STRONG SOLUTIONS OF EVOLUTION EQUATIONS GOVERNED BY $m$-ACCRETIVE OPERATORS AND THE RADON-NIKODYM PROPERTY

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Abstract. We construct, in every Banach space which fails the Radon-Nikodym property, a nonlinear operator $A$ which is $m$-accretive for some equivalent norm in $X$, such that the domain of $A$ is not a singleton and such that the only strong solutions of the equation $u' + Au \ni f$ are the constant ones.

1. Introduction

Let $X$ be a Banach space. An operator $A$ in $X$ is a map from $X$ into the power set $2^X$ of $X$. Let us denote

$$D(A) = \{x \in X : Ax \neq \emptyset\}$$

the domain of $A$, and

$$R(A) = \{y \in X : \exists x \in X \text{ such that } y \in Ax\}$$

the range of $A$. We shall freely identify $A$ with the subset of $X \times X$ of all couples $(x, y)$ such that $x \in D(A)$ and $y \in Ax$. The operator $A$ is called accretive if for every $(x, y), (x', y')$ in $A$ and for every $\lambda \geq 0$:

$$\|x - x' + \lambda(y - y')\| \geq \|x - y\|.$$  

The operator $A$ is said to be $m$-accretive if, moreover, for every $\lambda > 0$

$$R(I + \lambda A) = X,$$

where $I$ stands for the identity operator in $X$. It is well known that $A$ is $m$-accretive if and only if $R(I + \lambda A) = X$ for some $\lambda > 0$.

The operator $A$ is said to be trivial if there exists $a \in X$ such that $A = \{a\} \times X$.

Now let us consider the equation:

$$u' + Au \ni f,$$

where $A$ is $m$-accretive and $f \in L^1(0, T; X)$. We shall say that $u$ is a
strong solution of (1) if \( u \) is absolutely continuous and differentiable almost everywhere on \([0, T]\) and if

\[
u' + Au(t) \geq f(t) \quad \text{for almost every } t \in [0, T].
\]

Observe that if \((x, y) \in A\) and \(f(t) = y\) for almost every \(t \in [0, T]\) then the function \(u\) which is constant equal to \(x\) is a strong solution of (1) on \([0, T]\); in this case \(u\) is called a constant solution of (1) on \([0, T]\). We shall see below that in some cases the only strong solutions of (1) on \([0, T]\) are the constant ones.

A Banach space \(X\) has the Radon-Nikodym property if every bounded linear operator \(T\) from \(L^1([0, 1])\) into \(X\) is representable, which means that there exists \(g \in L^\infty([0, 1]; X)\) such that for \(f \in L^1([0, 1])\), \(Tf = \int_0^1 f(t)g(t)\,dt\). The classical Radon-Nikodym theorem expresses the fact that \(R\) has the Radon-Nikodym property. This property has been extensively studied in recent years (see e.g. [6] or [10]). Let us mention that every reflexive Banach space has the Radon-Nikodym property. On the other hand, \(L^1(\mathbb{R}^N)\) and \(BUC(\mathbb{R}^N)\) (the space of bounded uniformly continuous functions on \(\mathbb{R}^N\)) fail the Radon-Nikodym property.

Let us assume that \(A\) is an \(m\)-accretive operator in a Banach space \(X\). Then it is known that, if \(u_0\) belongs to \(D(A)\), \(f\) is in \(BV(0, T; X)\) and \(X\) has the Radon-Nikodym property, the equation (1) has a unique strong solution on \([0, T]\) satisfying \(u(0) = u_0\) (see [2] or [4]). It is also known that this result is no longer true in some concrete examples where \(X\) fails the Radon-Nikodym property. For instance, let us consider the Hamilton–Jacobi equation in \(\mathbb{R}^N\):

\[
(HJ) \quad u_t + H(\nabla u) = f.
\]

Under a suitable hypothesis on \(H\) we associate to this equation an \(m\)-accretive operator \(A\) in \(BUC(\mathbb{R}^N)\). However, even if the initial data \(u_0\) is smooth (\(u_0 \in D(A)\)) and \(f = 0\), there is no strong solution of (HJ) on \([0, T]\) for every \(T > 0\). A similar remark holds for the nonlinear conservation laws in \(\mathbb{R}^N\) to which one can associate an \(m\)-accretive operator in \(L^1(\mathbb{R}^N)\) (see e.g. [2, 4, 8]).

