PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 112, Number 4, August 1991

A NOTE ON CONTINUOUS MAPPINGS AND THE PROPERTY OF J. L. KELLEY

HISAO KATO

(Communicated by James E. West)

ABSTRACT. In this paper, it is proved that if X is a continuum and ω is any Whitney map for $C(X)$, then the following are equivalent:

(1) X has property [K].

(2) There exists a (continuous) mapping $F: X \times I \times [0, \omega(X)] \to C(X)$ such that $F(\{x\} \times I \times \{t\}) = \{A \in \omega^{-1}(t) | x \in A\}$ for each $x \in X$ and $t \in [0, \omega(X)]$, where $I = [0, 1]$.

(3) For each $t \in [0, \omega(X)]$, there is an onto map $f: X \times I \to \omega^{-1}(t)$ such that $f(\{x\} \times I) = \{A \in \omega^{-1}(t) | x \in A\}$ for each $x \in X$. Some corollaries are obtained also.

0. INTRODUCTION

By a *continuum* we mean a compact connected metric space. For a continuum X, $C(X)$ denotes the *hyperspace* of all nonempty subcontinua of X, with the topology induced by the *Hausdorff metric* d_H. Then the hyperspace $C(X)$ is a continuum, and in fact, is pathwise connected. In [6, 2.7. Theorem], Kelley proved that $C(X)$ is the continuous image of the Cantor fan, i.e., the cone over the Cantor set.

In this paper, we prove the following: Let X be a continuum and ω be any Whitney map for $C(X)$. Then the following are equivalent:

(1) X has property [K].

(2) There exists a map $F: X \times I \times [0, \omega(X)] \to C(X)$ such that

$$F(\{x\} \times I \times \{t\}) = \omega_x^{-1}(t)$$

for each $x \in X$, where $\omega_x^{-1}(t) = \{A \in \omega^{-1}(t) | x \in A\}$.

(3) For each $t \in [0, \omega(X)]$, there is an onto map $f: X \times I \to \omega^{-1}(t)$ such that $f(\{x\} \times I) = \omega_x^{-1}(t)$ for each $x \in X$.

Received by the editors April 18, 1988.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 54B20, 54C65; Secondary 54C05, 54C60.

Key words and phrases. Hyperspaces of continua, Whitney map, property [K], continuous selection, weakly chainable, uniformly pathwise connected.

Note that if F is a map that satisfies (2) as above, then $F(\{x\}\times I\times\{0\}) = \{x\}$ for each $x \in X$ and $F|X \times I \times \{t\} \to \omega^{-1}(t)$ is surjective. It is known that the hyperspace $C(X)$ is contractible if and only if there is a map $G\colon X \times [0, \omega(X)] \to C(X)$ such that $G(x, t) \in \omega_x^{-1}(t)$ for each $x \in X$ and $t \in [0, \omega(X)]$, where ω is any Whitney map for $C(X)$ (see [6]).

To prove the above theorem, we use a selection theorem of D. W. Curtis [2, Theorem 2.2] and an idea of S. Ferry [4, the proof of 3.1]. As a corollary, the property of being weakly chainable (or uniformly pathwise connected) is a Whitney property for the class of continua that have property [K]. The first is a partial answer to Roger's problem [11, pp. 384, 112]. Also, the following problem is considered: Is it true that the property [K] is a Whitney property?

We refer the reader to Nadler's monograph [13] for hyperspace theory.

1. Preliminaries

Let X be a continuum. A map $\omega\colon C(X) \to [0, \infty)$ is said to be a *Whitney map* for $C(X)$ provided that ω satisfies the following conditions:

(1) $\omega(\{x\}) = 0$ for each $x \in X$ and

(2) if $A, B \in C(X)$, $A \subset B$, and $A \neq B$, then $\omega(A) < \omega(B)$.

In [15] Whitney proved that there always exists a Whitney map on any continuum. Then $\omega^{-1}(t)$ $(0 \le t < \omega(X))$ is a continuum and it is called a *Whitney continuum.* Let X be a continuum with metric d. Then X has *property* [K] [6] if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every pair of points x, y of X with $d(x, y) < \delta$ and every subcontinuum A containing x, there exists a subcontinuum B containing y with $d_H(A, B) < \varepsilon$. In [14] Wardle proved that a continuum X has property [K] if and only if the set-valued function $\alpha\colon X \to C(X)$, where $\alpha(x) = \{A \in C(X) | x \in A\}$, is continuous. We may assume that X is naturally contained in $C(X)$.

