FULLY INDECOMPOSABLE EXPONENTS
OF PRIMITIVE MATRICES

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(Communicated by Thomas H. Brylawski)

Abstract. If $A$ is a primitive matrix, then there is a smallest power of $A$ (its fully indecomposable exponent) that is fully indecomposable, and a smallest power of $A$ (its strict fully indecomposable exponent) starting from which all powers are fully indecomposable. We obtain bounds on these two exponents.

1. Introduction

Let $B_n$ denote the set of all matrices of order $n$ over the Boolean algebra \{0, 1\} where, in particular, $1 + 1 = 1$. Then $B_n$ is a semigroup whose binary operation is ordinary matrix multiplication. Let $J_n$ denote the matrix in $B_n$ each of whose entries equals 1. A matrix $A \in B_n$ is primitive provided there is a positive integer $k$ such that $A^k = J_n$; the least positive integer $k$ satisfying $A^k = J_n$ is the exponent $e(A)$ of $A$. Exponents of primitive matrices have been well studied, and those numbers that are exponents of primitive matrices of order $n$ have been completely determined [4, 6, 7]. In particular the largest exponent of a primitive matrix of order $n$ is $e_n = n^2 - 2n + 2$. We denote the set of primitive matrices in $B_n$ by $P_n$.

A matrix $A \in B_n$ is partly decomposable provided for some positive integers $r$ and $s$ with $r + s = n$, $A$ has an $r$ by $s$ zero submatrix. A matrix in $B_n$ that is not partly decomposable is called fully indecomposable. It is well known that a fully indecomposable matrix is primitive and the exponents of fully indecomposable matrices have been studied [3]. We denote the set of fully indecomposable matrices in $B_n$ by $F_n$. Thus $F_n \subseteq P_n$. Moreover, it follows that

\begin{equation}
P_n = \{A : A \in B_n \text{ and } A^k \in F_n \text{ for some positive integer } k\}.
\end{equation}
There is a one-to-one correspondence between the set \( B_n \) and the set \( \Gamma_n \) of digraphs with vertex set \( \{1, \ldots, n\} \). This correspondence is given as follows: If \( A = [a_{ij}] \in B_n \), then \( \Gamma(A) \) is the digraph in which there is an arc \((i, j)\) from \( i \) to \( j \) if and only if \( a_{ij} = 1 \) \((i, j = 1, \ldots, n)\). We note that since \( A \) may have 1's on its main diagonal, the digraph \( \Gamma(A) \) may have loops. We call a vertex \( i \) of a digraph a loop-vertex provided \((i, i)\) is a loop of the digraph. We call a digraph in \( \Gamma_n \) a primitive digraph provided it corresponds to a primitive matrix. The following properties are direct consequences of the definitions:

(1.2) If \( A \) is primitive, \( \Gamma(A) \) is strongly connected (that is, for each pair of distinct vertices \( i \) and \( j \), there is a walk from \( i \) to \( j \));

(1.3) \( A \) is primitive if and only if there is an integer \( k \) such that for each pair of distinct vertices \( i \) and \( j \) there is a walk in \( \Gamma(A) \) of length \( k \) from \( i \) to \( j \);

(1.4) \([1] A^k \) is fully indecomposable if and only if for each set \( X \) of \( r \) vertices with \( 0 < r < n \), there are at least \( r + 1 \) different vertices that can be reached by a walk of length \( k \) which starts at a vertex in \( X \).

In addition the following is a well-known characterization of primitive matrices:

(1.5) \( A \) is primitive if and only if the greatest common divisor of the lengths of all (elementary) cycles of \( \Gamma(A) \) is 1.