In this note, we prove that in every Banach space which fails the Radon-Nikodym property, there exists a nontrivial operator \(A\) in \(X\), which is \(m\)-accretive for some equivalent norm in \(X\), such that the only strong solutions of (1) on \([0, T]\) are the constant ones. In particular, if \(f\) is not constant almost everywhere, or if \(f\) is constant equal to \(y \notin R(A)\) then equation (1) has no strong solution. The construction of the operator \(A\) relies on the existence, for every Banach space \(Y\) which fails the Radon-Nikodym property, of a Lipschitz function \(\phi\) from \(\mathbb{R}\) into \(Y\) such that the set of all points of differentiability of \(\phi\) is negligible. This last result will be proved in the appendix.

We also give various consequences of the existence of such an operator \(A\), some of them having been already announced in [9].
2. CONSTRUCTION OF AN \( m \)-ACCRETIVE OPERATOR

In this section we prove

**Theorem 2.1.** Let \( X \) be a Banach space. \( X \) has the Radon-Nikodym property if and only if for every equivalent norm \( \| \| \) on \( X \) and for every nontrivial \( m \)-accretive operator \( A \) in \( (X, \| \|) \) there exists \( f \in L^1(0, T; X) \) (where \( T > 0 \) is arbitrary) such that (1) has a nonconstant strong solution on \([0, T] \).

We need the following elementary lemma whose proof is left to the reader. This is for instance an immediate consequence of Theorem 8-16 and of corollary of Theorem 8-6 of [11].

**Lemma 2.2.** Let \( I \) be an interval of \( \mathbb{R} \) and \( f \) be an absolutely continuous function from \( I \) into \( \mathbb{R} \). If \( A = \{ x \in I : f'(x) = 0 \} \), then \( \mu(f(A)) = 0 \).

**Proof of Theorem 2.1.**

First, let assume that \( X \) has the Radon-Nikodym property. Let \( \| \| \) be an equivalent norm on \( X \) and \( A \) be a nontrivial \( m \)-accretive operator in \( (X, \| \|) \). Since \( A \) is \( m \)-accretive, \( D(A) \) is nonempty. Let us pick \( a \) in \( D(A) \). If \( Aa = X \) then, since \( A \) is accretive \( D(A) = \{ a \} \) and so \( A = \{ a \} \times X \), which contradicts the assumption \( A \) nontrivial. Hence, \( Aa \neq X \) and we can choose \( b \in X \setminus Aa \). Since \( X \) has the Radon-Nikodym property, the theory of nonlinear semigroups (see e.g. [2] or [4]) tells us that the initial value problem:

\[
\begin{cases}
u' + Au \ni b, \\
v(0) = a
\end{cases}
\]

has a strong solution and this solution is nonconstant since \( b \notin Aa \).

Conversely, let \( X \) be a Banach space which fails the Radon-Nikodym property. We shall construct an equivalent norm \( \| \| \) on \( X \) and a nontrivial \( m \)-accretive operator \( A \) in \( (X, \| \|) \) such that for every \( T > 0 \) and every \( f \in L^1(0, T; X) \), the only strong solutions of \( u' + Au \ni f \) are the constant ones.

Step 1. Construction of \( A \). Let \( | \cdot | \) be the original norm on \( X \), \( x_0 \in X \) and \( f \in X^* \) such that \( f(x_0) = \| x_0 \| = \| f \| = 1 \). Since \( Y = \text{Ker} f \) is a hyperplane of \( X \) and \( x_0 \notin Y \), for every \( z \in X \), there exists \( a \in \mathbb{R} \) and \( y \in Y \) such that \( z = ax_0 + y \), and we then define \( \| z \| = \max(|a|, |y|) \). Clearly \( \| \| \) is an equivalent norm on \( X \). By Corollary 3.3, there exists a Lipschitz function \( \phi \) from \( \mathbb{R} \) into \( Y \) such that the set of all points of differentiability of \( \phi \) is negligible. Without loss of generality, we can assume \( \phi(0) = 0 \) and \( \phi \) is 1-Lipschitz. Finally, define

\[
A = \{ (tx_0 + \phi(t), y) : t \in \mathbb{R}, \ y \in Y \}.
\]

