SPECTRAL PROPERTIES OF PERTURBED LINEAR OPERATORS
AND THEIR APPLICATION TO INFINITE MATRICES

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Abstract. In this paper, upper bounds for the difference between the eigenvalues and the eigenvectors of a closed linear operator $D$ and those of $D + F$, where $F$ is a bounded linear operator, are given in terms of the norm of $F$. These results are applied to approximate the eigenvalues and the eigenvectors of a diagonally infinite matrix by those of its corresponding diagonal matrix.

1. Introduction

Let $T$ be a closed operator in a Banach space $\mathcal{H}$ and $F$ be an operator on $\mathcal{H}$. One of the basic problems in perturbation theory for closed operators is to study the relation between the eigenvalues and the eigenvectors of $T$ and those of $T + F$. The operator $F$ is called the perturbation.

To the author's knowledge, the first explicit quantitative results regarding this problem were first published by Rosenbloom [5]. In [5], upper bounds for the norm of the perturbation are given to guarantee that the eigenvalues and the eigenvectors of the perturbed linear operator stay close to those of the unperturbed one. These results were established by the application of an elementary fixed point theorem (see [5, Lemma 1]).

In §2 we introduce the main theorem (Theorem 2.1) that improves the results of [5] (see Remark 2.3). The proof of the main theorem is based on the contraction mapping theorem [4, Theorem 1, p. 474].

In §3, we apply the results of §2 along with [2, Theorem 4.2] to the case of a diagonally dominant infinite matrix acting as a linear operator in the sequence space $l_2$, where it is established that the eigenvalues of such a matrix approach its main diagonal. The results of §3 are used to approximate the eigenvalues of Mathieu's equation in §4.

Before stating our first theorem, we introduce some notation that will be used throughout the paper.

The sets of positive integers and complex numbers are denoted by $\mathbb{N}$ and $\mathbb{C}$, respectively.

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The domain of an operator $A$ in a Banach space $S$ is denoted by $\mathcal{D}(A)$. For a matrix $A = (a_{ij})$ acting in $l_p$, $1 \leq p \leq \infty$, the symbol $\sum'_j$ will always denote the sum from one to infinity excluding the index $j = i$. We define the row and column sums of $A$:
\[ P_i = \sum'_j |a_{ij}|, \quad Q_i = \sum'_j |a_{ji}|, \]
and the corresponding Gersgorin discs (where they exist):
\[ \mathcal{R}_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq P_i \}, \quad \mathcal{E}_i = \{ z \in \mathbb{C} : |z - a_{ii}| \leq Q_i \}. \]

The algebra of bonded linear operators $A$ on a Banach space $S$ with domains $\mathcal{D}(A) = S$ is denoted by $L(S)$. The identity operator on $S$ is denoted by $I$.

2. The main theorem

In this section we show under certain conditions (see Theorem 2.1) that if the unperturbed system has an eigenvalue $\lambda_0$ with a corresponding eigenvector $x_0$, then an upper bound for the perturbation (in norm) can be given to guarantee that the perturbed system has an eigenvalue with a corresponding eigenvector that are both close to $\lambda_0$ and $x_0$, respectively.

**Theorem 2.1.** Let $D$ be a closed operator defined in a Banach space $S$ such that $\mathcal{D}(D)$ is dense in $S$ and assume that

1. The point $\lambda_0$ is an eigenvalue of both $D$ and $D'$ (the dual of $D$) with corresponding eigenvectors $x_0$ and $x'_0$, respectively. The vector $x_0$ satisfies $\|x_0\| = 1$.
2. The restriction of the operator $(D - \lambda_0 I)$ to the space $\mathcal{H}_1 = \{ x \in S : x'_0(x) = 0 \}$ has a bounded inverse $R$ mapping $\mathcal{H}_1$ into itself.
3. The vector $x'_0$ satisfies $x'_0(x_0) = 1$.

