VON NEUMANN ALGEBRAS WHICH ARE SECOND DUAL SPACES

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Abstract. A $\sigma$-finite von Neumann algebra is a second dual if and only if it is atomic.

Shtern [7] has shown that a $C^*$-algebra is a third dual if and only if it is the enveloping von Neumann algebra of a von Neumann algebra. $C^*$-algebras that are dual spaces are the von Neumann algebras. The remaining question appears to be: when is a $C^*$-algebra a second dual? This question is the same as asking when a von Neumann algebra is a second dual. Let $M$ be a von Neumann algebra and let $M = X^{**}$ for some Banach space $X$. By the Krein-Milman Theorem, $M$ has a pure normal state and hence $M$ has a type I summand. Can $M$ have type II and type III summands as well? The answer is yes if, for instance, $X$ is the quotient of the algebra $B(H)$ of bounded operators on a separable Hilbert space $H$ by the compact operators $K(H)$ [1, 6]. The above observation seems to suggest, without some restriction, that the question may be too general to have a fruitful answer. If, however, we restrict the question to the class of $\sigma$-finite von Neumann algebras, then a simple answer emerges.

Theorem. A $\sigma$-finite von Neumann algebra $M$ is a second dual space if and only if $M$ is atomic, that is, if and only if $M$ is a direct sum of the $B(H)$'s.

In the commutative case, the above is of course a well-known result of Pelczynski [4, Theorem 4] since a commutative $\sigma$-finite von Neumann algebra is just an $L_\infty(\mu)$ where $\mu$ is a $\sigma$-finite measure and in this case, Pelczynski has shown that $L_1(\mu)$ is (isomorphic to) a dual if and only if $\mu$ is purely atomic. Rosenthal [5, Corollary 2.2] has given a short proof of this result. Distel and Uhl [3, p. 83] gave a shorter proof using the Radon-Nikodym property and their approach works well in the noncommutative case.

Recall that the $\sigma$-finite von Neumann algebras are those that admit faithful normal states. Examples: (i) any von Neumann algebra acting on a separable Hilbert space; (ii) all von Neumann algebras appeared in quantum statistical mechanics and quantum field theory. A Banach space is called weakly compactly generated (WCG) if it is the (norm) closed linear span of one of its
weakly compact subsets. Infinite-dimensional von Neumann algebras are not WCG since a WCG von Neumann algebra has the Radon-Nikodym property [3, p. 87] and must be finite dimensional.

**Lemma.** A von Neumann algebra $M$ is $\sigma$-finite if and only if its predual $M_*$ is WCG.

**Proof.** Suppose $M$ has a faithful normal state $\varphi \in M_*$. By [8, Proposition 5.12], the set $K = \{\varphi(a) : a \in M, \|a\| \leq 1\}$ is a relatively $\sigma(M_*, M)$-compact set in $M_*$. The linear span $\text{lin} K$ is (norm) dense in $M_*$. Otherwise, there exists nonzero $b$ in $M$ with $\|b\| \leq 1$ such that $\langle \psi, b \rangle = 0$ for all $\psi \in \text{lin} K$. If $b = u|b|$ is the polar decomposition, then $\langle \varphi, \|b\| \rangle = \langle \varphi(u^*), b \rangle = 0$ as $\varphi(u^*) \in K$, contradicting the faithfulness of $\varphi$. So $M_*$ is WCG. Conversely, let $M_*$ be generated by a $\sigma(M_*, M)$-compact set $S$. By [8, Theorem 5.4], there is a normal state $\omega \in M_*$ with the property that for any $\varepsilon > 0$, there exists $\delta > 0$ such that $|\langle \psi, a \rangle| < \varepsilon$ for all $\psi \in S$ whenever $\|a\| \leq 1$ and $\langle \omega, a^*a + aa^* \rangle < \delta$. Hence $\langle \omega, a^*a \rangle = 0$ implies $\langle \psi, (a^*a)^{1/2} \rangle = 0$ for all $\psi \in S$ and so $\langle \psi, (a^*a)^{1/2} \rangle = 0$ for all $\psi \in M_*$, giving $a = 0$. That is, $\omega$ is faithful.

Now, if a von Neumann algebra $M$ is $\sigma$-finite as well as a second dual, then its predual $M_*$ is WCG and a dual. So $M_*$ has the Radon-Nikodym property [3, p. 87]. By [2, Theorem 4], $M$ is atomic.

**References**