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THE BLOCH CONSTANT OF BOUNDED ANALYTIC FUNCTIONS ON A MULTIPLY CONNECTED DOMAIN

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ABSTRACT. Let F be an analytic function on the bounded domain R, a multiply-connected region of the complex plane. The Bloch constant of F is defined by

$$\beta_F = \sup_{|z|<1} (1 - |z|^2)|(F \circ p)'(z)|,$$

where p is a conformal universal cover of R with domain Δ, the open unit disk. If F is bounded, then $\beta_F \leq \|F\|_\infty$, the sup-norm of F. In this paper we characterize those functions F for which $\beta_F = \|F\|_\infty$ in terms of the zeros of F when the boundary of R is the union of finitely many curves. We conclude this paper by showing the existence of extremal functions, and generalizing the results to bounded harmonic mappings on these domains.

1. INTRODUCTION

On any Riemannian manifold M, it is natural to ask which complex-valued functions are Lipschitz. If we are given a class of Lipschitz functions whose Lipschitz numbers are bounded, a further natural problem is to classify those functions in the class whose Lipschitz numbers are the maximum possible. In [3] the problem was solved for bounded analytic functions on the unit disk. Most of this paper is devoted to studying bounded analytic functions on planar domains (with some restrictions), although in §4 we briefly consider analogous results for complex–valued harmonic functions.

The techniques used here are somewhat different from those of [3]. There, the main result characterizes the extremal functions in terms of the distribution of their zeros, and at the same time provides a recipe for constructing examples. In the present paper, the statement of the main result (Theorem 2) has exactly the same flavor. The proof requires finding methods for handling the technical difficulties that arise during the process of pulling back the problem to the unit disk. It would certainly not be surprising if Theorem 2 were true in greater generality than we have done here, but it might very well be an empty condition: The methods we use in the case of the planar surfaces for constructing maximal

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bounded analytic functions are based on the work of Coifman and Weiss [2] and absolutely require this restriction on R. To generalize beyond, new methods would have to be developed to construct extremal functions. Even if Theorem 2 does generalize completely, it will be quite a challenge to produce examples.

Let f be analytic on Δ, the open unit disk. The *Bloch constant* of f is defined as

$$\beta_f = \sup_{|z|<1} (1 - |z|^2)|f'(z)|,$$

and f is called a *Bloch function* if β_f is finite. The correspondence $f \mapsto \beta_f$ is invariant under right composition by conformal automorphisms of Δ and defines a semi-norm. Bloch functions constitute a complex Banach space under the *Bloch norm* $\|f\| = |f(0)| + \beta_f$. Moreover a function f is Bloch if and only if it is Lipschitz with respect to the hyperbolic distance σ on Δ and the euclidean distance in $\boldsymbol{C}$ and β_f is the Lipschitz number of f; namely,

$$\beta_f = \sup_{z \neq w} \frac{|f(z) - f(w)|}{\sigma(z, w)}.$$

(Cf. [4, Theorem 10].) The density we use for σ is $dz/(1 - |z|^2)$, rather than the usual $2dz/(1 - |z|^2)$, to avoid an extra factor of 2 in a number of formulas.

In [3] it is observed that the Bloch constant of a bounded analytic function never exceeds its sup-norm, and those functions f for which $\beta_f = \sup_{|z|<1} |f(z)|$ are characterized. Normalizing, we may consider bounded functions as having image in Δ and the result becomes the following:

Theorem 1. *Let $f : \Delta \to \Delta$ be analytic. Then $\beta_f = 1$ if and only if either f is a conformal automorphism of Δ or the zeros of f form an infinite sequence $(z_n)_{n \in \boldsymbol{N}}$ such that*

$$\limsup_{n \to \infty} |g(z_n)| \prod_{k \neq n} \left| \frac{z_k - z_n}{1 - \overline{z}_k z_n} \right| = 1,$$

where g is a nonvanishing factor of f such that f/g is a Blaschke product.

The above condition can be expressed in terms of the pseudohyperbolic distance for the unit disk: $\Psi(z, w) = |(z - w)/(1 - \overline{z}w)|$, for $z, w \in \Delta$. The function Ψ is symmetric, invariant under the action of the group of conformal automorphisms of Δ, and $\lim_{|z| \to 1} \Psi(z, w) = 1$, for any $w \in \Delta$.

The notion of Bloch function can be generalized to analytic functions defined on other types of domains. The main purpose of this paper is to show that Theorem 1 may be generalized to functions on bounded planar domains with boundary a finite union of curves. In §3 we shall prove the existence of analytic functions $F : R \to \Delta$ satisfying the extremal condition $\beta_F = 1$.