Then for any $r \in (0, (\|S\|/\|x'_0\|\|R\|)^{1/2})$ and $F \in \mathcal{F}_r = \{ U \in L(S) : \|U\| \leq \delta \}$, where
\[ \delta = r/(\|x'_0\|\|R\| r^2 + (\|x'_0\|\|R\| + \|S\|)r + \|S\|) \]
and
\[ S = R(I - x_0x'_0), \]
the system of equations
\[ A x = (D + F)x = \lambda x, \quad x'_0(x) = 1 \]
has the unique solution $(x, \lambda)$ in the set $\{ z \in \mathcal{D}(D) : \|z - x_0\| \leq r \} \times \mathbb{C}$.

(Concerning the location of $\lambda$ in $\mathbb{C}$, see Remark 2.2.)

**Proof.** Define $P = I - x_0x'_0$. Hence $S = RP$.
Fix $r \in (0, (\|S\|/\|x_0\| \|R\|)^{1/2})$ and $F \in \mathcal{S}$. Writing $x = x_0 + y$ and $\lambda = \lambda_0 + \eta$, and substituting into equation (2.1), we find it is required to solve the equation:

\[(2.2) (D - \lambda_0 I)y = \eta(x_0 + y) - F(x_0 + y),\]

where $y \in \mathcal{Y} = \mathcal{X}_1 \cap \{z \in \mathcal{X} : \|z\| \leq r\} \cap \mathcal{D}(D)$. From the second hypothesis, the left-hand side of equation (2.2) is in $\mathcal{X}_1$, and so acting on both sides of the equation by $x_0'$ we get

\[(2.3) \eta = \eta x_0'(x_0 + y) = x_0'(F(x_0 + y)).\]

Therefore it is required to solve in $\mathcal{Y}$ the equation

\[(2.4) (D - \lambda_0 I)y = x_0'(F(x_0 + y))(x_0 + y) - F(x_0 + y).\]

From the definition of $P$, it is easy to see that solving equation (2.4) in $\mathcal{Y}$ is equivalent to solving the equation:

\[(2.5) (D - \lambda_0 I)y = x_0'(F(x_0 + y))x_0'F(x_0 + y)\]

in $\mathcal{Y}$. Now we may use hypothesis (2) and act on both sides of equation (2.5) by the operator $R$ to deduce that the equation

\[(2.6) y = x_0'(F(x_0 + y))Ry - SF(x_0 + y)\]

has the same solution set in $\mathcal{Y}$ as does equation (2.5).

Define the map $Q_F$ on the closed set $\mathcal{X}_1 \cap \{z \in \mathcal{X} : \|z\| \leq r\}$ by the equation:

\[Q_F(y) = x_0'(F(x_0 + y))Ry - SF(x_0 + y)\]

for all $y \in \mathcal{X}_1$. We prove that $Q_F$ is a contraction map mapping $\mathcal{X}_1$ into itself.

Let $y \in \mathcal{X}_1$. Since $P$ maps $\mathcal{X}$ onto $\mathcal{X}_1$ (for $x \in \mathcal{X}$, we have $x_0'(Px) = x_0'(x) - x_0'(x)x_0'(x_0) = x_0'(x) - x_0'(x) = 0$, by the third hypothesis), we have $PF(x_0 + y) \in \mathcal{X}_1$. Hence from the second hypothesis, both $SF(x_0 + y)$ and $Ry$ are in $\mathcal{X}_1$. Thus $Q_F(y) \in \mathcal{X}_1$. Also using the triangular inequality and the fact that $\|x_0\| = 1$ and $\|y\| \leq r$, we get

\[\|Q_F(y)\| \leq (\|x_0\| \|F\| \|R\|\|r + \|S\| \|F\|)(1 + r).\]

But since $\|F\| \leq r/(\|x_0\| \|R\|^2 + (\|x_0\| \|R\| + \|S\|)(r + \|S\|))$, we have $\|Q_F(y)\| \leq r$, and this proves that $Q_F$ maps $\mathcal{X}_1$ into itself. Now let $y_1$ and $y_2$ be in $\mathcal{X}_1$. We have

\[\|Q_F(y_1) - Q_F(y_2)\| \leq \|F\|(2\|x_0\| \|R\| + \|x_0\| \|R\| + \|S\|)\|y_1 - y_2\|
\leq \frac{2\|x_0\| \|R\|\|r + \|x_0\| \|R\| + \|S\|\|y_1 - y_2\|}{\|R\|\|r + \|x_0\| \|R\| + \|S\|}.\]

