A NOTE ON CONTINUOUS MAPPINGS AND THE PROPERTY OF J. L. KELLEY

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Abstract. In this paper, it is proved that if $X$ is a continuum and $\omega$ is any Whitney map for $C(X)$, then the following are equivalent:

1. $X$ has property $[K]$.
2. There exists a (continuous) mapping $F: X \times I \times [0, \omega(X)] \to C(X)$ such that $F(\{x\} \times I \times \{t\}) = \{A \in \omega^{-1}(t) | x \in A\}$ for each $x \in X$ and $t \in [0, \omega(X)]$, where $I = [0, 1]$.
3. For each $t \in [0, \omega(X)]$, there is an onto map $f: X \times I \to \omega^{-1}(t)$ such that $f(\{x\} \times I) = \{A \in \omega^{-1}(t) | x \in A\}$ for each $x \in X$. Some corollaries are obtained also.

0. Introduction

By a continuum we mean a compact connected metric space. For a continuum $X$, $C(X)$ denotes the hyperspace of all nonempty subcontinua of $X$, with the topology induced by the Hausdorff metric $d_H$. Then the hyperspace $C(X)$ is a continuum, and in fact, is pathwise connected. In [6, 2.7. Theorem], Kelley proved that $C(X)$ is the continuous image of the Cantor fan, i.e., the cone over the Cantor set.

In this paper, we prove the following: Let $X$ be a continuum and $\omega$ be any Whitney map for $C(X)$. Then the following are equivalent:

1. $X$ has property $[K]$.
2. There exists a map $F: X \times I \times [0, \omega(X)] \to C(X)$ such that

$$F(\{x\} \times I \times \{t\}) = \omega^{-1}_x(t)$$

for each $x \in X$, where $\omega^{-1}_x(t) = \{A \in \omega^{-1}(t) | x \in A\}$.
3. For each $t \in [0, \omega(X)]$, there is an onto map $f: X \times I \to \omega^{-1}(t)$ such that $f(\{x\} \times I) = \omega^{-1}_x(t)$ for each $x \in X$.

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Note that if $F$ is a map that satisfies (2) as above, then $F(\{x\} \times I \times \{0\}) = \{x\}$ for each $x \in X$ and $F|X \times I \times \{t\} \to \omega^{-1}(t)$ is surjective. It is known that the hyperspace $C(X)$ is contractible if and only if there is a map $G: X \times [0, \omega(X)] \to C(X)$ such that $G(x, t) \in \omega^{-1}_x(t)$ for each $x \in X$ and $t \in [0, \omega(X)]$, where $\omega$ is any Whitney map for $C(X)$ (see [6]).

To prove the above theorem, we use a selection theorem of D. W. Curtis [2, Theorem 2.2] and an idea of S. Ferry [4, the proof of 3.1]. As a corollary, the property of being weakly chainable (or uniformly pathwise connected) is a Whitney property for the class of continua that have property [K]. The first is a partial answer to Roger’s problem [11, pp. 384, 112]. Also, the following problem is considered: Is it true that the property [K] is a Whitney property?

We refer the reader to Nadler’s monograph [13] for hyperspace theory.

1. Preliminaries

Let $X$ be a continuum. A map $\omega: C(X) \to [0, \infty)$ is said to be a Whitney map for $C(X)$ provided that $\omega$ satisfies the following conditions:

1. $\omega(\{x\}) = 0$ for each $x \in X$ and
2. if $A, B \in C(X), A \subset B$, and $A \neq B$, then $\omega(A) < \omega(B)$.

In [15] Whitney proved that there always exists a Whitney map on any continuum. Then $\omega^{-1}(t) \ (0 \leq t < \omega(X))$ is a continuum and it is called a Whitney continuum. Let $X$ be a continuum with metric $d$. Then $X$ has property [K] [6] if for every $\varepsilon > 0$ there exists $\delta > 0$ such that, for every pair of points $x, y$ of $X$ with $d(x, y) < \delta$ and every subcontinuum $A$ containing $x$, there exists a subcontinuum $B$ containing $y$ with $d_H(A, B) < \varepsilon$. In [14] Wardle proved that a continuum $X$ has property [K] if and only if the set-valued function $\alpha: X \to C(X)$, where $\alpha(x) = \{A \in C(X)|x \in A\}$, is continuous. We may assume that $X$ is naturally contained in $C(X)$.