Let \( A \) be a primitive matrix. It follows from (1.1) that there is a smallest positive integer \( k \) such that \( A^k \) is fully indecomposable; we denote the smallest such integer by \( f(A) \). Schwarz [5] raised the question of determining the numbers

\[ f_n = \max\{f(A) : A \in P_n\} \quad (n \geq 1). \]

Before proceeding we observe an important difference that occurs in the investigation of the numbers \( e(A) \) and \( f(A) \) for primitive matrices \( A \). If \( A^k = J_n \), then because \( A \) can have no zero rows, \( A^i = J_n \) for all \( i \geq k \). However, if \( A^k \in F_n \), then it does not necessarily follow that \( A^i \in F_n \) for all \( i \geq k \). For example, let

\[ A = \begin{bmatrix} 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}. \]

Then \( A^i \notin F_7 \) \((i = 1, \ldots, 7)\); \( A^8, A^9 \in F_7 \); \( A^{10}, A^{11} \notin F_7 \); and \( A^i \in F_7 \) \((i \geq 12)\). (It follows (cf. (1.1)) that \( A \) is primitive since some power of \( A \) is...
fully indecomposable.) For a primitive matrix $A$ we define $f^*(A)$ to be the smallest positive integer $k$ such that $A^i$ is fully indecomposable for all $i \geq k$. We call $f(A)$ and $f^*(A)$, respectively, the fully indecomposable exponent and strict fully indecomposable exponent of the primitive matrix $A$. The matrix $A$ in (1.6) satisfies $f(A) = 8$ and $f^*(A) = 12$. In general we have

$$f(A) \leq f^*(A) \leq e(A) \quad (A \in \mathbb{P}_n).$$

We define

$$f_n = \max\{f^*(A) : A \in \mathbb{P}_n\} \quad (n \geq 1),$$

and hereby raise the question of determining the numbers $f_n^*$. Clearly $f_n \leq f_n^* (n \geq 1)$.

In what follows we obtain bounds for the numbers $f_n$ and $f_n^*$.

2. Bounds on the exponents

Let $A$ be a matrix in $B_n$, with associated digraph $\Gamma(A)$, and let $k$ be a nonnegative integer. For $X \subseteq \{1, \ldots, n\}$, $R_k(X)$ denotes the set of all those vertices that can be reached by a walk of length $k$ in $\Gamma(A)$ starting from a vertex in $X$. (If $k = 0$, $R_k(X) = X$.) A restatement of (1.4) is:

$$A^k \text{ is fully indecomposable if and only if } |R_k(X)| > |X| \text{ for all } X \text{ with } \phi \neq X \subseteq \{1, \ldots, n\}. \quad (2.1)$$

Chao and Zhang [2] showed that $f(A) \leq n$ if $A \in \mathbb{P}_n$ and $\text{trace}(A) \neq 0$. We refine this result to obtain an inequality for $f^*(A)$ in Theorem 2.2.

Lemma 2.1. Let $\Gamma$ be a strongly connected digraph with vertex set $\{1, \ldots, n\}$, let $s$ be a positive integer, and let $Z = \{i_1, \ldots, i_s\}$ be a set of $s$ loop-vertices of $\Gamma$. Then for each positive integer $t$,

$$|R_t(Z)| \geq \min\{s + t, n\}. \quad (2.2)$$

Proof. Suppose that $R_t(Z) \neq \{1, \ldots, n\}$. Since $\Gamma$ is strongly connected, there is an arc $(p, j)$ from some vertex $p \in R_t(Z)$ to a vertex $j \notin R_t(Z)$, because $j \notin R_t(Z)$ there is a vertex in $Z$, say $i_1$, such that the distance from $i_1$ to $p$ is $t$ and the distance from each of $i_2, \ldots, i_s$ to $p$ is at least $t$. Thus there is a walk $(i_1, \ldots, p)$ of length $t$ from $i_1$ to $p$ containing $t+1$ distinct vertices all of which are different from $i_2, \ldots, i_s$. Since $i_1, i_2, \ldots, i_s$ are loop-vertices, we conclude that

$$|R_t(Z)| \geq (s - 1) + (t + 1) = s + t,$$

and the lemma follows. □

Theorem 2.2. Let $s$ be a positive integer, and let $A$ be a matrix in $\mathbb{P}_n$ having $s$ 1's on its main diagonal. Then