Maurice Heins asked whether Theorem 1 might be generalized to the annulus. The author originally did this using results of [8, 9, 12] very specific to the annulus. John Garnett then suggested that smoothness of the boundary might

be enough to generalize the result to planar domains which turned out to be the case. The author wishes to thank them both for very helpful conversations.

2. Bloch functions on multiply-connected domains

Let R be a Riemann surface of hyperbolic type and let F be an analytic function with domain R. If $p_1, p_2 : \Delta \to R$ are conformal universal covers then $p_1 = p_2 \circ \gamma$, for some γ in the group of cover transformations. Hence $\beta_{F\circ p_2} = \beta_{F\circ p_1\circ\gamma} = \beta_{F\circ p_1}$. Thus we may define F to be a *Bloch function* if $\beta_{F\circ p}$ is finite for some, hence for every, conformal universal cover p. The number $\beta_F = \beta_{F\circ p}$ is called the *Bloch constant* of F.

Throughout we will let $a, b, \ldots$ be points of R and $z_a, z_b, \ldots$ will always represent chosen points of Δ such that $p(z_a) = a$, $p(z_b) = b, \ldots$.

Fix $a \in R$ and let $G(a, -)$ denote the Green's function on R with singularity at a. For $R = \Delta$ we have $G(a, b) = -\log|(a-b)/(1-\bar{a}b)|$. Hence $\Psi(a, b) = e^{-G(a,b)}$. This motivates the following:

Definition 1. Let R be a Riemann surface of hyperbolic type. For $a, b \in R$ define the *pseudohyperbolic distance* between a and b as the number $\Phi(a, b) = e^{-G(a,b)}$.

Using standard properties of the Green's function it follows that Φ satisfies the following properties:

(1) Φ is symmetric;
(2) Φ is invariant under composition by conformal automorphisms of R;
(3) $\sup_{b\in R}\Phi(a, b) = 1$, for any $a \in R$.

In this work we generalize Theorem 1 to the case of analytic functions $f : R \to \Delta$. In order to do so, however, we need to put some restrictions on the surface R. We require that R be a finitely-connected planar region with no isolated boundary points. By a theorem of Koebe (cf. [1, p. 211]), R is conformally equivalent to a bounded region with boundary a finite number of simple closed analytic curves (in fact they can be taken as circles). Keeping in mind that all our results are invariant under conformal equivalence, without loss of generality, we may assume in the proofs that R is a bounded planar region whose boundary is the union of simple closed analytic curves. Then fixing $a \in R$, the Green's function $G(a, -)$ vanishes continuously on the boundary of R. Hence, in analogy to the pseudohyperbolic distance on the unit disk we have that $\Phi(a, b) \to 1$, as b approaches a boundary point of R. Furthermore $\Phi(a, b)$ is closely related to the pseudohyperbolic distance of the points in the orbits of a and b by the formula

$$\Phi(a, b) = \prod_{\gamma\in\Gamma}\Psi(z_a, \gamma z_b), \tag{1}$$

where Γ is the group of covering transformations of R. This is equivalent to

the equality:

$$G[p(z), p(z')] = -\sum_{\gamma\in\Gamma} \log\left|\frac{z-\gamma z'}{1-\bar{z}\gamma z'}\right|$$

for $z, z' \in \Delta$, where p is a conformal universal cover of R with domain Δ (cf. e.g. [13, p. 529]).

It is well known that for $a \in R$ the sequence of points in the orbit of a satisfies the Blaschke condition $\sum_{p(z)=a}(1-|z|) < \infty$ [11, p. 338]. Let B_a be the corresponding Blaschke product. It follows that B_a is modulus automorphic; namely, $|B_a \circ \gamma| = |B_a|$, for all $\gamma \in \Gamma$. Note that $\Phi(a, b) = |B_a(z_b)|$.