Let $(Y, d)$ be a metric space, and for each positive integer $n$ let $P_n = \{(t_i) \in I^n|\sum_{i=1}^n t_i = 1\}$. A convex structure on $(Y, d)$ [2,2.1] is a sequence of subsets $M_n \subset Y^n$ and maps $k_n: M_n \times P_n \to Y$ satisfying the following conditions:

1. $k_n(y, \ldots, y; t_1, \ldots, t_n) = y$;
2. $k_n(y_1, y_2, \ldots, y_n; t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) = k_{n-1}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n)$;
3. for each $\varepsilon > 0$ there is $\delta > 0$ such that for every $n$ and $(t_i) \in P_n$,

   $d(k_n((y_i); (t_i)), k_n((y'_i); (t'_i))) < \varepsilon$ if $d(y_i, y'_i) < \delta$ for each $i$.

A subset $C$ of $Y$ is convex if for each $n, C^n \subset M_n$ and $k_n(C^n \times P_n) \subset C$. In [2, 2.2. Theorem] Curtis proved the following selection theorem:

1. Theorem (D. W. Curtis). Let $X$ be paracompact, $(Y, d)$ a metric space with a convex structure, and $\Phi: X \to Y$ a lower semicontinuous set-valued
function, with each $\Phi(x)$ a complete, convex subset of $Y$. Then $\Phi$ admits a continuous selection $s : X \to Y$.

2. Result

In this section, we prove the following main result of this paper:

(2.1) Theorem. Let $X$ be a continuum and $\omega$ be a Whitney map for $C(X)$. Then the following are equivalent:

(1) $X$ has property [K].

(2) There exists $F : X \times I \times [0, \omega(X)] \to C(X)$ such that $F(\{x\} \times I \times \{t\}) = \omega^{-1}_x(t)$ for each $x \in X$ and $t \in [0, \omega(X)]$, where $\omega^{-1}_x(t) = \{ A \in \omega^{-1}(t) \mid x \in A \}$.

(3) For each $t \in [0, \omega(X)]$, there is an onto map $f : X \times I \to \omega^{-1}(t)$ such that $f(\{x\} \times I) = \omega^{-1}_x(t)$ for each $x \in X$.

To prove (2.1) we need the following:

(2.2) Theorem (cf. [7, (2.3)]). Let $X$ be a continuum and $\omega$ be any Whitney map for $C(X)$. Then for any $\epsilon > 0$ there is $\delta > 0$ such that if $A, B \in C(X)$, $|\omega(A) - \omega(B)| < \delta$ and $B \subset U(A, \delta)$, then $d_H(A, B) < \epsilon$, where $U(A, \delta)$ denotes the $\delta$-neighborhood of $A$ in $X$.

A map $f : X \to B$ between metric spaces is said to be strongly regular if $f$ is proper and if for each $b \in B$ and $\epsilon > 0$ there is $\delta > 0$ such that if $d(b, b') < \delta$, then there are maps $g_{bb'} : f^{-1}(b) \to f^{-1}(b')$ and $g_{b'b} : f^{-1}(b') \to f^{-1}(b)$ and homotopies $h : f^{-1}(b) \to f^{-1}(b)$ and $k : f^{-1}(b') \to f^{-1}(b')$ such that

(i) $d(g_{bb'}(x), x) < \epsilon$ and $d(h_{t}(x), x) < \epsilon$ for all $x \in f^{-1}(b)$ and $0 \leq t \leq 1$,

(ii) $d(g_{b'b}(x), x) < \epsilon$ and $d(k_{i}(x), x) < \epsilon$ for all $x \in f^{-1}(b')$ and $0 \leq t \leq 1$,

(iii) $h_0 = g_{bb'} \cdot g_{b'b}$ and $h_1 = \text{id}$,

(iv) $k_0 = g_{bb'} \cdot g_{b'b}$ and $k_1 = \text{id}$.