$$f^*(A) \leq n - s + 1. \quad (2.3)$$
Proof. Let \( Z \) be the set of \( s \) loop-vertices of the digraph \( \Gamma(A) \), and let \( t \) be a positive integer. Let \( X \) be a set of vertices with \( \phi \neq X \subseteq \{1, \ldots, n\} \), and let \( k = |X| \). We show that

\[
|R_t(X)| \geq |X| + 1 \quad \text{for } t \geq n - s + 1.
\]

If \( |R_t(X)| = n \), then (2.4) holds. We now suppose that \( |R_t(X)| < n \). First assume that \( X \cap Z \neq \emptyset \). By Lemma 2.1

\[
|R_t(X)| \geq |R_t(X \cap Z)| \geq |X \cap Z| + t.
\]

Hence if \( t \geq n - s + 1 \),

\[
|R_t(X)| \geq |X \cap Z| + n - s + 1 \geq |X| + 1.
\]

Now assume that \( X \cap Z = \emptyset \). Let \( x^* \) be a vertex in \( X \) and let \( z^* \) be a vertex in \( Z \) such that \( x^* \) has the minimum distance \( d \) to \( z^* \) among all vertices \( x \) in \( X \) and \( z \) in \( Z \). Then

\[
d \leq n + 1 - |Z| - |X| = n + 1 - s - k.
\]

Because \( z^* \) is a loop-vertex, there are walks of length \( t \) from \( x^* \) to each vertex in \( R_k(z^*) \) for each integer \( t \geq (n + 1 - s - k) + k = n + 1 - s \). Using Lemma 2.1 we obtain that

\[
|R_t(X)| \geq |R_t(\{x^*\})| \geq |R_k(z^*)| \geq k + 1
\]

for each integer \( t \geq n + 1 - s \). Hence (2.4) holds, and using (1.4) we conclude that (2.3) holds. \( \Box \)

It is easy to construct matrices \( A \) satisfying the hypotheses of Theorem 2.2 for which equality holds in (2.3). Let \( s \) and \( n \) be positive integers with \( s \leq n \), and let \( \Gamma \) be the digraph obtained by including a loop at each of \( s \) consecutive vertices of a simple cycle of length \( n \). Then the matrix \( A \in B_n \) whose associated digraph is \( \Gamma \) is in \( P_n \) and satisfies \( f(A) = n + 1 - s \).

Corollary 2.3 (cf. [2]). Let the matrix \( A \) in \( P_n \) have nonzero trace. Then

\[
f(A) < f^*(A) < n.
\]

Corollary 2.4. Let \( A \) be a matrix in \( P_n \). Suppose that the digraph \( \Gamma(A) \) has a cycle of length \( r \) and that there are \( s \) vertices which belong to at least one cycle of length \( r \). Then

\[
f(A) \leq r(n - s + 1).
\]

Proof. The matrix \( A^r \) has \( s \) 1's on its main diagonal, and hence by Theorem 2.2 \( (A^r)^{n-s+1} \) is fully indecomposable. \( \Box \)

The matrix

\[
A = \begin{bmatrix}
0 & 1 & 0 & 1 \\
0 & 0 & 0 & 1 \\
0 & 1 & 0 & 0 \\
1 & 0 & 1 & 0
\end{bmatrix}
\]
satisfies the hypotheses of Corollary 2.4 with \( r = 3 \) and \( s = 4 \). Hence by (2.6), \( f(A) \leq 3 \). Since \( A^2 \neq J_4 \), \( f(A) = 3 \) and equality holds in (2.6).