Let F be analytic on R with $F(R) \subset \Delta$. Then $F \circ p = g\prod_n B_{a_n}$, where (a_n) constitutes the sequence of the zeros of F and $g : \Delta \to \Delta$ is nonvanishing. Since each B_{a_n} is modulus automorphic, so is g. Hence $|g|$ induces a function $\hat{g}$ on R by $\hat{g}(a) = |g(z_a)|$. Thus the modulus of F may be expressed as follows:

$$|F(a)| = \hat{g}(a)\prod_n |B_{a_n}(a)| = \hat{g}(a)\prod_n \Phi(a_n, a).$$

Theorem 2. *Let R be a finitely-connected domain with no isolated boundary points, and let F be an analytic function on R such that $|F(a)| < 1$ for all $a \in R$. Then $\beta_F = 1$ if and only if the zeros of F form an infinite sequence $(a_n)_{n\in N}$ satisfying the condition*

$$\limsup_{n\to\infty} \hat{g}(a_n)\prod_{k\neq n}\Phi(a_k, a_n) = 1. \tag{2}$$

Proof. Let p be a conformal universal cover of R with domain Δ, and let Γ be the group of covering transformations. Let $f = F \circ p : \Delta \to \Delta$. Since f cannot be a conformal automorphism, Theorem 1 implies that $\beta_f = 1$ if and only if the zeros of f form an infinite sequence (ζ_n) satisfying the condition

$$\limsup_{n\to\infty} |g(\zeta_n)|\prod_{k\neq n}\Psi(\zeta_k, \zeta_n) = 1, \tag{3}$$

where g is a nonvanishing factor of f such that f/g is a Blaschke product.

Let $\pi(\zeta_n) = \prod_{k\neq n}\Psi(\zeta_k, \zeta_n)$, for all $n \in N$. Assume that $p(\zeta_n) = p(\zeta_m)$. Thus $\zeta_n = \gamma\zeta_m$ for some $\gamma \in \Gamma$. Using the invariance of the pseudohyperbolic distance under the action of the group Γ, we then see that $\pi(\zeta_n) = \pi(\zeta_m)$. Thus $\pi(\zeta)$, which is less than 1, depends only on the value $p(\zeta)$. Hence the set $Z = \{p(\zeta_n) : n \in \boldsymbol{N}\}$ of the zeros of F must be infinite in order for β_F to be equal to 1. Let $a_1, a_2, \ldots$ be an enumeration of Z. Let $\pi_n = \pi(\zeta_t)$, where $p(\zeta_t) = a_n$. For each $j \in \boldsymbol{N}$, pick z_j in $p^{-1}(a_j)$. Since $p^{-1}(a_j) = \{\gamma z_j : \gamma \in \Gamma\}$, the sequence $(\zeta_n)_{n\in\boldsymbol{N}}$ of the zeros of f can be represented in the form $(\gamma z_j)_{j\in\boldsymbol{N},\gamma\in\Gamma}$. Fix a zero of f, say $\zeta_t = \gamma_o z_n$. Then $|g(\zeta_t)| = \hat{g}(a_n)$ and $\pi_n = \prod_{(k,\gamma)\neq(n,\gamma_o)}\Psi(\gamma z_k, \gamma_o z_n)$, which can be decomposed as follows:

$$\pi_n = \prod_{\gamma\neq\gamma_o}\Psi(\gamma z_n, \gamma_o z_n)\prod_{k\neq n,\gamma\in\Gamma}\Psi(\gamma z_k, \gamma_o z_n).$$

Note that $\prod_{k\neq n,\gamma\in\Gamma}\Psi(\gamma z_k, \gamma_o z_n) = \prod_{k\neq n}\Phi(a_k, a_n)$ by (1). For $a \in R$ let us define $\theta(a) = \prod_{1\neq\gamma\in\Gamma}\Psi(\gamma z_a, z_a)$, where 1 is the identity. Notice that this is independent of the choice of the point z_a in $p^{-1}(a)$. Then $\pi_n = \theta(a_n)\prod_{k\neq n}\Phi(a_k, a_n)$, and thus (3) becomes

$$\limsup_{n\to\infty} \hat{g}(a_n)\theta(a_n)\prod_{k\neq n}\Phi(a_k, a_n) = 1. \tag{4}$$

Since $\theta(a_n)$, $\hat{g}(a_n)$ and $\Phi(a_k, a_n)$ belong to the interval $(0, 1)$, formula (4) implies that

$$\limsup_{n\to\infty} \hat{g}(a_n)\prod_{k\neq n}\Phi(a_k, a_n) = 1.$$

So if $\beta_F = 1$ then F must possess infinitely many zeros a_n, $n \in \boldsymbol{N}$, such that (2) holds.

For the converse we need to show that if (2) holds then $\lim_{n\to\infty}\theta(a_n) = 1$, yielding (4). Since (4) is equivalent to (3), it follows that $\beta_F = \beta_f = 1$.

Note that, unless F is identically zero, the sequence (a_n) cannot have a limit point inside R, and thus as n tends to infinity, a_n approaches $\mathrm{fr}(R)$. Then the converse follows from the following result:

Theorem 3. *As a approaches the boundary of R, the number $\theta(a)$ tends to* 1.