Let (Y, d) be a metric space, and for each positive integer n let $P_n = \{(t_i) \in I^n | \sum_{i=1}^n t_i = 1\}$. A convex structure on (Y, d) [2,2.1] is a sequence of subsets $M_n \subset Y^n$ and maps $k_n\colon M_n \times P_n \to Y$ satisfying the following conditions:

(C1) $$k_n(y, \dots, y; t_1, \dots, t_n) = y;$$

(C2) $$\begin{aligned} &k_n(y_1, y_2, \dots, y_n; t_1, t_2, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \\ &\quad = k_{n-1}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n); \end{aligned}$$

(C3) for each $\varepsilon > 0$ there is $\delta > 0$ such that for every n and $(t_i) \in P_n$, $d(k_n((y_i); (t_i)), k_n((y_i'); (t_i)) < \varepsilon$ if $d(y_i, y_i') < \delta$ for each i.

A subset C of Y is *convex* if for each n, $C^n \subset M_n$ and $k_n(C^n \times P_n) \subset C$. In [2, 2.2. Theorem] Curtis proved the following selection theorem:

(1.1) **Theorem** (D. W. Curtis). *Let X be paracompact, (Y, d) a metric space with a convex structure, and $\Phi\colon X \to Y$ a lower semicontinuous set-valued*

function, with each $\Phi(x)$ *a complete, convex subset of* Y. *Then* Φ *admits a continuous selection* $s: X \to Y$.

2. Result

In this section, we prove the following main result of this paper:

(2.1) **Theorem.** *Let* X *be a continuum and* ω *be a Whitney map for* $C(X)$. *Then the following are equivalent*:

(1) X *has property* [K].

(2) *There exists* $F: X \times I \times [0, \omega(X)] \to C(X)$ *such that* $F(\{X\} \times I \times \{t\}) = \omega_x^{-1}(t)$ *for each* $x \in X$ *and* $t \in [0, \omega(X)]$, *where* $\omega_x^{-1}(t) = \{A \in \omega^{-1}(t) | x \in A\}$.

(3) *For each* $t \in [0, \omega(X)]$, *there is an onto map* $f: X \times I \to \omega^{-1}(t)$ *such that* $f(\{x\} \times I) = \omega_x^{-1}(t)$ *for each* $x \in X$.

To prove (2.1) we need the following:

(2.2) **Theorem** cf. [7, (2.3)]). *Let* X *be a continuum and* ω *be any Whitney map for* $C(X)$. *Then for any* $\varepsilon > 0$ *there is* $\delta > 0$ *such that if* $A, B \in C(X)$, $|\omega(A) - \omega(B)| < \delta$ *and* $B \subset U(A, \delta)$, *then* $d_H(A, B) < \varepsilon$, *where* $U(A, \delta)$ *denotes the* δ*-neighborhood of* A *in* X.

A map $f: X \to B$ between metric spaces is said to be *strongly regular* if f is proper and if for each $b \in B$ and $\varepsilon > 0$ there is $\delta > 0$ such that if $d(b, b') < \delta$, then there are maps $g_{bb'}: f^{-1}(b) \to f^{-1}(b')$ and $g_{b'b}: f^{-1}(b') \to f^{-1}(b)$ and homotopies $h_t: f^{-1}(b) \to f^{-1}(b)$ and $k_t: f^{-1}(b') \to f^{-1}(b')$ such that

(i) $d(g_{bb'}(x), x) < \varepsilon$ and $d(h_t(x), x) < \varepsilon$ for all $x \in f^{-1}(b)$ and $0 \le t \le 1$,

(ii) $d(g_{b'b}(x), x) < \varepsilon$ and $d(k_t(x), x) < \varepsilon$ for all $x \in f^{-1}(b')$ and $0 \le t \le 1$,

(iii) $h_0 = g_{b'b} \cdot g_{bb'}$ and $h_1 = \mathrm{id}$,

(iv) $k_0 = g_{bb'} \cdot g_{b'b}$ and $k_1 = \mathrm{id}$.

