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AN OPERATOR-VALUED MOMENT PROBLEM

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ABSTRACT. We link Carey's exponential representation of the determining function of a perturbation pair with the moment problem. We prove that an operator sequence represents the moments of a phase operator if and only if there is another positively defined sequence of operators satisfying a boundedness condition.

INTRODUCTION

The L-moment problem consists in characterizing the moment sequence

$$A_n = \int_{\mathbb{R}} t^n f(t)\, dt, \qquad n \in \mathbb{N}, \tag{1}$$

of a measurable function f (with prescribed support in $\mathbb{R}$) which satisfies $0 \le f \le L$ a.e. This problem was formulated and completely solved by Achiezer and Krein in the 1930s [1]. The problem may be formulated for operator-valued functions. On the other hand, R. W. Carey introduced in [2] a complete unitary invariant which occurs in the perturbation theory of self-adjoint operators. This invariant, called in [2] the "phase shift," is a direct generalization of the phase shift which has been encountered in perturbation situations of one-dimensional range.

Carey proved in [2] that, for z a complex number with $\operatorname{Im} z \ne 0$ and $R_z = (A - z)^{-1}$, the determining function

$$\phi(z) = I + KR_zK^*$$

can be represented in the form:

$$\phi(z) = \exp\left(\int_{\mathbb{R}} B(\lambda)/(\lambda - z)\, d\lambda\right),$$

where B is a summable operator function $B\colon \mathbb{R} \to \mathscr{L}(\mathscr{X})$ with $0 \le B(\lambda) \le 1$. In the same paper he also proved the converse result: suppose $B(\lambda)$ is a

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summable function with values in $\mathscr{B}(\mathscr{X}, \mathscr{X})$ such that $0 \leq B(\lambda) \leq 1$; then

$$\phi(z) = \exp\left(\int_{\mathbb{R}} B(\lambda)/(\lambda - z)\, d\lambda\right)$$

is the determining function of a certain perturbation pair $\{A, K\}$ with A a selfadjoint operator acting on the Hilbert space $\mathscr{H}$, and $K: \mathscr{H} \to \mathscr{X}$ with $\mathscr{X}$ another Hilbert space. The function $B(\lambda)$ is called in [2] the phase operator corresponding to the perturbation pair $\{A, K\}$. $B(\lambda)$ is a complete unitary invariant associated with the perturbation problem $A \to A + K^*K$.

This paper studies the relationship between the two previously mentioned concepts. Hence we prove that the operator sequence $(A_n)_{n=0}^{\infty}$ represents the moments of a summable function

$$B: \mathbf{R} \to \mathscr{L}(\mathscr{H}), \qquad 0 \leq B(t) \leq 1,$$

with $\operatorname{supp} B$ compact if and only if we can find another sequence of operators $(A'_m)_{m=0}^{\infty}$ positively defined (see the definition given in the next section), with a certain boundedness condition and for which we have the relations

$$\exp\left(-\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1}.$$

The main ingredients of this proof are the two results contained in [2].

The main result

Our aim is to prove the following:

Theorem. *The sequence $(A_n)_{n=0}^{\infty}$ represents the successive moments of a summable operator function $B(\lambda)$, $0 \leq B(\lambda) \leq 1$, with* $\operatorname{supp} B$ *compact if and only if there is an operator sequence $(A'_m)_{m=0}^{\infty}$ such that*

$$\exp\left(-\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1},$$

with both sequences $(A'_m)_{m=0}^{\infty}$ and $(-A'_{m+k+2} + CA'_{m+k})_{m,k}$ positively defined for C a positive constant.

Remark. The condition of positive definition for $(-A'_{m+k+2} + CA'_{m+k})_{m,k}$ can be reformulated as a boundedness condition.

Proof. We assume first that $(A_n)_{n=0}^{\infty}$ represents the sequence of successive moments of a summable operator function $B: \mathbf{R} \to \mathscr{L}(\mathscr{H})$, $0 \leq B(\lambda) \leq 1$, i.e.,

$$A_n = \int_{\mathbb{R}} t^n B(t)\, dt = \int_{\operatorname{supp} B} t^n B(t)\, dt.$$

The first step is a reduction of the power series of the moments (1.1) to a Cauchy integral formula $(*)$.