(2.3) Lemma. Let $\Lambda(X) \subset C(C(X))$ denote the space of maximal order arcs in $C(X)$, and let $e : \Lambda(X) \to X$ be the map defined by $e(\alpha) = \alpha(0)$ for $\alpha \in \Lambda(X)$ (see [2]). If $X$ has property [K], then $e$ is a strongly regular mapping.

Proof. Let $\omega$ be a Whitney map for $C(X)$. Also, let $\epsilon > 0$ and $x_0 \in X$. By (2.2) there exist open subsets $U_{1}^*, U_{2}^*, \ldots, U_{n}^*$ of $\Lambda(X)$ such that $\bigcup U_i^* \supset e^{-1}(x_0)$ and $\text{diam} U_i^* < \epsilon/2$ for each $i$, where $U_i^*$ is of the following form:

$U_i^* = \{ \alpha \in \Lambda(X) \mid \alpha(t_j^i) \in V_j^i \text{ for some } 0 \leq t_j^i < t_1^i < \cdots < t_n^i \leq \omega(X) \}$ and open subsets $V_0^i, V_1^i, \ldots, V_n^i$ of $X$, where $\alpha(t_j^i) \in \alpha$ with $\omega(\alpha(t_j^i)) = t_j^i$.  

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Let \( W^*_j \in \{ \alpha \in \Lambda(X) | \alpha(i_j^j) \subset \text{Cl} \ V^j \} \) for each \( j = 0, 1, \ldots, n \). Note that \( W^*_j \) is closed in \( \Lambda(X) \) and \( U^*_j \subset W^*_j \). Note that if \( V^j \) \((i = 1, 2, \ldots, n)\) is a neighborhood of \( \alpha \) such that \( \text{Cl} \ V^j+1 \subset V^j \), \( \cap \text{Cl} V^j = \{ \alpha \} \), and \( U^*_j = \{ \alpha \in \Lambda(X) | \alpha(0) \in V^j \} \), then \( \cap U^*_j = e^{-1}(x_0) \).

We may assume that \( \text{diam} W^*_j < \varepsilon \) and \( \cap_{j \in J} U^*_j(x_0) = \emptyset \) if and only if \( \cap_{j \in J} U^*_j \neq \emptyset \), where \( J \) is a subset of \( \{0, 1, \ldots, n\} \) and \( U^*_j(x_0) = e^{-1}(x_0) \cap U^*_j \). Since \( e \) is an open map, there is a neighborhood \( O \) of \( \alpha \) such that \( e^{-1}(x) \subset \bigcup_{j} U^*_j \) and \( e^{-1}(x) \cap \cap_{j \in J} U^*_j \neq \emptyset \) if and only if \( \cap_{j \in J} U^*_j(x_0) \neq \emptyset \). For each \( x \in O \), let \( U^*_j(x) = e^{-1}(x) \cap W^*_j \).