We note the following consequences of Corollary 2.4:

\[
\text{(2.7) If } \Gamma(A) \text{ has a Hamilton cycle (so } r = s = n\text{), then } f(A) \leq n. \]

\[
\text{(2.8) If } A \text{ is symmetric (so } r = 2 \text{ and } s = n\text{), then } f(A) \leq 2. \]

**Corollary 2.5.** Let \( A \) be a matrix in \( P_n \). Suppose that the digraph \( \Gamma(A) \) has diameter \( d \). Then \( f(A) \leq 2d(n - d) \).

**Proof.** Since \( \Gamma(A) \) is strongly connected, there is a cycle of length \( r \leq 2d \) containing \( s \geq d + 1 \) distinct vertices. Hence by (2.6), \( f(A) \leq 2d(n - d) \).

We now use Corollary 2.4 to obtain an upper bound on the numbers \( f_n(n \geq 1) \).

**Theorem 2.6.** \( f_n \leq \lfloor \frac{1}{2}(n - 1)(n + 3) \rfloor \) for \( n \geq 1 \).

**Proof.** Let \( A \) be a matrix in \( P_n \). Since \( \Gamma(A) \) is strongly connected, \( \Gamma(A) \) has a simple cycle of length \( r \) for some \( r \) with \( 1 \leq r \leq n \). Let \( s \) be the number of vertices which belong to at least one cycle of length \( r \). Then \( s \geq r \) and by Corollary 2.4,

\[
\text{(2.9) } f(A) \leq r(n - s + 1) \leq r(n - r + 1). \]

For integral \( r \), \( r(n - r + 1) \) achieves its maximum when \( r = (n + 1)/2 \) (\( n \) odd) and \( r = n/2, \, n/2 + 1 \) (\( n \) even). Since \( A \) is primitive, then by (1.5) the greatest common divisor of the lengths of the cycles is 1. Hence if \( n \) is odd and \( r = (n + 1)/2 \), \( \Gamma(A) \) has a cycle of length different from \( (n + 1)/2 \).

We now obtain from (2.6) that

\[
f(A) \leq \begin{cases} (n^2 + 2n)/4, & (n \text{ even}) \\ (n^2 + 2n - 3)/4, & (n \text{ odd}) \end{cases}
\]

and the theorem follows.

The upper bound for \( f_n \) in Theorem 2.6 probably is not of the same order of magnitude as \( f_n \). The example (1.6) can be generalized to all \( n \geq 5 \) to show that \( f_n \geq 2n - 4 \). Indeed it is tempting to conjecture that \( f_n = 2n - 4 \) (\( n \geq 5 \)).

We now consider the maximum strict fully indecomposable exponent \( f^*_n \).

First we obtain by example a lower bound for \( f^*_n \).

Suppose that \( k \) and \( n \) are integers with \( n \geq 5 \) and \( 2 \leq k \leq n - 3 \). Let \( A \) be the matrix in \( B_n \) whose associated digraph is pictured in Figure 1 (p. 1198). This digraph has cycles of lengths \( n - k + 1 \) and \( n - k \), and hence by (1.5) \( A \in P_n \). Let \( X_k = \{n - k + 1, \ldots, n\} \). One easily checks that

\[
|R_{i(n-k)-1}(X_k)| = i \quad (i = 1, \ldots, k).
\]

Hence it follows from (2.1) that

\[
\text{(2.10) } f^*_n(A) \geq k(n - k). \]
Indeed one may show that $f^*(A) = k(n - k)$. Taking $k = \lfloor n/2 \rfloor$ in (2.10) we obtain

\[(2.11) \quad f^*_n \geq \lfloor n/2 \rfloor \lceil n/2 \rceil \quad (n \geq 5).\]

For a primitive matrix $A$ in $B_n$, $f^*(A) \leq e(A) \leq e_n = n^2 - 2n + 2$ and hence it follows that

\[(2.12) \quad f^*_n \leq n^2 - 2n + 2.\]

Let $\lambda(A)$ denote the number of distinct lengths of the cycles of $\Gamma(A)$. It follows from (1.5) that $\lambda(A) \geq 2$ if $n > 1$. We now turn our attention to obtaining an improved bound for $f^*(A)$ in the case that $\lambda(A) = 2$.