Step 2. \( A \) is \( m \)-accretive in \( (X, \| \|) \). Let \( (x, y), (x', y') \in A \) and \( \lambda > 0 \). From the definition of \( A \), there exists \( t, t' \in \mathbb{R} \) such that \( x = tx_0 + \phi(t) \),
\(x' = t'x_0 + \phi(t')\), and \(y, y' \in Y\). Thus
\[
\|x - x'\| = \|(t - t')x_0 + \phi(t) - \phi(t')\| = \sup(|t - t'|, |\phi(t) - \phi(t')|) = |t - t'|
\]
and
\[
\|x - x' + \lambda(y - y')\| = \|(t - t')x_0 + \phi(t) - \phi(t') + \lambda(y - y')\| = \sup(|t - t'|, |\phi(t) - \phi(t') + \lambda(y - y')|) \geq |t - t'|,
\]
therefore, \(A\) is accretive. On the other hand, since, for \(\lambda > 0\):
\[
I + \lambda A = \{(tx_0 + \phi(t), tx_0 + \phi(t) + \lambda y) : t \in \mathbb{R}, \ y \in Y\} = \{(tx_0 + \phi(t), tx_0 + y) : t \in \mathbb{R}, \ y \in Y\}
\]
we have that, if \(z = tx_0 + y\) with \(t \in \mathbb{R}\) and \(y \in Y\), then \(z \in (I + \lambda A)(tx_0 + \phi(t))\); thus \(R(I + \lambda A) = X\) and \(A\) is \(m\)-accretive.

Step 3. The only strong solutions of (1) are the constant solutions. Let \(u : [0, T] \to X\) be a strong solution of (1) that is not constant. Thus there exists \(v \in \text{AC}(0, T; \mathbb{R})\) (the space of absolutely continuous functions from \([0, T]\) into \(\mathbb{R}\)) and \(w \in \text{AC}(0, T; Y)\) such that \(u(t) = v(t)x_0 + w(t)\). Since for every \(t \in [0, T]\), \(u(t) \in \text{D}(A) = D(A) = \{tx_0 + \phi(t) : t \in \mathbb{R}\}\), we have \(w(t) = \phi(v(t))\). Since \(w(t) = u(t) - v(t)x_0\) for every \(t \in [0, T]\) and since \(u\) and \(v\) are differentiable almost everywhere, \(w\) is differentiable almost everywhere. On the other hand, if \(B = \{t \in [0, T] : \text{either } v'(t) \text{ does not exist or } v'(t) = 0\}\) then by Lemma 2.2, \(\mu(v(B)) = 0\). So, since \(v\) is not constant, if \(C = \{t \in [0, T] : v'(t) \exists \text{ and } v'(t) \neq 0\}\), then \(\mu(C) \neq 0\) and \(\mu(v(C)) \neq 0\). Therefore the set
\[
A = \{t \in C : v(t) \text{ is not a point of differentiability of } \phi\}
\]
is not negligible and we claim that \(w\) is not differentiable at any point of \(A\) which contradicts the fact that \(w\) is differentiable almost everywhere. To prove our claim, let \(t \in A\). Since for every \(\varepsilon > 0\), \(\{v(t + h) : |h| < \varepsilon\}\) is a neighborhood of \(v(t)\) (use the continuity of \(v\) and the fact that \(v'(t) \neq 0\)), we have that
\[
\lim_{h \to 0} \frac{\phi(v(t + h)) - \phi(v(t))}{v(t + h) - v(t)}
\]
does not exist. Now we write
\[
\frac{w(t + h) - w(t)}{h} = \frac{\phi(v(t + h)) - \phi(v(t))}{v(t + h) - v(t)} \cdot \frac{v(t + h) - v(t)}{h}.
\]
This makes sense if \(|h|\) is small enough (since then \(v(t + h) - v(t) \neq 0\)) and observing that \(\lim_{h \to 0} (v(t + h) - v(t))/h\) exists and is nonzero, we get that \(\lim_{h \to 0} (w(t + h) - w(t))/h\) does not exist, which proves our claim.

Remarks 2.3. 1. The idea of the proof of Theorem 2.1 is actually very simple: we have constructed an \(m\)-accretive operator \(A\) in \((X, \|\|)\) such that the domain of \(A\) is the graph of a Lipschitz function which is differentiable only
on a set of Lebesgue measure zero. A solution of (1) is then forced to follow this graph and thus cannot be differentiable almost everywhere if it is not constant.