Hence we have

\[\|Q_F(y_1) - Q_F(y_2)\| \leq \alpha\|y_1 - y_2\|,
\]

where $\alpha = \frac{2\|x_0\| \|R\|\|r + \|x_0\| \|R\| + \|S\|}{\|x_0\| \|R\|\|r + \|x_0\| \|R\| + \|S\|\|r}. $
But since
\[ 0 < r < \left( \frac{||S||}{(||x_0'|| ||R||)} \right)^{1/2} \]
then \( \alpha \in (0, 1) \) and this completes the proof that \( Q_F \) is a contraction map mapping \( \mathcal{Y}_1 \) into itself. Thus from [4, Theorem 1, p. 474], \( Q_F \) has the unique fixed point \( y^* = Q_F(y^*) \in \mathcal{Y}_1 \), and from hypothesis (2) and the definition of \( Q_F \), \( y^* \in \mathcal{Y} \). Hence equation (2.6) has the unique solution \( y^* \in \mathcal{Y} \). But since the two equations (2.4) and (2.6) have the same solution set in \( \mathcal{Y} \), then equation (2.2) has a unique solution \((y^*, \eta^*)\) in \( \mathcal{Y} \times C \) where \( y^* \) is the unique solution of equation (2.4) and \( \eta^* \) is given by the equation:
\[ \eta^* = x_0'(F(x_0' + y^*)) \]
Thus the system (2.1) has the unique solution \((x, \lambda) \in \{ z \in C : \| z - x_0' \| \leq r \} \times C \), where \( x = x_0' + y^* \) and \( \lambda = \lambda_0 + \eta^* \). This completes the proof of the theorem.

Remark 2.2. In Theorem 2.1, by choosing \( r \) and \( \delta \) small enough, the solution \((x, \lambda)\) of the system (2.1) can come close to \( x_0 ', \lambda_0 \) to any degree of accuracy. To explain this point, let \( \varepsilon > 0 \). If we let \( r \) vary in the interval \((0, \left( \frac{||S||}{(||x_0'|| ||x'|| ||R||)} \right)^{1/2})\), then
\[ \delta = \delta(r) = \frac{r}{(||x_0'|| ||S|| r^2 + (||x_0'|| ||R|| r + ||S||))} \]
is a function of \( r \). This function has a greatest lower bound equal to zero, and is increasing since the derivative:
\[ \delta'(r) = \frac{||S|| - ||x_0'|| ||R|| r^2}{(||x_0'|| ||R|| r^2 + (||x_0'|| ||R|| + ||S||))} > 0 \]
for all \( r \in (0, \left( \frac{||S||}{(||x_0'|| ||R||)} \right)^{1/2}) \). So if we choose \( r < \varepsilon \) such that
\[ \delta = \delta(r) < \varepsilon/(||x_0'||(1 + r)) \]
the unique solution \((x, \lambda)\) of the system (2.1) in the product space \( \{ z \in \mathcal{D}(D) : \| z - x_0' \| \leq r \} \times C \) satisfies
\[ \| x - x_0' \| < \varepsilon \]
and
\[ |\lambda - \lambda_0| \leq \| x_0' ||(1 + ||y'||)\|F\| \leq \frac{\varepsilon ||x_0'||(1 + ||y'||)}{||x_0'||(1 + r)} \leq \varepsilon \]

Remark 2.3. In [5] it is proved that if the hypotheses of Theorem 2.1 hold, \( r > 0 \) and
\[ \| F \| \leq \psi(r) = \frac{r}{(2 ||x_0'|| ||R|| r^2 + (||x_0'|| ||R|| + ||S||))} \]
then the conclusion of the theorem follows. Although the domain of the function \( \delta(r) \) in Theorem 2.1 is restricted to the interval \((0, r_0)\), where \( r_0 = \left( \frac{||S||}{(||x_0'|| ||R||)} \right)^{1/2} \), we can prove that given \( r_1 \in (0, \infty) \), there exists \( r_2 \in (0, r_0) \) such that \( r_2 \leq r_1 \) and \( \delta(r_2) > \psi(r_1) \). This assertion is proved by
considering the cases \( r_1 < r_0 \) and \( r_1 < r_0 \) separately.