Computing the sum for N-indices, we obtain

$$\begin{aligned}\sum_{n=0}^{\infty} A_n z^{-n-1} &= \sum_{n=0}^{\infty} z^{-n-1} \int_{\mathbb{R}} t^n B(t)\,dt \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{R}} 1/z(t/z)^n B(t)\,dt\end{aligned} \tag{1.1}$$

for $|z| > |t|$ outside the supp of B. From this we obtain

$$-\sum_{n=0}^{\infty} \int_{\mathbb{R}} 1/z(t/z)^n B(t)\,dt = -\int_{\mathbb{R}} (B(t)/(z-t))\,dt. \tag{$*$}$$

Next we build the sequence $(A'_m)_{m=0}^{\infty}$ using Carey's result on the phase shift [2]:

> If B is a $\mathscr{B}(\mathscr{X}, \mathscr{X})$-valued operator function, so that $0 \le B(\lambda) \le 1$, then
>
> $$\phi(z) = \exp\left(\int_{\mathbb{R}} (B(t)/(t-z))\,dt\right)$$
>
> is the determining function of the perturbation pair (A, K) on $\{\mathscr{H}, \mathscr{K}\}$, where A is a selfadjoint operator acting on the Hilbert space $\mathscr{H}$, and K an operator from $\mathscr{H}$ into another Hilbert space $\mathscr{X}$, i.e.,
>
> $$\phi(z) = 1 + KR_zK^* \quad \text{with } R_z = (A-z)^{-1}.$$

From the above theorem it follows that

$$\begin{aligned}KR_zK^* &= K(A-z)^{-1}K^* = -Kz^{-1}\left(\sum_{n=0}^{\infty}(A/z)^n\right)K^* \\ &= -K\left(\sum_{n=0}^{\infty} A^n z^{-n-1}\right)K^* = -\sum_{n=0}^{\infty} z^{-n-1} KA^nK^*.\end{aligned}$$

We identify $A'_m = KA^mK^*$; then $\phi(z)$ becomes

$$\phi(z) = I + KR_zK^* = I - \sum_{n=0}^{\infty} z^{-n-1}KA^nK^* = I - \sum_{m=0}^{\infty} A'_m z^{-m-1}.$$

Applying the preceding result to the function $B(\cdot)$ obtained from the sequence of moments in (1.1), we get the required equality:

$$\exp\left(-\sum_{n=0}^{\infty} A_n z^{-n-1}\right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1}.$$

We shall prove that the obtained Hankel quadratic form $\sum_{m,k} A'_{m+k} x_m x_k$ is *positively defined* (i.e., $\sum_{m,k} A'_{m+k} x_m x_k \ge 0$ for every $(x_m) \subset \mathscr{H}$ with finite support) and satisfies the next boundedness condition. More precisely,

Positivity

$$\sum_{k,m} \langle A'_{k,m} x_k , x_m \rangle \geq 0 \quad \forall (x_k) \subset \mathscr{H} ,$$

a family with finite support. Indeed,

$$\begin{aligned}\sum_{k,m} \langle K^* A^{m+k} K x_k , x_m \rangle &= \sum_{k,m} \langle A^k K x_k , A^m K x_m \rangle \\ &= \left\langle \sum_{k=0}^{\infty} A^k K x_k , \sum_{m=0}^{\infty} A^m K x_m \right\rangle = \left\| \sum_{m=0}^{\infty} A^m K x_m \right\| > 0.\end{aligned}$$

Boundedness condition. We shall prove that there exists a constant $C > 0$ so that

$$\sum_{m,n=0}^{\infty} \langle K A^{m+n+2} K^* x_n , x_m \rangle \leq \sum_{m,n=0}^{\infty} C \langle K A^{m+n} K^* x_n , x_m \rangle .$$

Indeed,

$$\left\| A \sum_{n=0}^{\infty} A^n K^* x_n \right\| \leq C \left\| \sum_{n=0}^{\infty} A^n K^* x_n \right\| ,$$

which is true for $\sqrt{C} = \|A\| > 0$.

Conversely, assume that we have the representation formula

$$\exp\left(- \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = I - \sum_{m=0}^{\infty} A'_m z^{-m-1} , \tag{1.2}$$

with $(A'_m)_{m=0}^{\infty}$ positively defined and satisfying the boundedness conditions above. Then we shall prove the existence of a summable operator-valued function $B : \mathbb{R} \to \mathscr{L}(\mathscr{H})$, with $0 \leq B(\lambda) \leq 1$ $\operatorname{supp} B$ compact, the function that will furnish the moments of the prescribed $(A_n)_{n=0}^{\infty}$ operator sequence, i.e.,

$$A_n = \int_{\mathbb{R}} B(t) t^n \, dt.$$

We shall consider the operator sequence $(A'_m)_{m=0}^{\infty}$, to be doubly indexed. With this assumption, $(A'_m)_{m=0}^{\infty}$ can be represented as an operator-valued, positively defined function

$$A' : \mathbb{N} \times \mathbb{N} \to \mathscr{L}(\mathscr{H}) , \quad A'(m , n) = A'_{m+n} .$$

The classical Kolmogorov theorem gives a decomposition for positively defined kernels:

> Let $K : I \times I \to \mathscr{L}(\mathscr{H})$ be a positively defined operator-valued function (i.e., $\sum_{i,j} \langle K(i , j) x_i , x_j \rangle \geq 0$ for every family $(x_i)_i$ with finite support). Then $K(i , j)$ admits a decomposition of the form $K(i , j) = h_i^* h_j$, with $h_i \in \mathscr{L}(\mathscr{H})$.

