THE SPACE \((l_\infty/c_0, \text{weak})\) IS NOT A RADON SPACE

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Abstract. Talagrand [10] gives an example of a Banach space with weak topology which is not a Radon space, independently of their weight. This result gives an answer to a question formulated by Schwartz [9]. In this paper, following the papers of Drewnowski and Roberts [1] and Talagrand [10], we prove that the classical space \((l_\infty/c_0, \text{weak})\) is not a Radon space.

Introduction. A Hausdorff topological space \(E\) is said to be a Radon space if every finite Borel measure is a Radon measure; i.e.,
\[
\mu(A) = \{\mu(K) : K \subseteq A, \ K \text{ compact}\}
\]
for each Borel subset \(A\) of \(E\).

We shall say that a cardinal \(\alpha\) is of measure zero (resp. nonmeasurable) if there is not a real-valued, diffuse, nontrivial measure (resp. \(\{0, 1\}\)-valued), on the power set of a set \(A\) with cardinal \(\alpha\).

The weight (density character) of a topological space \(E\) is the smallest cardinal such that there exists in \(E\) a dense subset \(A\) with this cardinal.

A topological space \(E\) has the \(\alpha\)-property of Lindelöf, \(\alpha\) a transfinite cardinal, if for each family \((G_i)_{i \in I}\) of open subsets of \(E\) there exists \(J \subseteq I\) such that \(\text{card}(J) \leq \alpha\) and \(\bigcup_{i \in I} G_i = \bigcup_{i \in J} G_i\). The smallest cardinal \(\alpha\) such that \(E\) has the \(\alpha\)-property of Lindelöf is called the \(L\)-weight of \(E\).

Likewise, \(E\) is a Flock space if for every well-ordered family \((G_i)_{i \in I}\) of open sets, with \(H_\alpha = G_\alpha \setminus \bigcup_{\beta < \alpha} G_\beta\), every union \(\bigcup_{a \in J} H_a\) \((J \subseteq I)\) is universally Borel-measurable. This property is an extrapolation of the Montgomery lemma [7], which proves that every metrizable space is Flock. Also, the strongly Lindelöf spaces are Flock spaces.

Let \(I\) be a noncountable set and \(x \notin I\). Consider the topology \(T\) on \(I \cup \{x\}\) such that the discrete topology is on \(I\) and the neighborhoods of \(x\) are the complementary sets of countable subsets of \(I\). Let \(K\) be the Stone-Čech compactification of this space. Then, as Talagrand [10] proved, \((C(K), \text{weak})\) is not a Radon space.
Talagrand also proved that \((Z, \text{weak})\) is not a Radon space when \(Z\) is the subspace of \(l^\infty(I)\) of the countable support functions and \(\text{card}(I) = \aleph_1\). Following that proof and the work of Drewnowski and Roberts \([1]\), this paper is devoted to proving that the classical space \((l^\infty/c_0, \text{weak})\) is not a Radon space. Note that this fact could be proved if \(Z\) were a subspace of \(l^\infty/c_0\), but this is not yet known. Let us remark that it is easily proved that \((E, \text{weak})\) is not a Radon space if the weight of \(E\) is real-measurable.

It is easily deduced that every Borel measure on \((l^\infty, \text{weak})\) valued on \(\{0, 1\}\) is a \(\delta_x\).

In Theorem 18 \([2]\) we proved that a Banach space with the weak topology is a Radon space if and only if it is Flock and its \(L\)-weight is of measure zero. Thus, in the last part of this paper, we prove that \(l^\infty/c_0\) is not a Flock space. Conversely, in \([4]\), we improved Schachermayer’s theorem, proving that a subset \(\Omega\) of a WCG Banach space with the weak topology whose weight is of measure zero is a Radon space of type \((\mathcal{F})\); that is, every finite Borel measure on \((\Omega, \text{weak})\) is \(\tau\)-additive.

1. **Theorem.** On \((l^\infty/c_0, \text{weak})\) there exists a Borel measure \(\mu \neq 0\) valued on \(\{0, 1\}\) such that the space is not a Radon measure.

**Proof.** Let \(\beta\omega\) be the Stone-Čech compactification of \(\omega = \mathbb{N}\) with discrete topology. Let \(\mathcal{A}\) be the clopen algebra of \(K = \beta\omega \setminus \omega\) and \(\mathcal{A}_0 = \mathcal{A} \setminus \{\emptyset\}\). \(\mathcal{A}\) has the following property (called the Cantor separability property in \([11]\)):

For every decreasing sequence \((A_n)\) in \(\mathcal{A}_0\) there exists an \(A \in \mathcal{A}_0\) such that \(A \subset \bigcap_{n=1}^{\infty} A_n\).