2. If we replace $A$ by $B$:

$$B = \{(tx_0 + \phi(t), -x_0 + y) : t \in \mathbb{R}, y \in Y\}$$

then $B$ is also $m$-accretive in $(X, \|\|)$ and the only strong solutions of $u' + Bu \ni f$ are the constant solutions. And since $0 \notin R(B)$ the equation $u' + Bu \ni 0$ has no strong solution: this result was obtained in [9]. Let us notice that since the equation $u' + Bu \ni 0$ always has at least one strong solution whenever $X$ has the Radon-Nikodym property and $B$ is $m$-accretive, we have obtained a new characterization of the Radon-Nikodym property.

3. Note that the Radon-Nikodym property is an isomorphic property i.e. does not depend on the equivalent norm on $X$. This justifies the fact that we have chosen an equivalent norm on $X$ in the statement of Theorem 2.1. In the linear case, the operators which are $m$-accretive for some equivalent norm are precisely the generators of uniformly bounded semigroups. The class of nonlinear operators which are $m$-accretive for some equivalent norm does not seem to be well known.

4. If $X$ is a Banach space, $A$ an $m$-accretive operator in $X$, $u_0 \in \overline{D(A)}$ and $f \in L^1(0, T; X)$, we say that $u \in \mathcal{C}(0, T; X)$ is a weak solution of

$$\begin{cases} 
  u' + Au \ni f, \\
  u(0) = u_0
\end{cases}$$

if there exists $u_{0,n} \in X$, $f_n \in L^1(0, T; X)$, and $u_n \in \mathcal{C}(0, T; X)$ such that $u_n$ is a strong solution of $u'_n + Au_n \ni f_n$ on $[0, T]$ satisfying $u_n(0) = u_{0,n}$, $\lim_{n \to \infty} u_{0,n} = u_0$ in $X$, $\lim_{n \to \infty} f_n = f$ in $L^1(0, T; X)$, and $\lim_{n \to \infty} u_n = u$ in $\mathcal{C}(0, T; X)$. This notion of solution has been considered e.g. in [3]. If $A$ is the operator constructed in Theorem 2.1, the only weak solution of $u' + Au \ni f$ on $[0, T]$ are the constant solutions. Thus strong solutions, and more generally weak solutions, are not suitable to solve (2) in Banach spaces which fail the Radon-Nikodym property.

5. In order to solve (2) in a general Banach space $X$, one has to introduce the notion of mild solution: a function $u \in \mathcal{C}(0, T; X)$ is called a mild solution of (2) on $[0, T]$ if for every $\varepsilon > 0$, there exists $n \in \mathbb{N}$, $t_0 = 0 < t_1 < \cdots < t_{n-1} < t_n \leq T$ and $x_0, \ldots, x_n, y_1, \ldots, y_n \in X$ such that

$$\begin{align*}
  (x_i - x_{i-1})/(t_i - t_{i-1}) + Ax_i \ni y_i & \quad \text{for } i = 1, \ldots, n, \\
  |t_i - t_{i-1}| \leq \varepsilon & \quad \text{for } i = 1, \ldots, n, \\
  \sum_{i=1}^{n} \int_{t_{i-1}}^{t_i} |f(t) - y_i| dt \leq \varepsilon, \\
  |u_0 - x_0| \leq \varepsilon, & \quad |u(t) - x_{i-1}| \leq \varepsilon \quad \text{for every } t \in [t_{i-1} - t_i], \ i = 1, \ldots, n.
\end{align*}$$

It is known that (2) has a unique mild solution $u \in \mathcal{C}(0, T; X)$ for every $u_0 \in \overline{D(A)}$ and every $f \in L^1(0, T; X)$. This shows existence and uniqueness.
of a solution of certain concrete partial differential equations (for instance, for
the Hamilton–Jacobi equations which can be solved in $X = \text{BUC}(\mathbb{R}^n)$ or for
the nonlinear conservation law equations which can be solved in $X = L^1(\mathbb{R}^N)$).

In a Banach space that has the Radon-Nikodym property, $u$ is a mild solution
of (2) on $[0, T]$ if and only if $u$ is a weak solution of (2) on $[0, T]$. In a
general Banach space, if $u$ is a weak solution of (2) on $[0, T]$, then $u$ is
a mild solution of (2) on $[0, T]$. We refer the reader to [4] for these results.