(i) If \( r_1 < r_0 \), then from the definitions of \( \delta(r) \) and \( \psi(r_1) \), it is clear that
\[
\delta(r_1) > \psi(r_1).
\]

(ii) If \( r_1 \geq r_0 \), then since \( \psi(r) \) attains its maximum at
\[
r_0^* = \left( \frac{\|S\|}{(2\|x_0\| \|R\|)} \right)^{1/2} < r_0
\]
and \( \psi(r) \) is continuous, there exists \( r_2 < r_0^* \) such that \( \psi(r_2) = \psi(r_1) \).
So from case (i), \( \delta(r_2) > \psi(r_2) = \psi(r_1) \).

3. Diagonally dominant infinite matrices

In this section we show that for a diagonally dominant infinite matrix operator acting in \( l_2 \) and having a discrete spectrum consisting of simple eigenvalues that diverge to infinity, the eigenvalues of \( A \) approach the centers of the Gersgorin discs and their corresponding eigenvectors approach the unit coordinate vectors \( e_i, i \in \mathbb{N} \).

In applying Theorem 2.1 to an infinite matrix operator \( A = (a_{ij}) \) acting in \( l_p, 1 \leq p < \infty \), we consider the diagonal matrix \( d = \text{diag}(a_{11}, a_{22}, \ldots) \) as the unperturbed operator and regard \( F = ((1 - \delta_{ij})a_{ij}) \), where \( \delta_{ij} \) denotes the Kronecker delta, as the perturbation. Since \( e_i \in \mathcal{D}(D) \) for all \( i \in \mathbb{N} \), then \( \mathcal{D}(D) \) is dense in \( l_p \). Hence from [7, Theorem VII.1.1], the dual \( D' \) of \( D \) exists. Also \( D' \) has the same eigenvalues \( a_{11}, a_{22}, \ldots \) as \( D \) has. This follows from the following theorem.

**Theorem 3.1.** The dual \( D' \) of the matrix operator \( D = \text{diag}(a_{11}, a_{22}, \ldots) \) in \( l_p, 1 \leq p < \infty \), exists and has the same eigenvalues \( a_{ii}, i \in \mathbb{N} \), as \( D \) has.

**Proof.** As shown above, the dual \( D' \) of \( D \) exists.

Let \( \lambda \) be an eigenvalue of \( D' \) with a corresponding eigenvector \( y' \). Hence \( 0 \neq y' \in l_p' \) and \( D'y' = \lambda y' \). We have the equation:

\[
(3.1) \quad \lambda y'(x) = y'(Dx)
\]
for all \( x = (x_1, x_2, \ldots) \in \mathcal{D}(D) \).

Let \( y = (y_1, y_2, \ldots) \) be the vector in \( l_q \) \((1/p + 1/q = 1)\) which satisfies the equation:

\[
(3.2) \quad y'(x) = \sum_{i=1}^{\infty} x_i y_i
\]
for all \( x = (x_1, x_2, \ldots) \in l_p \) (see [6, Theorem III.5.2]). The vector \( y \neq 0 \) since \( \|y\| = \|y'\| \) and \( y' \neq 0 \). Substituting \( x = e_i, i \in \mathbb{N} \), in equations (3.1) and (3.2), we get the equation:

\[
(3.3) \quad \lambda y_i = a_{ii} y_i
\]
for all \( i \in \mathbb{N} \).
Since $y \neq 0$, then from equation (3.3) it follows that there is only one nonzero component of the vector $y$; denote it by $y_j$. At this component $y_j$ we have $\lambda = a_{jj}$. Thus $\lambda$ is an eigenvalue of $D$.

If $\lambda = d_{ii}$, $i \in \mathbb{N}$, then the linear functional $y' \in l_p'$ defined by the equation:

$$y'(x) = x_i,$$

for all $x = (x_1, x_2, \ldots) \in l_p$, satisfies

$$(D'y')(x) = y'(Dx) = a_{ii}x_i = \lambda x_i = \lambda y'(x)$$

for all $x = (x_1, x_2, \ldots) \in \mathcal{D}(D)$. This proves the $D'y' = \lambda y'$. Since $y' \neq 0$ ($\|y'\| = 1$) then $\lambda = a_{ii}$ is an eigenvalue of $D'$ and this completes the proof of the theorem.