(2.3) **Lemma.** *Let* $\Lambda(X) \subset C(C(X))$ *denote the space of maximal order arcs in* $C(X)$, *and let* $e: \Lambda(X) \to X$ *be the map defined by* $e(\alpha) = \alpha(0)$ *for* $\alpha \in \Lambda(X)$ (*see* [2]). *If* X *has property* [K], *then* e *is a strongly regular mapping.*

Proof. Let ω be a Whitney map for $C(X)$. Also, let $\varepsilon > 0$ and $x_0 \in X$. By (2.2) there exist open subsets $U_1^*, U_2^*, \dots, U_n^*$ of $\Lambda(X)$ such that $\bigcup U_i^* \supset e^{-1}(x_0)$ and diam $U_i^* < \varepsilon/2$ for each i, where U_i^* is of the following form:

$$U_i^* = \{\alpha \in \Lambda(X) | \alpha(t_j^i) \subset V_j^i \text{ for some } 0 \le t_0^i < t_1^i < \cdots < t_{n_i}^i \le \omega(X) \text{ and open subsets } V_0^i, V_1^i, \dots, V_{n_i}^i \text{ of } X\},$$

where $\alpha(t_j^i) \in \alpha$ with $\omega(\alpha(t_j^i)) = t_j^i$.

Let $W_i^* = \{\alpha \in \Lambda(X) | \alpha(t_j^i) \subset \mathrm{Cl}\, V_j^i$ for each $j = 0, 1, \ldots, n_i\}$. Note that W_i^* is closed in $\Lambda(X)$ and $U_i^* \subset W_i^*$. Note that if V^i $(i = 1, 2, \ldots,)$ is a neighborhood of x_0 such that $\mathrm{Cl}\, V^{i+1} \subset V^i$, $\cap \mathrm{Cl}\, V^i = \{x_0\}$, and $U_i^* = \{\alpha \in \Lambda(X) | \alpha(0) \in V^i\}$, then $\bigcap W_i^* = e^{-1}(x_0)$.

We may assume that $\operatorname{diam} W_i^* < \varepsilon$ and $\bigcap_{j\in J} U_j^*(x_0) = \varnothing$ if and only if $\bigcap_{j\in J} U_j^* \neq \varnothing$ if and only if $\bigcap_{j\in J} W_j^* \neq \varnothing$, where J is a subset of $\{0, 1, \ldots, n\}$ and $U_i^*(x_0) = e^{-1}(x_0) \cap U_i^*$. Since e is an open map, there is a neighborhood O of x_0 in X such that if $x \in O$, then $e^{-1}(x) \subset \bigcup U_i^*$ and $e^{-1}(x) \cap \bigcap_{j\in J} U_j^* \neq \varnothing$ if and only if $\bigcap_{j\in J} U_j^*(x_0) \neq \varnothing$. For each $x \in O$, let $W_i^*(x) = e^{-1}(x) \cap W_i^*$.

We show that $W_i^*(x)$ is convex. Consider $\alpha_1, \alpha_2, \ldots, \alpha_n \in e^{-1}(x)$ and $(t_1, t_2, \ldots, t_n) \in P_n$. Suppose first that each $t_i > 0$. Define $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) = \alpha$, where $\alpha = \{\alpha_1(\tau_1) \cup \cdots \cup \alpha_{i-1}(\tau_{i-1}) \cup \alpha_i(t) | 1 \leq i \leq n, 0 \leq t \leq \tau_i\} \subset C(X)$ and $\tau_i = t_i/(t_i + \cdots + t_n)$. In the case that $t_i = 0$ for some i, $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n)$ is defined by the boundary condition (C2) (see the proof of [2, (4.1)]). Clearly, if $\alpha_1, \alpha_2, \ldots, \alpha_n \in W_i^*(x)$, then $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) \in W_i^*(x)$ and $\{k_n\}$ satisfies all the conditions for a convex structure. Hence $W_i^*(x)$ is convex, in particular, an AR. If $\bigcap_{j\in J} W_j^*(x) \neq \varnothing$ for a subset J of $\{0, 1, \ldots, n\}$, then $\bigcap_{j\in J} W_j^*(x)$ is an AR. Note that $\bigcap_{j\in J} W_j^*(x_0) \neq \varnothing$ if and only if $\bigcap_{j\in J} W_j^*(x) \neq \varnothing$. Hence we can see that there is a map $f: e^{-1}(x) \to e^{-1}(x_0)$ such that $f(W_i^*(x)) \subset W_i^*(x_0)$ for each i. Also, there is a map $g: e^{-1}(x_0) \to e^{-1}(x)$ such that $g(W_i^*(x_0)) \subset W_i^*(x)$ for each i. Clearly, $f \circ g(W_i^*(x_0)) \subset W_i^*(x_0)$ and $g \cdot f(W_i^*(x)) \subset W_i^*(x)$ for each i. We can easily see that f and g satisfy the desired conditions. Thus $e: \Lambda(X) \to X$ is a strongly regular mapping with AR fibers.