Proof. We first need to study the Green's function near the boundary of R. For $a \in R$ let $b \mapsto \mu(a, b)$ be the solution to the Dirichlet problem with boundary values $\log|a - b|$. So

$$\mu(a, b) = G(a, b) + \log|a - b| \quad \text{for } b \neq a.$$

Let V denote a simply connected subregion of R on which an inverse q of p is defined: $p[q(a)] = a$ for all $a \in V$. Since we are only interested in the behavior of θ near the boundary we add the assumption that the common boundary $C = \mathrm{fr}\, V \cap \mathrm{fr}\, R$ be an analytic arc. Furthermore, by shrinking V if necessary, we may assume that q is defined in a neighborhood of V in R.

For $a, b \in V$, $a \neq b$, we define

$$\xi(a, b) = \mu(a, b) - \log|1 - \overline{q(a)}q(b)| + \log\left|\frac{q(a) - q(b)}{a - b}\right|.$$

Letting $\xi(a, a) = \mu(a, a) - \log(1 - |q(a)|^2) + \log|q'(a)|$, we have that ξ is continuous on $V \times V$. Later we shall show that $\xi(a, a) = -\log[\theta(a)]$. Since

$$\xi(a, b) = G(a, b) + \log\Psi[q(a), q(b)], \tag{5}$$

for $a \neq b$, it follows that for $a \in V$ the function $\xi(a, -)$ is harmonic on $V - \{a\}$ and hence on V. We also note that fixing $b \in V$, $\lim_{a\to\zeta} G(a, b) = 0$ and $\lim_{a\to\zeta}\Psi[q(a), q(b)] = 1$ for all $\zeta \in C$. Thus setting $\xi(b, \zeta) = \xi(\zeta, b) = 0$, we get that ξ is continuous on $\{b\} \times \overline{V} \cup \overline{V} \times \{b\}$. Now let S be a subset of V which has positive distance from the boundary of R. Then by the regularity of the solution to the Dirichlet problem, μ is continuous in $S \times \overline{R}$. Thus ξ

is continuous in $(S \cap V) \times \overline{V}$. Our aim is to show that for every point $\zeta \in C$, $\lim_{a \to \zeta} \xi(a, a) = 0$.

Since C is an analytic arc, the function q can be analytically extended across C. In addition, fixing b the function $\xi(-, b)$ vanishes on C. Thus $\xi(-, b)$ can also be extended harmonically across C, and thus μ can be also. Call $\widetilde{V}$ the domain of the extended functions with $\widetilde{V} \cap R = V$. Fix $\zeta \in C$ and let r be chosen so that $\{a : |a - \zeta| \leq r\} \subset \widetilde{V}$. Let $s \in (0, r)$ and set $D = \{a : |a - \zeta| < s\} \cap V$. Thus for any $b \in D$ we have the representation

$$\xi(a, b) = \frac{1}{2\pi} \int_0^{2\pi} \xi(\zeta + re^{i\theta}, b) P(\theta, a)\, d\theta, \tag{6}$$

where P denotes the Poisson kernel in the circle $C_r = \{|a - \zeta| = r\}$. It would be nice to use (6) to conclude the continuity of ξ, but we can only be sure of the continuity of ξ away from $C \times C$. The rest of the proof is aimed at eliminating this problem. Let $M > \log(\operatorname{diam} R)$. For $b \in R$ define $u_b(a) = M - \log|a - b|$, $a \in R$. Then u_b is positive superharmonic on R and the function $a \mapsto u_b(a) + \log|a - b| = M$ is trivially harmonic. By the minimality property of the Green's function we have $G(-, b) \leq u_b$. In particular, then, $\mu(a, b) \leq M$, and so $\xi(a, b) \leq M - \log|a - b| + \log \Psi[q(a), q(b)]$, for $a \in \widetilde{V}$, and $b \in V$, and thus ξ is bounded on $C_r \times D$. Recall that $\lim_{a \to \zeta} \xi(a, b) = 0$. Since the integrand in (6) is bounded for all $a, b \in D$, and $0 \leq \theta \leq 2\pi$, for every $\varepsilon > 0$ there exists a subset $J_\varepsilon \subset [0, 2\pi]$ such that the set $\{\zeta + re^{i\theta} : \theta \in J_\varepsilon\}$ has positive distance from C and

$$\left| \frac{1}{2\pi} \int_{[0, 2\pi] \setminus J_\varepsilon} \xi(\zeta + re^{i\theta}, b) P(\theta, a)\, d\theta \right| < \varepsilon.$$