From Theorem 2.1 and Theorem 4.2 of [2], we have

**Theorem 3.2.** Let $A = (a_{ij})$ be a matrix operator in $l_2$ and assume that

1. $\{a_{ii}\}$ is a strictly increasing sequence of real numbers diverging to infinity.
2. For every $i \in \mathbb{N}$,

$$P_i = \sum_j |a_{ij}| = \sigma_i |a_{ii}|, \quad \text{where } \sigma_i \in [0, 1].$$

3. $F = ((1 - \delta_{ij})a_{ij}) \in L(l_2)$.
4. $|a_{ii} - a_{jj}| > P_i + P_j$ for all $i, j \in \mathbb{N}$, $i \neq j$.

Then the spectrum of $A$ consists of a discrete, countable set of eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$ and for every $i \in \mathbb{N}$, $\lambda_i$ is a simple eigenvalue lying in the set $\mathcal{R}_i$.

Furthermore, if $a_{i+1,i+1} - a_{ii} \to \infty$ as $i \to \infty$ then $|\lambda_i - a_{ii}| \to 0$ as $i \to \infty$, and for every $i \in \mathbb{N}$ there exists an eigenvector $z_i$ of $A$ corresponding to the eigenvalue $\lambda_i$ such that $\|z_i - e_i\| \to 0$ as $i \to \infty$.

**Proof.** Let $k = \|F\|$ and $k_1 = k + 1$. Since $\|(D + k_1 I)^{-1}\| \leq 1/k_1$, then $\|F(D + k_1 I)^{-1}\| < 1$. Thus the operator $A_1 = (a_{ij}^{(1)}) = A + k_1 I$ satisfies the hypotheses of [2, Theorem 4.2] (in the case $p = 2$). So the spectrum of $A_1$ consists of a discrete, countable set of eigenvalues $\{\mu_i : i \in \mathbb{N}\}$ and for every $i \in \mathbb{N}$, $\mu_i$ is a simple eigenvalue of $A_1$ lying in the set $\{z \in \mathbb{C} : |z - a_{ii}^{(1)}| \leq P_i\}$. Since $\lambda_i = \mu_i - k_1$, $i \in \mathbb{N}$, there are eigenvalues of $A$, each of algebraic multiplicity equal to one, and a point $\mu$ is in the spectrum of $A_1$ if and only if $\mu - k_1$ is in the spectrum of $A$, then the spectrum of $A$ is a discrete, countable set of eigenvalues $\{\lambda_i : i \in \mathbb{N}\}$, and for every $i \in \mathbb{N}$, $\lambda_i$ is a simple eigenvalue lying in the set $\mathcal{R}_i$. This establishes the first part of the theorem. Now suppose $a_{i+1,i+1} - a_{ii} \to \infty$ as $i \to \infty$. We show that the matrix operator $D = \text{diag}(a_{11}, a_{22}, \ldots)$ in $l_2$ satisfies the hypotheses of Theorem 2.1. It is clear that $a_{ii}$, $i \in \mathbb{N}$, are the eigenvalues of $D$ with corresponding eigenvectors $e_i$, $i \in \mathbb{N}$. From Theorem 3.1, it follows that the dual $D'$ of $D$ has the same
eigenvalues as does $D$ and for every $i \in \mathbb{N}$, the linear functional $x'_i \in l'_2$ defined by the equation:

$$x'_i(x) = x_i,$$

for all $x = (x_1, x_2, \ldots) \in l_2$, is an eigenvector of $D'$ corresponding to the eigenvalue $a_{ii}$.