We show that \( W^*_j(x) \) is convex. Consider \( \alpha_1, \alpha_2, \ldots, \alpha_n \in e^{-1}(x) \) and \((t_1, t_2, \ldots, t_n) \in P_n \). Suppose first that each \( t_i > 0 \). Define \( k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) = \alpha \), where \( \alpha = \{ \alpha_i(t) \mid 1 \leq i \leq n \} \subset C(X) \) and \( t_i = t_i/(t_i + \cdots + t_n) \). In the case that \( t_i = 0 \) for some \( i \), \( k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) \) is defined by the boundary condition (C2) (see the proof of [2, (4.1)]). Clearly, if \( \alpha_1, \alpha_2, \ldots, \alpha_n \in W^*_j(x) \), then \( k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) \in W^*_j(x) \) and \( \{k_n \} \) satisfies all the conditions for a convex structure. Hence \( W^*_j(x) \) is convex, in particular, an AR. If \( \cap_{j \in J} W^*_j(x) \neq \emptyset \) for a subset \( J \) of \( \{0, 1, \ldots, n\} \), then \( \cap_{j \in J} W^*_j(x) \) is an AR. Note that \( \cap_{j \in J} W^*_j(x_0) \neq \emptyset \) if and only if \( \cap_{j \in J} W^*_j(x) \neq \emptyset \). Hence we can see that there is a map \( f: e^{-1}(x) \to e^{-1}(x_0) \) such that \( f(W^*_j(x)) \subset W^*_j(x_0) \) for each \( i \). Also, there is a map \( g: e^{-1}(x_0) \to e^{-1}(x) \) such that \( g(W^*_j(x_0)) \subset W^*_j(x) \) for each \( i \). Clearly, \( f \circ g(W^*_j(x_0)) \subset W^*_j(x_0) \) and \( g \circ f(W^*_j(x)) \subset W^*_j(x) \) for each \( i \). We can easily see that \( f \) and \( g \) satisfy the desired conditions. Thus \( e: \Lambda(X) \to X \) is a strongly regular mapping with AR fibers.

**Proof of (2.1).** First, we prove that (1) implies (2). We use Ferry's idea as in the proof of [4, 3.1]. In [2] it was proved that \( \Lambda(X) \) has a convex structure. By \( (2.3) \) \( e \) is a strongly regular mapping with AR fibers. Note that \( e^{-1}(x) \) is convex (see [2]). Since \( \Lambda(X) \) is a compact metric space, it can be embedded in the Hilbert cube \( Q = [0, 1]^\infty \). Let \( F(Q, \Lambda(X)) \) be the space of maps from \( Q \) to \( \Lambda(X) \) in the sup norm and let \( H \subset F(Q, \Lambda(X)) \) be the subspace \( \{ g | g \text{ retracts } Q \text{ onto some } e^{-1}(x) \} \). Note that \( H \) is complete.

Define a convex structure on \( H \) as follows: Suppose that \( M_0 \subset \Lambda(X)^n \) and \( k_n : M_0 \times P_n \to \Lambda(X) \) satisfy the conditions of convex structure on \( \Lambda(X) \). By \( [2, 4.1] \) \( e^{-1}(x) \) is a convex subset for each \( x \in X \). Let \( q: H \to X \) be the map defined by \( q(r) = x \), if \( r \) retracts \( Q \) onto \( e^{-1}(x) \). Let \( M^*_n = \{(r_1, \ldots, r_n) \in H^n | q(r_1) = \cdots = q(r_n) \} \) and let \( k_n^*: M^*_n \times P_n \to H \) be the map defined by \( k_n^*(r_1, \ldots, r_n; t_1, \ldots, t_n)(z) = k_n(r_1(z), \ldots, r_n(z); t_1, \ldots, t_n) \) for each \( z \in Q \). Clearly, \( H \) has a convex structure. By [4, Step I, p. 376] \( q \) is an open map. Since \( q^{-1}(x) \) is convex in \( H \), by (1.1) there is a section \( s: X \to H \) of \( q \), i.e., \( q \circ s = \text{id}_X \).
Define a map \( G: X \times Q \rightarrow A(X) \) by \( G(x, z) = s(x)(z) \) for each \( x \in X \) and \( z \in Q \). Define a map \( G': X \times Q \times [0, \omega(X)] \rightarrow C(X) \) by \( G'(x, z, t) = G(x, z) \cap \omega^{-1}(t) \). Finally, choose an onto map \( h: I \rightarrow Q \) and then define \( F: X \times I \times [0, \omega(X)] \rightarrow C(X) \) by \( F(x, s, t) = G'(x, h(s), t) \). Clearly, \( F \) satisfies the desired conditions.

Clearly, (2) implies (3). Also, (3) implies (1) (see [13, (16.14)]).