Let \(\mathcal{F} = (V_a)_{a \in A}\), a maximal well-ordered family in \(\mathcal{A}_0\). Then every countable intersection \(\bigcap_{n=1}^{\infty} V_n\) contains a \(V \in \mathcal{F}\). We denote by \(F\) the set of continuous functions from \(K\) in \(\{0, 1\}\) which vanish in \(H = \bigcap\{V : V \in \mathcal{F}\}\).

Like Talagrand, we are going to make two classes \(\mathcal{C}\) and \(\mathcal{D}\) of weak-Borel sets (Borel with the weak topology) in \(F \subset C(K)\) such that

(i) the smallest \(\sigma\)-algebra containing \(\mathcal{C}\) is the class \(\mathcal{B}\) of the weak-Borel sets of \(F\);

(ii) if \(C \in \mathcal{C}\), then either \(C \in \mathcal{D}\) or \(F \setminus C \in \mathcal{D}\);

(iii) every countable intersection of elements of \(\mathcal{D}\) is not empty; and

(iv) for every \(t \in K \setminus H\), \(\{f \in F : f(t) = 1\} \in \mathcal{D}\).

Now, we define a Borel measure \(\mu\) on \((F, \text{weak})\) by \(\mu(B) = 1\) if \(B \in \mathcal{B}\) and contains a countable intersection of members of \(\mathcal{D}\), and \(\mu(B) = 0\) in the other case. The measure \(\mu\) is not \(\tau\)-additive, because \(F\) is the union of the open sets \(G_t = \{f \in F : f(t) = 0\}\) when \(t \in K \setminus H\), \(\mu(F) = 1\), and each \(\mu(G_t) = 0\). It follows that there exists a Borel measure \(\mu_0\) on \((C(K), \text{weak}) \approx (l^\infty/c_0, \text{weak})\) valued in \(\{0, 1\}\) which is not a Radon measure.

Let \(k\) be an integer, \(J\) a set, \((\mu_p^p)_{p \leq k}\), of Radon measures on \(K\), and \((\alpha^p)_{p \leq k}, (\beta^p)_{p \leq k}\) rational numbers such that \(\alpha^p < \beta^p\). By definition, \(\mathcal{C}\) is the
class of sets \( C = \bigcup_{j \in J} U_j \), where
\[
U_j = \{ f \in F : \forall p \leq k, \mu_j^p(f) \in (a^p, b^p) \},
\]
varying over all \( k, J \) and measures \( \mu_j^p \).

The class \( \mathcal{D} \) is built up by the sets \( C \in \mathcal{E} \) such that, for every \( V \in \mathcal{Y} \), there exist \( j \in J \) and \( f \in U_j \) with \( f = 1 \) in \( K \setminus V \), and with the complementaries \( F \setminus C \) of the sets \( C \in \mathcal{E} \) such that they do not have this condition. Only (iii) has to be proved. To do this, it is enough that, for every sequence \( (C_n) \subset \mathcal{E} \cap \mathcal{D} \) and every \( V_0 \in \mathcal{V} \), there exists \( f \in \bigcap_n C_n \) such that \( f = 1 \) in \( K \setminus V_0 \). It is easily proved (with the evident notations) that for every \( n \in \mathbb{N} \), there exists \( \varepsilon_n > 0 \) such that, if \( W \in \mathcal{Y} \), then there exist \( j \in J_n \) and \( f \in U_j \) with \( f = 1 \) on \( K \setminus W \), where
\[
U_j = \{ f \in F : \forall p \leq k_n, \mu_j^p(f) \in (a^p_n + \varepsilon_n, b^p_n - \varepsilon_n) \}.
\]

Let \( \mathcal{U}_n \) be an ultrafilter on \( J_n \) which contains all sets
\[
\{ j \in J_n : \exists f \in U_j, f = 1 \text{ on } K \setminus V \}
\]
when \( V \in \mathcal{Y} \). Given \( p \leq k_n \), let \( \nu_n^p = \lim_{\mathcal{U}_n} \mu_j^p \). Then there exists \( V \in \mathcal{Y} \), \( V \subset V_0 \), such that \( \nu_n^p(V \setminus W) = 0 \) for every \( W \in \mathcal{Y} \) and \( p \leq k_n, n \in \mathbb{N} \). Similarly, by means of an easy induction, there exist \( f_n \in F, j_n \in J_n, V_\alpha \), \( V'_\alpha \in \mathcal{Y} \) and clopen sets \( H_{n0}, H_{n1} \in \beta \omega \) such that