Now fix $i \in \mathbb{N}$ and let $\mathcal{H}_i = \{x \in l_2 : x'_i(x) = 0\}$. It is clear that $e_j \in \mathcal{H}_i$ for all $j \neq i$ and $x'_i(e_i) = 1$. Hence from the continuity of $x'_i$, $\mathcal{H}_i$ is the closure of the set $\text{span}\{e_j : j \in \mathbb{N} - \{i\}\}$. Also it is clear that the matrix operator $R_i = (r^{(i)}_{jk})$ defined by

$$r^{(i)}_{jk} = \begin{cases} c & \text{if } k = j = i, \\ (a_{jj} - a_{ii})^{-1} & \text{if } k = j \neq i, \\ 0 & \text{otherwise,} \end{cases}$$

where $c$ is an arbitrary complex number, maps $\mathcal{H}_i$ into itself and is the inverse of the restriction of the operator $(D - a_{ii}I)$ to the space $\mathcal{H}_i$. From equation (3.4), $R_i$ is bounded with norm:

$$(3.5) \|R_i\| = \max \left\{ \frac{1}{a_{ii} - a_{i-1,i-1}}, \frac{1}{a_{i+1,i+1} - a_{ii}} \right\}$$

if $i \geq 2$ (if $i = 1$, then $\|R_i\| = (a_{22} - a_{11})^{-1}$). Hence $D$ satisfies all the hypotheses of Theorem 2.1.

Now we show that for all $i \in \mathbb{N}$, the operator $S_i = R_i(I - e_i'x'_i)$ has the same norm as $R_i$ (from the proof of Theorem 2.1, the operator $I - e_i'x'_i$ maps $l_2$ onto $\mathcal{H}_i$. Hence $S_i$ is well defined with domain $l_2$). Since $S_i x = R_i x$ for all $x \in \mathcal{H}_i$, then

$$(3.6) \|R_i\| \leq \|S_i\|.$$ 

On the other hand let $x = \sum_{j=1}^{\infty} \alpha_j e_j \in l_2$ and $\|x\| \leq 1$. We have

$$(I - e_i'x'_i)(x) = x - x'_i \left( \sum_{j=1}^{\infty} \alpha_j e_j \right) e_i = x - x'_i \left( \sum_{j=1}^{\infty} \alpha_j e_j \right) e_i = x - x'_i \left( \sum_{j=1}^{\infty} \alpha_j e_j \right) e_i = x - \sum_{j=1}^{\infty} \alpha_j e_j - \alpha_i e_i$$
which is an element in $\mathcal{F}$; denote it by $x$. From [3, Theorem I.4.1] we have $\|x\| \leq 1$. Therefore $\|S_jx\| = \|R_jx\| \leq \|R_i\|$. Hence we have

\[(3.7) \quad \|S_i\| \leq \|R_i\|.
\]

Equations (3.6) and (3.7) prove $\|S_i\| = \|R_i\|$. From the hypothesis $a_{i+1,i+1} - a_{ii} \to \infty$ and equation (3.5), it follows that $\|R_i\| \to 0$ as $i \to \infty$. Hence there exists $i_0 \in \mathbb{N}$ such that for all $i \geq i_0$, \[(3.8) \quad k\|R_i\| < 1/4.
\]

For all $i \geq i_0$, we solve (in $r_i$):

\[(3.9) \quad k = r_i/((r_i^2 + 2r_i + 1)\|R_i\|).
\]

The solution of equation (3.9) is given by

\[(3.10) \quad r_i = \frac{(1 - 2k\|R_i\|) \pm \sqrt{1 - 4k\|R_i\|}}{2k\|R_i\|}.
\]

From equation (3.8), both values of $r_i$ in equation (3.10) are positive results and the smaller value, which we denote by $r_i$, is given by the equation:

\[(3.11) \quad r_i = \frac{2k\|R_i\|}{1 - 2k\|R_i\| + \sqrt{1 - 4k\|R_i\|}}.
\]

This value $r_i$ satisfies $r_i \in (0, 1)$ for all $i \geq i_0$. Since $\|x_i\| = \|e_i\| = 1$ and $\|S_i\| = \|R_i\|$ for all $i \in \mathbb{N}$, then from Theorem 2.1 (where we take $\delta = \|F\| = k$), the system of equations

\[A^jz_i = \lambda_iz_i, \quad x'_i(z_i) = 1,
\]

where $i \geq i_0$, has a unique solution $(z_i, \lambda_i)$ in the set $\{z \in \mathcal{D}(D) : \|z - e_i\| \leq r_i\} \times C$, where $r_i$ is the smaller solution of equation (3.9) and is given by equation (3.11). Since $\|R_i\| \to 0$ as $i \to \infty$ then from equation (3.11), $r_i \to 0$ as $i \to \infty$. Hence $\|z_i - e_i\| \to 0$ as $i \to \infty$. Also from equation (2.3),

\[(3.12) \quad \lambda_i - a_{ii} = x'_i(F(z_i)).
\]