Thus A'_{m+n} can be represented as $A'_{m+n} = K_n^* K_m$.

From the boundedness and positivity conditions we can find a constant $C > 0$ such that $(-A'_{m+k+2} + CA'_{m+k})$ is positively defined. From the Kolmogorov decomposition, it follows that $(-K_{m+1}^* K_{n+1} + CK_m^* K_n)$ is positively defined. According to this, for $(x_k)_k$ an arbitrary family of vectors of finite support we have

$$C \sum_{k,m} \langle K_m^* K_k x_k , x_m \rangle \geq \sum_{k,m} \langle K_{m+1}^* K_{k+1} x_k , x_m \rangle ,$$

an inequality which becomes:

$$\sqrt{C} \left\| \sum_{k=0}^{\infty} K_k x_k \right\| \geq \left\| \sum_{k=0}^{\infty} K_{k+1} x_k \right\| .$$

We take by definition

$$A \left(\sum_{k=0}^{\infty} K_k x_k \right) := \sum_{k=0}^{\infty} K_{k+1} x_k .$$

Since the K_n are linear, so is A and, from our previous remarks, A is continuous. Taking $x_0 = (1, 0, \dots)$, $x_i = 0$, $i > 1$, we obtain $AK_0 = K_1$, and using the induction method for a suitable choice of $(x_n)_{n=0}^{\infty}$, we obtain $K_n = A^n K_0$. We prove now that A is a selfadjoint operator: For x in a dense subset of $\mathscr{H}$, we can find $(x_k)_k$ such that $x = \sum_{k=0}^{\infty} K_k x_k$. In this case,

$$\langle Ax , x \rangle = \left\langle \sum_{k=0}^{\infty} K_{k+1} x_k , \sum_{k=0}^{\infty} K_k x_k \right\rangle \in \mathbb{R} ,$$

because $K_k^* K_{k+1}$ are positively defined. Thus, $\langle Ax , x \rangle \in \mathbb{R}$, and so A is a selfadjoint operator.

With this assumption, (1.2) will be rewritten in the form

$$\begin{aligned} \exp \left(- \sum_{n=0}^{\infty} A_n z^{-n-1} \right) &= I - \sum_{n=0}^{\infty} K_0^* A^n K_0 z^{-n-1} \\ &= I - \sum_{n=0}^{\infty} 1/z K_0^* (A/z)^n K_0 = I + K_0^* R_z K_0 = \phi(z) , \end{aligned} \tag{1.3}$$

where $R_z = (A - z)^{-1}$ and $A = A^*$, $K = K_0^*$.

From this calculation we obtain the determining function of the perturbation pair $\{A, K\}$:

$$\phi(z) = I + K_0 R_z K_0^* \quad \text{and} \quad \exp \left(- \sum_{n=0}^{\infty} A_n z^{-n-1} \right) = \phi(z).$$

The perturbation function is a holomorphic operator-valued function satisfying

(1.1) $\phi(z)^* = \phi(\overline{z})$.

(1.2) $1/2i \{ \phi(z) - \phi(z)^* \} = \operatorname{Im} \phi(z) \geq 0$ for $\operatorname{Im} z > 0$.

(1.3) $\| \phi(z) - 1 \| = O(1/\operatorname{Im} z)$ as $|\operatorname{Im} z| \to \infty$.

We are now in the situation of applying the main theorem from [2]:

> Suppose $\phi(z)$ is the determining function of a perturbation pair $\{A, K\}$ on the Hilbert spaces, A a selfadjoint operator acting on $\mathscr{H}$, and $K: \mathscr{H} \to \mathscr{X}$. There exists a summable function $B(\lambda)$ with values in the set of positive operators of the unit ball of $\mathscr{B}(\mathscr{X}, \mathscr{X})$ such that
>
> $$\phi(z) = \exp\left(\int_{\mathbb{R}} (B(\lambda)/(\lambda - z))\, d\lambda\right), \qquad \operatorname{Im} z \neq 0.$$

Representing $\phi(z)$ in power series, we have

$$\begin{aligned}\phi(z) &= \exp\left(-\int_{\mathbb{R}} (B(\lambda)/z(1-\lambda/z))\, d\lambda\right) \\ &= \exp\left(-\sum_{n=0}^{\infty} z^{-n-1} \int_{\mathbb{R}} B(\lambda)\, \lambda^n\, d\lambda\right).\end{aligned} \tag{1.4}$$

From both representations (1.3) and (1.4),

$$A_n = \int_{\mathbb{R}} B(\lambda)\lambda^n\, d\lambda,$$

which ends the proof of the main theorem.

Final remarks

The support of function B can be characterized in terms of its moments, as in the scalar case (cf. [1]). Our approach also gives a necessary condition for the L-problem of moments with operator values, without restriction on the support.

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