(i) \( \mu_j^p_n(V \setminus V_\alpha) \leq \varepsilon_n/2 \), for every \( n \) and \( p \leq k_n \);
(ii) \( f_n \in U_{j_n}' \) and \( f_n = 1 \) on \( K \setminus V_\alpha \);
(iii) \( f_n = 0 \) on \( V'_\alpha \subseteq V_\alpha \) and \( \mu_{j_n}^p(V_\alpha \setminus H) = 0 \) for every \( n \) and \( p \leq k_n \);
(iv)
\[
\begin{align*}
A_{00} &= \emptyset, & A_{01} &= K \setminus V, \\
A_{n0} &= \{ t \in V_\alpha \setminus V'_\alpha : f_n(t) = 0 \}, \\
A_{n1} &= \{ t \in V_\alpha \setminus V'_\alpha : f_n(t) = 1 \};
\end{align*}
\]
(v) \( H, H_{n0}, H_{n1} \) are pairwise disjoint sets such that \( H_{ni} \supseteq A_{ni} \cup H_{n-1, i} \), \( H_{0i} = A_{0i} \) for \( i = 0, 1 \);
(vi) The set \( V_{\alpha n+1} \) verifies that \( V_{\alpha n+1} \subset V'_\alpha \) and \( V_{\alpha n+1} \cap (H_{n0} \cup H_{n1}) = \emptyset \).

Clearly the open sets \( G_0 = \bigcup_{n=1}^{\infty} H_{n0} \) and \( G_1 = \bigcup_{n=1}^{\infty} H_{n1} \) are disjoint, but \( G_0 \cap G_1 = \emptyset \) also is, because if \( t \in G_0 \cap G_1 \), then \( t \in \beta \omega \) would be a cluster point of two disjoint sets of integer numbers contained in \( G_0 \) and \( G_1 \), respectively. As \( (G_0 \cup G_1) \cap H = \emptyset \), then there exists a clopen \( H_1 \) in \( K \) such that \( \overline{G_1} \cap K \subset H_1 \), \( G_0 \cap K \cap H_1 = \emptyset \), and \( H \cap H_1 = \emptyset \).

Function \( f = \chi_{H_1} \in F \) satisfies \( f = f_n \) on \( A_n = A_{n0} \cup A_{n1} \). Moreover, since for every \( n \) and every \( p \leq k_n \), we have that \( \mu_j^p(f \neq f_n) < \varepsilon_n \). It follows that \( f \in U_{j_n} \) for every \( n \). Then \( f \in \bigcap_{n=1}^{\infty} C_n \) and \( f = 1 \) on \( K \setminus V_0 \).
2. **Corollary.** \((l_\infty/c_0, \text{ weak})\) is not a Radon space.

3. **Theorem.** Let \(\Omega\) be a Flock space whose \(L\)-weight is nonmeasurable. Then every finite perfect Borel measure \(\mu\) on \(\Omega\) is \(\tau\)-additive.

**Proof.** It is enough to prove that, for every family of open sets \((G_\alpha)_{\alpha \in A}\) in \(\Omega\),

\[
\mu \left( \bigcup_{\alpha \in A} G_\alpha \right) = \sup_{J} \mu \left( \bigcup_{\alpha \in J} G_\alpha \right),
\]

where the supremum is taken over all finite subsets \(J\) in \(A\).

From Zermelo's theorem, and since the \(L\)-weight of \(\Omega\) is nonmeasurable, then we can suppose \(A\) is well ordered and cardinally nonmeasurable. Let \(H_\alpha = G_\alpha \setminus \bigcup_{\beta < \alpha} G_\beta\). Since \(\Omega\) is a Flock space, the union \(\bigcup_{\alpha \in A} G_\alpha\) of every subfamily of \((H_\alpha)_{\alpha \in A}\) is a universally Borel-measurable set. By [5, Theorem 2.5], if \(S = \{ \alpha \in A : \mu(H_\alpha) \neq 0 \}\), then \(S\) is countable and \(\mu(\bigcup_{\alpha \in S} H_\alpha) = 0\).