**Lemma 2.7.** Let $\Gamma$ be a digraph in $\Gamma_n$, and let $\gamma$ be a cycle of $\Gamma$ of length $r$. If $X$ is a set of vertices belonging to the cycle $\gamma$, then

\[R_{ir+j}(X) \subseteq R_{(i+1)r+j}(X) \quad (i \geq 0; \quad 0 \leq j \leq r-1).\]

**Proof.** If from some $x$ in $X$ there is a path of length $ir+j$ to a vertex $z$, then there is also a walk from $x$ to $z$ of length $(i+1)r+j$. $\square$

**Lemma 2.8.** Let $\Gamma$ be a digraph in $\Gamma_n$, and let $\gamma$ be a cycle of $\Gamma$ of length $r$. Let $X$ be the set of all vertices belonging to the cycle $\gamma$. Then

\[R_i(X) \subseteq R_{i+1}(X) \quad (i \geq 0).\]

**Proof.** If from some vertex $x$ of $X$ there is a walk of length $i$ to a vertex $z$, then there is a walk of length $i+1$ to $z$ from the vertex which precedes $x$ in $\Gamma$. $\square$

**Corollary 2.9.** If $Z$ is the set of vertices of $\Gamma$ consisting of those vertices $z$ for which there is a walk of length at most $d$ to $z$ from some vertex of the cycle $\gamma$, then $R_d(X) = Z$.

**Lemma 2.10.** Let $r$ and $s$ be relatively prime positive integers with $r > s$. Let $\Gamma$ be a digraph in $\Gamma_n$ with exactly two cycles $\gamma$ and $\mu$ where $\gamma$ has length $r$, and $\mu$ has length $s$. Then

\[R_{ir+j}(X) \subseteq R_{(i+1)r+j}(X) \quad (i \geq 0; \quad 0 \leq j \leq r-1).\]}
Figure 2. A digraph satisfying the hypotheses of Lemma 2.10 with $n = 10$, $r = 8$, and $s = 5$.

$\mu$ has length $s$, and $\gamma$ and $\mu$ intersect. Let $X$ be a nonempty set of vertices of the cycle $\gamma$. Then

$$|R_i(X)| \geq \min\{n, |X| + l\} \quad \text{if } i \geq lr \text{ and } l \geq 1. \quad (2.13)$$

**Proof.** A digraph $\Gamma$ satisfying the hypotheses is pictured in Figure 2. Let $Z$ be the set of vertices of $\gamma$, so that $\phi \neq X \subseteq Z$. For $i \geq 1$ let $X^{(i)}$ denote the set of vertices of $\gamma$ that are reachable from $X$ by a walk in $\gamma$ of length $i$. It follows that $X^{(i)} = X^{(j)}$ if $i \equiv j \pmod{r}$. We first show that for $X \neq Z$, we have $|R_i(X) \cap Z| > |X|$. Since $X = X^{(r)} \subseteq R_r(X) \cap Z$, it suffices to show that $(R_r(X) \setminus X) \cap Z$ is nonempty. Suppose this set were empty. Then making use of the cycle $\mu$ we see that $X^{(r-s)} \subseteq R_r(X)$ and hence $X^{(r-s)} = X$. This implies that $X = X^{(s)}$ which contradicts the fact that $r$ and $s$ are relatively prime. It follows that (2.12) holds if $i = lr$ and $l \geq 1$. It now suffices to show that (2.12) holds for $l = 1$ and $r + 1 \leq i < 2r$.