Proof of (2.1). First, we prove that (1) implies (2). We use Ferry's idea as in the proof of [4, 3.1]. In [2] it was proved that $\Lambda(X)$ has a convex structure. By (2.3) e is a strongly regular mapping with AR fibers. Note that $e^{-1}(x)$ is convex (see [2]). Since $\Lambda(X)$ is a compact metric space, it can be embedded in the Hilbert cube $Q = [0, 1]^\infty$. Let $F(Q, \Lambda(X))$ be the space of maps from Q to $\Lambda(X)$ in the sup norm and let $H \subset F(Q, \Lambda(X))$ be the subspace $\{g | g$ retracts Q onto some $e^{-1}(x)\}$. Note that H is complete.

Define a convex structure on H as follows: Suppose that $M_n \subset \Lambda(X)^n$ and $k_n: M_n \times P_n \to \Lambda(X)$ satisfy the conditions of convex structure on $\Lambda(X)$. By [2, 4.1] $e^{-1}(x)$ is a convex subset for each $x \in X$. Let $q: H \to X$ be the map defined by $q(r) = x$, if r retracts Q onto $e^{-1}(x)$. Let $M_n^* = \{(r_1, \ldots, r_n) \in H^n | q(r_1) = \cdots = q(r_n)\}$ and let $k_n^*: M_n^* \times P_n \to H$ be the map defined by $k_n^*(r_1, \ldots, r_n; t_1, \ldots, t_n)(z) = k_n(r_1(z), \ldots, r_n(z); t_1, \ldots, t_n)$ for each $z \in Q$. Clearly, H has a convex structure. By [4, Step I, p. 376] q is an open map. Since $q^{-1}(x)$ is convex in H, by (1.1) there is a section $s: X \to H$ of q, i.e., $q \circ s = \mathrm{id}_X$.

Define a map $G\colon X \times Q \to \Lambda(X)$ by $G(x, z) = s(x)(z)$ for each $x \in X$ and $z \in Q$. Define a map $G'\colon X \times Q \times [0, \omega(X)] \to C(X)$ by $G'(x, z, t) = G(x, z) \cap \omega^{-1}(t)$. Finally, choose an onto map $h\colon I \to Q$ and then define $F\colon X \times I \times [0, \omega(X)] \to C(X)$ by $F(x, s, t) = G'(x, h(s), t)$. Clearly, F satisfies the desired conditions.

Clearly, (2) implies (3). Also, (3) implies (1) (see [13, (16.14)]).

3. Some corollaries

In this section, we give some applications of (2.1). In [10, 3] Lelek and Fearnley respectively defined the notion "weakly chainable," and they proved that a continuum X is weakly chainable if and only if X is a continuous image of the pseudo-arc. Clearly, the product of two weakly chainable continua is weakly chainable. By (2.1) we have:

(3.1) **Corollary.** *The property of being weakly chainable is a Whitney property for the class of continua that have property* [K].

(3.1) is a partial answer to Rogers problem [11, p. 384].

In [9] Kuperberg defined the notion "uniformly pathwise connected" and he proved that a continuum X is uniformly pathwise connected if and only if X is a continuous image of the Cantor fan. By [9, (3.7)] we easily see that the product of two uniformly pathwise connected continua is uniformly pathwise connected. Hence we have:

(3.2) **Corollary.** *The property of being uniformly pathwise connected is a Whitney property for the class of continua that have property* [K].

In [12, p. 558; 14, p. 295], Nadler and Wardle respectively asked the following problem:

(*) If a continuum X has property [K], is it true that $C(X)$ or $\omega^{-1}(t)$ has property [K]?

(See [5] for a partial answer.) Here, we consider the following problem:

(**) If a continuum X has property [K], is it true that $X \times I$ has property [K]?