Now,

\[
\mu \left( \bigcup_{\alpha \in A} G_\alpha \right) = \mu \left( \bigcup_{\alpha \in A} H_\alpha \right) = \mu \left( \bigcup_{\alpha \in S} H_\alpha \right) \leq \sup_{J} \mu \left( \bigcup_{\alpha \in J} G_\alpha \right),
\]

so

\[
\mu \left( \bigcup_{\alpha \in A} G_\alpha \right) = \sup_{J} \mu \left( \bigcup_{\alpha \in J} G_\alpha \right).
\]

4. **Corollary.** \((l_\infty/c_0, \text{ weak})\) is not a Flock space.

**Proof.** Follows from Theorems 1 and 3, since cardinal of \(l_\infty/c_0\) is nonmeasurable.

**Remark.** Since [3, Theorem 6] says that every Radon space is a Flock space, Corollary 4 improves Corollary 2. Even this theorem proves that a regular space \(E\) is Flock and its weight is of measure zero if and only if \(E\) is a Radon space of type \((\mathcal{F})\).

5. **Theorem.** If \((\varphi_n)\) is a sequence of continuous functions which separates points on a topological space \(E\), then every Borel measure \(\mu \neq 0\) on \(E\) which takes values in \(\{0, 1\}\) is concentrated in a point.

**Proof.** Let \(\nu\) be the Borel measure on \(\mathbb{R}\) defined by

\[
\nu(B) = \mu(\varphi^{-1}_1(B))
\]

on the class \(\mathcal{B}\) of Borel measure on \(\mathbb{R}\). Then, as \(\nu\) takes only the values 0 and 1, there exists \(\alpha_1 \in \mathbb{R}\) such that \(\mu(\varphi^{-1}_1(\alpha_1)) = 1\). Now, by induction, we can construct a sequence \((\alpha_n)\) in \(\mathbb{R}\) such that

\[
\mu \left( \bigcap_{k=1}^{n} \varphi^{-1}_k(\alpha_k) \right) = 1
\]

for every \(n\). To do this, it is enough to consider the following measure:

\[
\nu(B) = \mu \left( \bigcap_{k=1}^{n} \varphi^{-1}_k(\alpha_k) \cap \varphi^{-1}_{n+1}(B) \right) \quad (B \in \mathcal{B}).
\]
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Then

\[ \mu \left( \bigcap_{n=1}^{\infty} \varphi_n^{-1}(\alpha_n) \right) = 1. \]

Finally, as \((\varphi_n)\) separates points of \(E\), then \(\bigcap_{n=1}^{\infty} \varphi_n^{-1}(\alpha_n)\) contains only a point \(x\). So \(\mu\) is concentrated in \(x\).

6. Corollary. Every Borel measure \(\mu \neq 0\) on \((l_{\infty}, \text{weak})\) taking values in \(\{0, 1\}\) is concentrated in a point.

This allows us to formulate the following open question: Is \((l_{\infty}, \text{weak})\) a Flock space?

Remark. Drewnowski and Roberts [1] have proved that for every Schauder decomposition (finite or infinite) \(C = l_{\infty}/c_0 = X_1 + X_2 + \cdots\), at least one of \(X_n\) contains a space isomorphic to \(C\), complemented in \(C\). It can easily be proved, as in Theorem 5, that if \(C\) is a Banach space with a Schauder decomposition and there exists a nontrivial Borel measure \(\mu\) on \(C\) valued in \(\{0, 1\}\), then there exist \(n \in \mathbb{N}\) and a nontrivial Borel measure \(\mu_n\) on \(X_n\) taking values in \(\{0, 1\}\). So, taking into account that \(C \approx C \oplus C\) and \(C \approx l_{\infty} \oplus C\) \((C = l_{\infty}/c_0)\) the following question naturally appears:

Open question. If \(X\) is a complemented subspace in \(l_{\infty}/c_0\) on which there exists a nontrivial measure taking values in \(\{0, 1\}\), then does \(X\) have a complemented copy of \(C\)?

Let \(L = l_{\infty}([0, 1]^\varepsilon)\), where \(\varepsilon = 2^{\aleph_0}\) and \([0, 1]\) is considered with the product Lebesgue measure. If we suppose \(\varepsilon = \aleph_1(CH)\), then \(L\) is isometric to a complemented and closed subspace \(X\) of \(C\).

Another question arises:

Open question. Does there exist on \(L\) a nontrivial Borel measure \(\mu\) which only takes 0, 1 values?

References

1. L. Drewnowski and J. W. Roberts, On the primariness of the Banach space \(l_{\infty}/c_0\), preprint.
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