If $X = Z$, then using Lemma 2.8 we see that

$$|R_i(X)| \geq \min\{n, |X| + i\}$$

for all $i \geq 1$. Hence we may assume that $X \neq Z$. If $R_i(X) \notin Z$, then the desired conclusion holds. We now assume that $R_i(X) \subseteq Z$. Contrary to what we wish to prove, we assume that $|R_i(X)| = |X|$, from which it follows that $R_i(X) = X^{(i)}$. Since $R_i(X) \subseteq Z$, we have that $R_{i-s}(X) \subseteq R_i(X)$. It follows that $X^{(i-s)} \subseteq X^{(i)}$ and hence that $X^{(i-s)} = X^{(i)}$, implying that the set $Y = X^{(i-s)}$ of vertices of $\gamma$ satisfies $Y^{(s)} = Y$. Again we contradict the assumption that $r$ and $s$ are relatively prime. The proof of the lemma is now complete. \qed

We now obtain a bound for the strict fully indecomposable exponent $f^*(A)$ of a primitive matrix $A$ for which the number $\lambda(A)$ of distinct cycle lengths of the digraph $\Gamma(A)$ equals 2.

**Theorem 2.11.** Let $A$ be a matrix in $P_n$ satisfying $\lambda(A) = 2$. Then

$$f^*(A) \leq \lfloor (n + 1)^2/4 \rfloor.$$
Proof. Let the lengths of the cycles of $\Gamma(A)$ be $r$ and $s$ where $r > s$, and $r$ and $s$ are relatively prime. There exist cycles $\gamma$ and $\mu$ of $\Gamma$ with lengths $r$ and $s$, respectively, such that $\gamma$ and $\mu$ intersect. Let the digraph $\Gamma^*$ consist of the vertices and arcs of $\gamma$ and $\mu$, and let $m$ be the number of vertices of $\Gamma^*$. Let $Y$ be a subset of the vertices of $\Gamma$ with $1 \leq k = |Y| \leq n - 1$. First suppose that $\gamma$ and $\mu$ above can be chosen so that $Y$ contains $p \geq 1$ vertices of $\gamma$. Applying Lemma 2.10 and Corollary 2.9, we see that

$$|R_i(Y)| \geq k + 1 \quad (i \geq (k - p + 1)r).$$

Since $r \leq n - (k - p)$, an easy calculation shows that

$$(k - p + 1)r \leq \left[ \frac{1}{4}(n + 1)^2 \right].$$

Now suppose that $\gamma$ and $\mu$ cannot be chosen so that $\gamma$ contains a vertex of $Y$. In this case $r \leq n - k$. There is a walk of length $t$ from some vertex in $Y$ to some vertex $x$ of $\gamma$ where $t \leq n - r - k + 1$. Applying Lemma 2.10 and Corollary 2.9 again, we see that

$$|R_i(\{x\})| \geq k + 1 \quad (i \geq kr).$$

Hence

$$|R_i(Y)| \geq k + 1 \quad (i \geq kr + n - r - k + 1).$$

An easy calculation now shows that

$$kr + n - r - k + 1 \leq \left[ \frac{n^2}{4} \right] + 1.$$ 

It follows that for all $Y$ with $\phi \neq Y \subseteq X$,

$$|R_i(Y)| \geq |Y| + 1 \quad (i \geq \left[ \frac{(n + 1)^2}{4} \right]).$$

Hence $A^t$ is fully indecomposable for all integers $i \geq \left[ \frac{(n + 1)^2}{4} \right].$ \hfill $\Box$

We conclude with the following remark. Let $A$ be a primitive matrix of order $n$. The exponent $e(A)$ of $A$ satisfies $e(A) \leq n^2 - 2n + 2$. If $\lambda(A) \geq 3$, then by a theorem of Lewin and Vitek [4], $e(A) \leq \left[ (n^2 - 2n + 2)/2 \right] + 1$. In particular, the largest exponent for matrices in $P_n$ occurs among those matrices with $\lambda(A) = 2$. We conjecture that the strict fully indecomposable exponent behaves in a similar way and thus that

$$f_n^* \leq \left[ \frac{(n + 1)^2}{4} \right].$$

In view of (2.11) the validity of this latter inequality would imply that $f_n^*$ is roughly $n^2/4$.

References


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