Remark 4, together with the above discussion, shows that in every Banach space
which fails the Radon-Nikodym property, there exists an operator $A$ which is
$m$-accretive for some equivalent norm in $X$, and a mild solution of (2) that
is not a weak solution. Observe that in the example constructed in the proof
of Theorem 2.1, if we set for $t \in [0, T]$, $f(t) = g(t)x_0 + h(t)$ with $g(t) \in \mathbb{R}
and $h(t) \in Y$, the mild solution of $u' + Au \ni f$ such that $u(0) = u_0$ is given
by $u(t) = v(t)x_0 + \phi(v(t))$ where $v$ satisfies $u(0) = v(0)x_0 + \phi(v(0))$ and
$v(t) = v(0) = \int_0^t g(t)dt$.

6. The operator $B$ as constructed in Theorem 2.1 is such that $B$ and $-B$
are $m$-accretive. It is easy to check that if $z = tx_0 + y$ with $t \in \mathbb{R}$ and $y \in Y$,
then if $\lambda > 0$:

$$(I + \lambda A)^{-1}(z) = (t + \lambda)x_0 + \phi(t + \lambda)$$

thus for every $n \in \mathbb{N}$:

$$(I + \lambda A/n)^{-n}(z) = (t + \lambda)x_0 + \phi(t + \lambda)$$

and so, by the exponential formula, the mild solution of $u' + Bu \ni 0$ such that
$u(0) = 0$ is given by $u(t) = tx_0 + \phi(t)$. Similarly, the mild solution of $v' - Bu \ni 0$
such that $v(0) = T x_0 + \phi(T)$ is given by $v(t) = (T - t)x_0 + \phi(T - t) = u(T - t)$;
thus we have time reversibility for the equation $u' + Bu \ni 0$. This phenomenon
is not always true (see e.g. [5]). On the other hand, it is known that if $u$ is a
weak solution of $u' + Au \ni 0$ on $[0, T]$ then $v$ defined by $v(t) = u(T - t)$ is
a weak solution of $v' - Av \ni 0$ on $[0, T]$. In our example, $u$ and $v$ are mild
solutions which are not weak solutions. So time reversibility for the equation
$u' + Bu \ni 0$ does not force the solutions to be weak solutions.

3. Appendix

In this section, we construct, for every Banach space $X$ that fails the Radon-
Nikodym property, a Lipschitz mapping $\phi : \mathbb{R} \to X$ which is almost nowhere
differentiable. The following result is shown in [1]:

**Proposition 3.1.** Let $X$ be a Banach space. Then $X$ has the Radon-Nikodym
property if and only if every Lipschitz function from $[0, 1]$ into $X$ is differen-
tiable almost everywhere.

Thus, if $X$ fails the Radon-Nikodym property, there exists a Lipschitz map
$\phi : [0, 1] \to X$ such that $\mu(N_\phi) > 0$, where $\mu$ denotes the Lebesgue measure
on $[0, 1]$ and $N_\phi$ is the set of points of nondifferentiability of $\phi$. In this
section we prove:

**Theorem 3.2.** Let $X$ be a Banach space which does not have the Radon-Nikodym property. Then there exists a Lipschitz mapping $\psi : [0, 1] \to X$ such that $\mu(N_\psi) = 1$.

**Proof.** Let $Y = \text{Lip}([0, 1], X)$ be the space of all Lipschitz mappings from $[0, 1]$ into $X$ which vanish at zero. For $f \in Y$ let us define

$$
||f||_Y = \sup \left\{ \frac{||f(y) - f(y)||}{y - x} : x, y \in [0, 1], x \neq y \right\}.
$$

Then $(Y, || ||_Y)$ is a Banach space. For $n \geq 1$, let us consider the set

$$
U_n = \{ f \in Y, \mu(N_f) > 1 - \frac{1}{n} \}.
$$

$U_n$ is open. Let $f \in U_n$ and $x \in [0, 1]$. Let us denote

$$
D_f(x) = \lim_{\varepsilon \to 0} \text{diam} \left\{ \frac{f(x + \varepsilon) - f(x)}{\varepsilon} : |\varepsilon| < \varepsilon \right\}.
$$

Since $x \in N_f$ if and only if $D_f(x) > 0$, we have

$$
N_f = \bigcup \{ A_k : k \geq 1 \}, \quad \text{where } A_k = \{ x \in [0, 1] : D_f(x) \geq \frac{1}{k} \}.
$$

Since $(A_k)$ is increasing, there exists $k \geq 1$ such that $\mu(A_k) > 1 - 1/n$. We claim that if $u \in Y$ and if $||u||_Y < 1/2k$ then $A_k \subseteq N_{f+u}$, thus showing that $f + u \in U_n$ and hence, that $U_n$ is open.