Defining $\xi_\varepsilon(a, b) = \frac{1}{2\pi} \int_{J_\varepsilon} \xi(\zeta + re^{i\theta}, b) P(\theta, a) d\theta$, we then have $|\xi(a, b) - \xi_\varepsilon(a, b)| < \varepsilon$. Since for $\theta \in J_\varepsilon$, the points $\zeta + re^{i\theta}$ are bounded away from C, both $\xi(\zeta + re^{i\theta}, b)$ and $P(\theta, a)$ are continuous for $\theta \in J_\varepsilon$, and $a, b \in \overline{D}$. Thus ξ_ε is continuous on $\overline{D} \times \overline{D}$. Since $\lim_{b \to \zeta} \xi(a, b) = 0$, for $a \in D$ fixed, $|\xi_\varepsilon(a, \zeta)| \leq \varepsilon$, hence $|\xi_\varepsilon(\zeta, \zeta)| = \lim_{a \to \zeta} |\xi_\varepsilon(a, \zeta)| \leq \varepsilon$. By the continuity of ξ_ε on $\overline{D} \times \overline{D}$, there exists a neighborhood W of ζ such that $|\xi_\varepsilon(a, b) - \xi_\varepsilon(\zeta, \zeta)| < \varepsilon$, for all $a, b \in W$. So $|\xi_\varepsilon(a, b)| < 2\varepsilon$, and therefore $|\xi(a, b)| < 3\varepsilon$, for $a, b \in W$. Since ε was arbitrary, we conclude that $\lim_{a \to \zeta} \xi(a, a) = 0$. To conclude our argument note that for $a \in V$ we can write

$$\begin{aligned} \theta(a) &= \prod_{1 \neq \gamma \in \Gamma} \Psi(\gamma q(a), q(a)) = \lim_{b \to a} \frac{\prod_{\gamma \in \Gamma} \Psi(\gamma q(a), q(b))}{\Psi(q(a), q(b))} \\ &= \lim_{b \to a} \frac{\Phi(a, b)}{\Psi(q(a), q(b))} = \lim_{b \to a} e^{-\xi(a, b)} = e^{-\xi(a, a)} \end{aligned}$$

by (5). Therefore $\theta(a)$ approaches 1 as a tends to a boundary point of R.

3. Existence of extremal functions

Under the usual hypotheses on R we are going to prove the existence of an analytic function $F : R \to \Delta$ which maximizes the Bloch constant. For the purpose we shall make use of the definition of a Blaschke product for R given by Coifman and Weiss in [2].

Let $fr(R) = \bigcup_{k=1}^{m} C_k$, where each C_k is a simple closed analytic curve and C_m is the exterior boundary component. The unbounded connected component of the complement of C_m does not contain any of the curves C_k, $1 \le k \le m-1$. Let $\mathrm{ds}(\tau)$ be the element of arc length on $\mathrm{fr}(R)$. For $z \in R$, $\tau \in \mathrm{fr}(R)$, Coifman and Weiss defined a Poisson kernel $P(z, \tau)$ for R and observed that, for a fixed, the function with domain R defined by

$$b(z, a) = (z - a) \int_{fr(R)} P(z, \tau) \log|\tau - a| \, \mathrm{ds}(\tau),$$

is a conformal mapping of R onto Δ with $m-1$ concentric circular slits removed, or onto an annulus $\{r < |z| < 1\}$ minus $m-2$ concentric circular slits, depending on whether a is an interior or a boundary point of R. They observed that if $a' \in C_k$, $1 \le k < m$, then $b(z, a')$ maps C_m onto the outer boundary and C_k onto the inner boundary of the annulus $\{r < |z| < 1\}$. If $a' \in C_m$, then $b(z, a')$ is a constant of modulus one. In addition, assuming that $a \in R$ is closer to C_k than to any other boundary component, if a' is the closest point on the boundary to a, then $|b(z, a)| \to 1$, as z approaches a'.

For a sequence $(a_n)_{n \in \boldsymbol{N}}$ in R, let a_n' be a point of $\mathrm{fr}(R)$ such that $|a_n - a_n'| = \min_{\zeta \in \mathrm{fr}(R)} |a_n - \zeta|$ and assume that $\sum_{n=1}^{\infty} |a_n - a_n'| < \infty$. This is the analogue of the Blaschke condition. Then the function

$$B(z, \{a_n\}) = \prod_{n=1}^{\infty} \frac{b(z, a_n)}{b(z, a_n')}$$

is analytic on R. Coifman and Weiss named $B(z, \{a_n\})$ a *Blaschke product* relative to $(a_n)_{n \in \boldsymbol{N}}$.

