qT$, we must have

$$\|T(1_A)\| \geq \|q(T(1_A))\| = \|\widetilde{T}(Q1_A))\| \geq \eta$$

for every set $A \subset \omega$ that has an infinite intersection with at least one of the sets N_k. It follows that $\|T(1_{M_k})\| \geq \eta$ for every k, contradicting $(**)$.

4.4. **Proposition** (CH). $l_\infty(C)/C_0(C) \cong C$.

Proof. As in the proof of Proposition 3.2, let A be the union of a sequence (A_n) of pairwise disjoint nonempty clopen subsets of ω^*. Then, identifying $l_\infty(C)$ with $(\sum C(A_n))_{l_\infty}$, there is a natural isometric isomorphism between $l_\infty(C)$ and $C(\overline{A})$ in which the function $f \in C(\overline{A})$ corresponding to an element (f_n) of $l_\infty(C)$ is such that $f|A_n = f_n$ for all n. Under this isomorphism the subspace $C_0(C)$ of $l_\infty(C)$ is mapped onto the subspace in $C(\overline{A})$ consisting of functions vanishing on the boundary $\partial A = \overline{A} \sim A$ of A. It follows that $l_\infty(C)/C_0(C) \cong C(\partial A)$. Finally, by a result of Gillman [14; p. 161], assuming CH we have that ∂A is homeomorphic with ω^* and therefore $C(\partial A) \cong C$.

We conclude with a few remarks.

We used CH only to show that $l_\infty/c_0 \approx (\sum \oplus l_\infty/c_0)_{l_\infty}$.

It may be possible to show that l_∞/c_0 is primary without this result and therefore without CH. However, it is likely that l_∞/c_0 being primary implies this decomposition. For example, in [2] it it shown that $X = (\Sigma \oplus E_n)_{l_p}$, $\dim E_n < \infty$, is primary implies $X \approx (\Sigma \oplus X)_{l_p}$. A variation of this technique may show that the l_∞-decomposition of l_∞/c_0 is necessary (as well as sufficient) for l_∞/c_0 to be primary. A famous open problem in Banach space theory is the Schroeder–Bernstein problem: If two Banach spaces X and Y embed complementably into one another, must X and Y be isomorphic? We have shown that whenever $l_\infty/c_0 \approx X \oplus Y$ then l_∞/c_0 embeds complementably into either X or Y (without using CH). So if l_∞/c_0 fails to be primary without CH, then it is also a counterexample to the Schroeder–Bernstein property. This leaves open the possibility that the Schroeder–Bernstein problem has a negative answer in ZF but a positive answer with CH.

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