To prove the claim, it is enough to notice that for every $x \in [0, 1] :$

$$
\lim_{\varepsilon \to 0} \text{diam} \left\{ \frac{u(x + \varepsilon) - u(x)}{\varepsilon} : |\varepsilon| < \varepsilon \right\} < \frac{1}{k};
$$

thus, for every $x \in A_k :$

$$
\lim_{\varepsilon \to 0} \text{diam} \left\{ \frac{(f(x + \varepsilon) + u(x + \varepsilon)) - (f(x) + u(x))}{h} : |h| < \varepsilon \right\} > 0.
$$

$U_n$ is dense. Let $x \in (0, 1)$ be such that $\mu(I \cap N_\phi)/\mu(I)$ tends to 1 when the diameter of the interval $I$ containing $x$ tends to 0. Let $I = [a, b]$ be an interval contained in $[0, 1]$ and containing $x$ such that $\mu(I \cap N_\phi)/\mu(I) > 1 - 1/2n$. The map $\psi$ defined by $\psi(t) = \phi(a + t(b - a)) - \phi(a)$ belongs to $Y$ and $\mu(N_\psi) > 1 - 1/2n$. Let $f \in Y$. We claim that if $\alpha > 0$ is small enough, then $f + \alpha \psi \in U_n$ which proves that $U_n$ is dense. Indeed, let $\gamma > 0$ sufficiently small to have:

$$
\mu(\{ x \in N_f : D_f(x) > \gamma \}) > \mu(N_f) - 1/2n.
$$

If we take $\alpha = \gamma/||\psi||_Y$, we get that:

$$
N_{f+\alpha \psi} \supseteq (0, 1) \setminus N_f \cap N_\psi = A,
$$

$$
N_{f+\alpha \psi} \supseteq \{ x \in N_f : D_f(x) > \gamma \} = B,
$$


and
\[ \mu(A \cup B) = \mu(A) + \mu(B) > (1 - \mu(N_f) - \frac{1}{2^n}) + (\mu(N_f) - \frac{1}{2^n}) = 1 - \frac{1}{n}. \]

Now, by Baire theorem, we have that \( G = \bigcap_{n \in \mathbb{N}} U_n \) is a dense \( \mathcal{G}_\delta \) in \( Y \) and any \( \psi \in G \) satisfies Theorem 3.2.

For proving Theorem 2.1, we have used the following consequence of Theorem 3.2.

**Corollary 3.3.** Let \( X \) be a Banach space which fails the Radon-Nikodym property and \( Y \) be a hyperplane of \( X \). Then there exists a Lipschitz mapping \( \phi : \mathbb{R} \to Y \) such that the set of all points of differentiability of \( \phi \) is negligible.

**Proof.** Since \( Y \) fails the Radon-Nikodym property, there exists by Theorem 2.2 a Lipschitz mapping \( \psi : [0, 1] \to Y \) such that \( \mu(N_\psi) = 1 \). Now, if for \( x \in \mathbb{R} \) we denote \([x]\) the greatest integer \( \leq x \), the function \( \phi : \mathbb{R} \to Y \) defined by \( \phi(x) = [x](\psi(1) - \psi(0)) + \psi(x - [x]) \) is Lipschitz continuous on \( \mathbb{R} \) and satisfies Corollary 3.3.

**Remark 3.4.** If \( X = L^1([0, 1]) \) the function \( \phi : [0, 1] \to X \) defined by \( \phi(t) = 1_{[0, 1]}(t) \) where \( 1_{[0, 1]}(t) \) is the indicator function of \([0, 1]\), is nowhere differentiable. The following improvement of Theorem 3.2 seems to be unknown:

If \( X \) is a Banach space which does not have the Radon-Nikodym property, does there exist a Lipschitz map \( \phi : [0, 1] \to X \) which is nowhere differentiable?

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