Let $(a_n)_{n \in \boldsymbol{N}}$ be an infinite sequence of points of R satisfying the following conditions:

(1) The cluster points of $(a_n)_{n \in \boldsymbol{N}}$ are contained in C_m.
(2) For each n, $\mathrm{dist}(a_n, \mathrm{fr}(R)) = \mathrm{dist}(a_n, C_m)$.
(3) $\sum_n \mathrm{dist}(a_n, \mathrm{fr}(R)) < \infty$.

It follows that for every integer n, there exists $a_n' \in C_m$ such that $\sum_n |a_n - a_n'| = \sum_n \mathrm{dist}(a_n, \mathrm{fr}(R)) < \infty$, and the Blaschke product relative to $(a_n)_{n \in \boldsymbol{N}}$ has the form $B(z, \{a_n\}) = \prod_{n \in \boldsymbol{N}} \lambda_n b(z, a_n)$, where λ_n are constants of modulus one. Thus the image of B is contained in the unit disk. From B we shall extract an extremal function by taking a suitable subsequence of its zeros. But we first need a technical lemma.

Lemma. *Let* $0 < \alpha_{n,j} < 1$ *for* $n \ne j$, *and* $0 < \alpha_{n,n} \le 1$. *Assume that for all* j, $\lim_{n \to \infty} \alpha_{n,j} = 1$ *and for all* n, $A_n = \prod_j \alpha_{n,j} > 0$. *Then there exists a*

sequence $(k_n)_{n\in\mathbf{N}}$ *of positive integers such that*

$$\lim_{n\to\infty}\prod_{j\neq k}\alpha_{k_n,k_j}=1.$$

Proof. The result follows by constructing k_n inductively to satisfy the following conditions:

$$(H_t)\qquad \begin{aligned}&\alpha_{k_{t+1},k_1}\alpha_{k_{t+1},k_2}\alpha_{k_{t+1},k_3}\cdots\alpha_{k_{t+1},k_t}>1-\frac{1}{2^{t+1}}\text{ and}\\ &\alpha_{k_t,k_1}\alpha_{k_t,k_2}\alpha_{k_t,k_3}\cdots\alpha_{k_t,k_{t-1}}\prod_{j\geq k_{t+1}}\alpha_{k_t,j}>1-\frac{1}{2^t}.\end{aligned}$$

We first observe that if $(a_j)_{j\in\mathbf{N}}$ is a sequence of points of the interval (0,1) and δ is a positive constant such that $\prod a_j<\delta$, then $\prod_{j=1}^{k-1}a_j<\delta$, for all k sufficiently large.

Now let $k_1=1$. Then let k_2 be any integer k such that

$$\alpha_{k,1}>\frac{3}{4}\quad\text{and}\quad\alpha_{1,2}\alpha_{1,3}\cdots\alpha_{1,k-1}<2\prod_{j\geq 2}\alpha_{1,j}.$$

The first inequality is satisfied from some point on by the hypothesis, and the second because of the observation. We now get

$$\prod_{j\geq k_2}\alpha_{1,j}=\frac{\prod_{j\geq 2}\alpha_{1,j}}{\alpha_{1,2}\alpha_{1,3}\cdots\alpha_{1,k_2-1}}>\frac{1}{2}$$

which gives the stated conditions for $t=1$. Continuing now, since $\alpha_{k_2,k_1}>\frac{3}{4}$, we have $\prod_{j\geq k_2}\alpha_{k_2,j}<\frac{4}{3}\alpha_{k_2,k_1}\prod_{j\geq k_2}\alpha_{k_2,j}$. Thus we can find some k_3 such that

$$\alpha_{k_3,k_1}\alpha_{k_3,k_2}>\frac{7}{8}\quad\text{and}\quad\alpha_{k_2,k_2+1}\alpha_{k_2,k_2+2}\cdots\alpha_{k_2,k_2+k_3-1}<\frac{4}{3}\alpha_{k_2,k_1}\prod_{j\geq k_2}\alpha_{k_2,j}.$$