But since $Fe_i$ is in the closure of the set $\text{span}\{e_j : j \neq i\}$ and $x'_i(e_j) = \delta_{ij}$, then $\lambda_i - a_{ii} = x'_i(F(z_i - e_i))$. Thus we have

\[(3.13) \quad |\lambda_i - a_{ii}| \leq \|x'_i\| \|F\| \|z_i - e_i\| \leq kr_i.
\]

But since $r_i \to 0$ as $i \to \infty$, then $|\lambda_i - a_{ii}| \to 0$ as $i \to \infty$ and this completes the proof of the theorem.

Remark 3.3. If hypothesis (2) of Theorem 3.2 is replaced by

\[Q_i = \sum_j |a_{ji}| = \sigma_i |a_{ii}|, \quad \text{where} \quad \sigma_i \in [0, 1],
\]
for all \( i \in \mathbb{N} \), we get a similar conclusion to that given in the theorem where the sets \( \mathcal{R}_i \) are replaced by \( \mathcal{C}_i \), \( i \in \mathbb{N} \).

### 4. Approximation of the Eigenvalues of Mathieu's Equation

In [1], we have shown that the eigenvalues of Mathieu's equation:

\[
d^2y/d\theta^2 + (a - 2\cos 2\theta)y = 0
\]

corresponding to the eigenfunctions \( c_2n(\theta, 1) \) are the eigenvalues of the infinite matrix operator \( B = (b_{ij}) \) defined in \( l_2 \) by:

\[
b_{12} = b_{21} = \sqrt{2}, \quad b_{23} = 1
\]

and

\[
b_{ij} = \begin{cases} 
1 & \text{if } j = 1 \pm 1, \ i > 3, \\
4(i-1)^2 & \text{if } j = i, \ i \geq 1, \\
0 & \text{otherwise}.
\end{cases}
\]

Let \( A = (a_{ij}) = B + 2I \). It is clear that \( A \) satisfies all the hypotheses of Theorem 3.2. Some calculations show that the operator \( F = ((1-\delta_{ij})a_{ij}) \) has norm equal to 2. Using the same notations introduced in Theorem 3.2, the operator \( R_i \) (which is the inverse of the restriction of \( D - a_{ii}I \) to the space \( \mathcal{H} \) and \( D = \text{diag}(a_{11}, a_{22}, \ldots) \)) has a norm:

\[
\|R_i\| = \frac{1}{4[(i-1)^2-(i-2)^2]} = \frac{1}{4(2i-3)}
\]

for all \( i \geq 2 \) and \( \|R_i\| = 1/4 \). For all \( i \geq 3 \), \( R_i \) satisfies equation (3.8). Hence from Theorem 3.2, the set of the eigenvalues of \( A \) is a discrete, countable set \( \{\lambda_i : i \in \mathbb{N}\} \), where for all \( i \geq 3 \):

(4.1) \[|\lambda_i - (2 + 4(i-1)^2)| \leq 2r_i\]

(see equation (3.13) and notice that \( \|F\| = k = 2 \)) and \( r_i \) is given by

(4.2) \[r_i = \frac{1}{\sqrt{\left(2(i-1) + \sqrt{4(i-2)^2 - 1}\right)}}\]

(see equation (3.11)). Since the eigenvalues of \( B \) are \( \mu_i = \lambda_i - 2, \ i \in \mathbb{N} \), then from equations (4.1) and (4.2), we have for all \( i \geq 3 \):

\[|\mu_i - 4(i-1)^2| \leq 2r_i\]

and \( r_i \to 0 \) as \( i \to \infty \). For example,

for \( i = 3 \):

\[r_3 = 2 - \sqrt{3} < 0.268\]

Hence \(|\mu_3 - 16| < 0.536\).

For \( i = 4 \):

\[r_4 = 4 - \sqrt{15} < 0.1271\]

For \( i = 10 \):

\[r_{10} = 16 - \sqrt{255} < 0.032\]

Hence \(|\mu_{10} - 324| < 0.064\).
REFERENCES


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