Then we have:

(3.3) **Corollary.** *If problem* (**) *has an affirmative answer, then problem* (*) *has an affirmative answer.*

Proof. Suppose that (**) has an affirmative answer. Then $X \times I^n$ has property [K]. Now we show that $X \times Q$ has property [K]. Let $\varepsilon > 0$. Consider the projection $p_n\colon X \times Q \to X \times I^n$. For some integer n, p_n is a monotone and $\varepsilon/2$-mapping. Choose $\delta > 0$ such that if A is a subset of $X \times I^n$ with diam $A < \delta$, then $\operatorname{diam} p_n^{-1}(A) < \varepsilon/2$. Then there is $\delta' > 0$ such that if w, $w' \in X \times I^n$, $d(w, w') < \delta'$, $A \in C(X \times I^n)$, and $w \in A$, then there is $B \in C(X \times I^n)$ such that $w' \in B$ and $d_H(A, B) < \delta$. Choose $\delta'' > 0$ such that if z, $z' \in X \times Q$ and $d(z, z') < \delta''$, then $d(p_n(z), p_n(z')) < \delta'$. Let z, $z' \in X \times Q$, $d(z, z') < \delta''$,

and $z \in A \in C(X \times Q)$. Then there is $B' \in C(X \times I^n)$ such that $p_n(z') \in B'$ and $d_H(p_n(A), B') < \delta'$. Let $B = p_n^{-1}(B')$. Then $z' \in B$ and $B \in C(X \times Q)$. We also see that

$$\begin{aligned} d_H(A, B) &= d_H(A, p_n^{-1}(B')) \\ &\leq d_H(A, p_n^{-1}(p_n(A))) + d_H(p_n^{-1}(p_n(A)), p_n^{-1}(B')) \\ &< \varepsilon/2 + \varepsilon/2 = \varepsilon. \end{aligned}$$

Hence $X \times Q$ has property [K]. As in the proof of (2.1), $\Lambda(X)$ can be embedded in the Hilbert cube Q. Consider the embedding $i\colon \Lambda(X) \to X \times Q$ defined by $i(\alpha) = (\alpha(0), \alpha)$. Then the map G satisfies $G \cdot i = \mathrm{id}$ (see the proof of (2.1)). By [14, (2.9)], $\Lambda(X)$ has property [K]. Then $T\colon \Lambda(X) \times [0, \omega(X)] \to C(X)$, which is defined by $T(\alpha, t) = \alpha \cap \omega^{-1}(t)$, is monotone. By [14, (4.3)], $C(X)$ has property [K]. Similarly, we obtain that $\omega^{-1}(t)$ has property [K] for any Whitney map ω for $C(X)$.

(3.4) **Problem.** If X has property [K], is it true that $X \times I$ has property [K]?

It is known that there is a continuum X such that X has property [K], but $X \times X$ does not have property [K] (see [14, (4.7)]).

References

1. W. J. Charatonik, *Hyperspaces and the property of Kelley*, Bull. Acad. Polon. Sci. Sér. Sci. Tech. **30** (1982), 457–459.
2. D. W. Curtis, *Application of a selection theorem to hyperspace contractibility*, Canad. J. Math. **37** (1985), 747–759.
3. L. Fearnley, *Characterizations of the continuous images of the pseudo-arc*, Trans. Amer. Math. Soc. **111** (1964), 380–399.
4. S. Ferry, *Strongly regular mappings with compact ANR fibers are Hurewicz fiberings*, Pacific J. Math. **75** (1978), 373–382.
5. H. Kato, *On the property of Kelley in the hyperspace and Whitney continua*, Topology Appl. **30** (1988), 165–174.
6. J. L. Kelley, *Hyperspaces of a continuum*, Trans. Amer. Math. Soc. **52** (1942), 22–36.
7. J. Krasinkiewicz, *On the hyperspaces of snake-like and circle-like continua*, Fund. Math. **83** (1974), 155–164.
8. P. Krupski, *The property of Kelley in circularly chainable and in chainable continua*, Bull. Acad. Polon. Sci. Sér. Sci. Tech. **29** (1981), 377–381.
9. W. Kuperberg, *Uniformly pathwise connected continua*, Studies in Topology, Academic Press, New York, 1975, pp. 315–324.
10. A. Lelek, *On weakly chainable continua*, Fund. Math. **51** (1963), 271–283.
11. W. Lewis, *Continuum theory problems*, Topology Proc. **8** (1983), 361–394.
12. S. B. Nadler, Jr., *Some basic connectivity properties of Whitney map inverses in $C(X)$*, Studies in Topology, Academic Press, New York, 1975, pp. 393–410.
13. ——, *Hyperspaces of sets*, Dekker, New York and Basel, 1978.
14. R. W. Wardle, *On a property of J. L. Kelley*, Houston J. Math. **3** (1977), 291–299.
15. H. Whitney, *Regular families of curves*. I, Proc. Nat. Acad. Sci. U.S.A. **18** (1932), 275–278.

Faculty of Integrated Arts and Sciences, Hiroshima University, Higashisenda-machi, Naka-ku, Hiroshima 730, Japan