For this value of k_3 we have that

$$\alpha_{k_2,k_1}\alpha_{k_2,k_2}\prod_{j\geq k_3}\alpha_{k_2,j}>\frac{3}{4}$$

which gives us the relations (H_t) for $t=2$. Assume that we have found $k_1,\dots,k_t$ satisfying the sets of conditions $(H_1),\dots,(H_{t-1})$. Now from the first condition of (H_{t-1}), we see that $\mu=2^t\alpha_{k_t,k_1}\cdots\alpha_{k_t,k_{t-1}}/(2^t-1)>1$. So by the observation following the definition of (H_t), we deduce that $\prod_{j=1}^{k-1}\alpha_{k_t,j}<\mu\prod_{j=1}^{\infty}\alpha_{k_t,j}$, for all k sufficiently large. This inequality can be rewritten as $\mu\prod_{j=k}^{\infty}\alpha_{k_t,j}>1$, which implies that $\prod_{j=k}^{\infty}\alpha_{k_t,j}>1-1/2^t$. On the other hand, since $\lim_{k\to\infty}\alpha_{k,k_1}\cdots\alpha_{k,k_t}=1$, the inequality $\alpha_{k,k_1}\cdots\alpha_{k,k_t}>1-1/2^{t+1}$ holds for all k sufficiently large. Let k_{t+1} be any k large enough to satisfy

both conditions, and notice that this yields both conditions of (H_t). This sequence will now satisfy the conclusion of the Lemma.

Relative to each pair of integers (n, j) we define $\alpha_{n,j} = \Phi(a_n, a_j)$ for $n \neq j$ and $\alpha_{n,n} = 1$. This sequence satisfies the hypotheses of the lemma. Let $(k_n)_{n \in \boldsymbol{N}}$ be the sequence of integers given by the lemma. Thus the Blaschke product for R relative to the sequence $(a_{k_n})_{n \in \boldsymbol{N}}$ satisfies the condition

$$\lim_{n \to \infty} \prod_{j \neq n} \Phi(a_{k_n}, a_{k_j}) = 1. \tag{7}$$

For simplicity of notation we shall rename as a_n the elements of this subsequence and B_o the corresponding Blaschke product for R. Let $\hat{g}_o$ be the factor of $|B_o|$ induced by the modulus automorphic non-vanishing factor of the decomposition $B_o \circ p = b_o g_o$, where b_o is a Blaschke product. Note that if $\hat{g}_j$ is the corresponding factor of the modulus of the function $b_j : a \in R \mapsto b(a, a_j)$, then $\hat{g}_o = \prod_j \hat{g}_j$. Set $\hat{\alpha}_{n,j} = \hat{g}_j(a_n)$. Since $|B_o(a)| \to 1$ as a approaches C_m, $\hat{g}_o(a)$ does also, whence every factor does and so $\hat{g}_j(a_n) \to 1$. Thus we may reapply the lemma to the sequence $(\hat{\alpha}_{n,j})$: there exists a sequence $(s_n)_{n \in \boldsymbol{N}}$ of positive integers such that

$$\lim_{n \to \infty} \prod_j \hat{g}_{s_j}(a_{s_n}) = 1. \tag{8}$$

Let B_1 be the Blaschke product relative to the sequence $(b_n)_{n \in \boldsymbol{N}}$, where $b_n = a_{s_n}$. Setting $\hat{g}(b) = \prod_j \hat{g}_{s_j}(b)$, from (7) and (8) we obtain

$$\lim_{n \to \infty} \hat{g}(b_n) \prod_{j \neq n} \Phi(b_n, b_j) = 1,$$

which proves that $\beta_{B_1} = 1$.

4. Bloch harmonic mappings on multiply connected domains

A *harmonic mapping* is a complex-valued function of the form $u + iv$, where u and v are harmonic functions. In particular a harmonic mapping is analytic if v is a harmonic conjugate of u. The characterization of a Bloch function as a Lipschitz function of the hyperbolic disk into the euclidean plane suggests a way of defining *Bloch* for classes of functions other than analytic.

Definition 2. A harmonic mapping h with domain Δ is called *Bloch* if it satisfies the Lipschitz condition with a global constant when regarded as a function from the hyperbolic disk into $\boldsymbol{C}$, endowed with the euclidean metric. The Lipschitz number of h

$$\beta_h = \sup_{z \neq w} \frac{|h(z) - h(w)|}{\sigma(z, w)}$$

is called the *Bloch constant of h.*

For a further reference to Bloch harmonic mappings, cf. [5].

A harmonic mapping can be decomposed into a sum of the form $f+\overline{g}$, where f and g are analytic functions. Furthermore this decomposition is unique up to an additive constant.

In [5] it was shown that the Bloch constant of a harmonic mapping $h = f+\overline{g}$ can be expressed in terms of the derivatives of f and g by

$$\beta_h = \sup_{|z|<1}(1-|z|^2)[|f'(z)|+|g'(z)|].$$

In particular, if u is a harmonic function and v is a harmonic conjugate of u and $f = u+iv$ is the corresponding analytic function, then $\beta_u = \beta_v = \beta_f$.

As in the corresponding analytic case, it is possible to generalize the notion of Bloch harmonic mapping on the disk to other types of domains. A harmonic mapping H with domain R, a hyperbolic Riemann surface, is called *Bloch* if $h = H \circ p$ is a Bloch harmonic mapping in the sense of Definition 2, where p is a conformal universal cover of R with domain Δ. Of course, this definition is independent of the choice of p. Furthermore, we define the Bloch constant of H by $\beta_H = \beta_h$. This may be taken as the definition of β_H, or equivalently, we may define β_H by the formula in Definition 2, letting σ be the hyperbolic metric on R such that p is a local isometry.

In [4] it was shown that a harmonic mapping $h : \Delta \to \Delta$ is necessarily Bloch with Bloch constant $\beta_h \leq 4/\pi$, with equality holding if and only if there exist a constant ζ of modulus one and an extremal analytic function $f : \Delta \to \Delta$ thought of as a map of the hyperbolic disk into itself such that

$$\mathrm{Re}(\zeta h) = \frac{2}{\pi}\,\mathrm{Arg}\left(\frac{1+f}{1-f}\right),$$

where Arg denotes the principal branch of the argument. This result is only partially extendable to non-simply connected domains. Under the usual conditions that R be a bounded domain whose boundary is a finite union of simple closed analytic curves, an immediate consequence of this definition and [5] is the following:

Theorem 4. *Let H be a harmonic mapping with domain R and such that $|H(a)| < 1$ for $a \in R$. Then $\beta_H \leq 4/\pi$. Let $F : R \to \Delta$ be analytic with $\beta_F = 1$. Then for*

$$H = \mathrm{Arg}\left(\frac{1+F}{1-F}\right),$$

we have $\beta_H = 4/\pi$.

But the converse to this result, which holds in the case of the disk, is false in general, even in the simplest case of a doubly-connected region. We now construct a function H on an annulus with $\beta_H = 4/\pi$ but for which there does not exist any function F analytic on the annulus such that $Re(\zeta H) = \frac{2}{\pi}\,\mathrm{Arg}\left(\frac{1+F}{1-F}\right)$. (Actually since H is real-valued we may dispense with the ζ and the taking of the real part.)

Let $0 < s < 1$ and let $\gamma(z) = (s+z)/(1+sz)$. Then γ generates a discrete subgroup Γ of the Möbius transformations taking Δ into itself. The quotient Δ/Γ is topologically an annulus which may be identified with the set

$$R = \{r < |z| < \tfrac{1}{r}\}, \quad \text{where } r = \exp\left(-\frac{\pi^2}{2\log\left(\frac{1+s}{1-s}\right)}\right).$$

Let $p : \Delta \to R$ be a conformal universal cover. The harmonic function

$$h(z) = \frac{2}{\pi}\operatorname{Arg}\left(\frac{1+z}{1-z}\right),$$

with domain Δ, is invariant under the action of γ so, there is a unique harmonic function H with domain R such that $H \circ p = h$. Thus $\beta_H = 4/\pi$. Assume that

$$H(z) = \frac{2}{\pi}\operatorname{Arg}\left(\frac{1+F}{1-F}\right)$$

for some analytic function $F : R \to \Delta$, and let $f = F \circ p$. Then f is analytic on Δ and $f \circ \tau = f$, for all $\tau \in \Gamma$. In particular f is not a conformal automorphism of the disk. But

$$\frac{2}{\pi}\operatorname{Arg}\left(\frac{1+f(z)}{1-f(z)}\right) = \frac{2}{\pi}\operatorname{Arg}\left(\frac{1+z}{1-z}\right),$$

so for all $z \in \Delta$, there exists a real number $r(z)$ such that

$$\frac{1+f(z)}{1-f(z)} = r(z)\frac{1+z}{1-z}.$$

Thus

$$r(z) = \left(\frac{1+f(z)}{1-f(z)}\right)\left(\frac{1-z}{1+z}\right)$$

is analytic on Δ, but it is also real-valued. Hence $r(z)$ is a constant c. So letting $a = (c-1)/(c+1)$, we solve the above equation to get $f(z) = (a+z)/(1+az)$, which is a conformal automorphism of Δ, contradicting the assumption.

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