3. SOME COROLLARIES

In this section, we give some applications of (2.1). In [10, 3] Lelek and Fearnley respectively defined the notion “weakly chainable,” and they proved that a continuum \( X \) is weakly chainable if and only if \( X \) is a continuous image of the pseudo-arc. Clearly, the product of two weakly chainable continua is weakly chainable. By (2.1) we have:

\( (3.1) \) Corollary. The property of being weakly chainable is a Whitney property for the class of continua that have property \([K]\).

\( (3.1) \) is a partial answer to Rogers problem [11, p. 384].

In [9] Kuperberg defined the notion “uniformly pathwise connected” and he proved that a continuum \( X \) is uniformly pathwise connected if and only if \( X \) is a continuous image of the Cantor fan. By [9, (3.7)] we easily see that the product of two uniformly pathwise connected continua is uniformly pathwise connected. Hence we have:

\( (3.2) \) Corollary. The property of being uniformly pathwise connected is a Whitney property for the class of continua that have property \([K]\).

In [12, p. 558; 14, p. 295], Nadler and Wardle respectively asked the following problem:

\( (*) \) If a continuum \( X \) has property \([K]\), is it true that \( C(X) \) or \( \omega^{-1}(t) \) has property \([K]\)?

(See [5] for a partial answer.) Here, we consider the following problem:

\( (**) \) If a continuum \( X \) has property \([K]\), is it true that \( X \times I \) has property \([K]\)?

Then we have:

\( (3.3) \) Corollary. If problem \( (**) \) has an affirmative answer, then problem \( (*) \) has an affirmative answer.

\textit{Proof.} Suppose that \( (**) \) has an affirmative answer. Then \( X \times I^n \) has property \([K]\). Now we show that \( X \times Q \) has property \([K]\). Let \( \varepsilon > 0 \). Consider the projection \( p_n: X \times Q \rightarrow X \times I^n \). For some integer \( n \), \( p_n \) is a monotone and \( \varepsilon/2 \)-mapping. Choose \( \delta > 0 \) such that if \( A \) is a subset of \( X \times I^n \) with \( \operatorname{diam} A < \delta \), then \( \operatorname{diam} p_n^{-1}(A) < \varepsilon/2 \). Then there is \( \delta' > 0 \) such that if \( w, w' \in X \times I^n \), \( d(w, w') < \delta' \), \( A \in C(X \times I^n) \), and \( w \in A \), then there is \( B \in C(X \times I^n) \) such that \( w' \in B \) and \( d_H(A, B) < \delta \). Choose \( \delta'' > 0 \) such that if \( z, z' \in X \times Q \) and \( d(z, z') < \delta'' \), then \( d(p_n(z), p_n(z')) < \delta' \). Let \( z, z' \in X \times Q \), \( d(z, z') < \delta'' \),
and $z \in A \in C(X \times Q)$. Then there is $B' \in C(X \times I^n)$ such that $p_n(z') \in B'$
and $d_{H}(p_n(A), B') < \delta'$. Let $B = p_n^{-1}(B')$. Then $z' \in B$ and $B \in C(X \times Q)$. We also see that

$$d_{H}(A, B) = d_{H}(A, p_n^{-1}(B'))$$

$$\leq d_{H}(A, p_n^{-1}(p_n(A))) + d_{H}(p_n^{-1}(p_n(A)), p_n^{-1}(B'))$$

$$< \varepsilon/2 + \varepsilon/2 = \varepsilon.$$ 

Hence $X \times Q$ has property [K]. As in the proof of (2.1), $\Lambda(X)$ can be embedded in the Hilbert cube $Q$. Consider the embedding $i: \Lambda(X) \to X \times Q$ defined by $i(\alpha) = (\alpha(0), \alpha)$. Then the map $G$ satisfies $G \cdot i = \text{id}$ (see the proof of (2.1)). By [14, (2.9)], $\Lambda(X)$ has property [K]. Then $T: \Lambda(X) \times [0, \omega(X)] \to C(X)$, which is defined by $T(\alpha, t) = \alpha \cap \omega^{-1}(t)$, is monotone. By [14, (4.3)], $C(X)$ has property [K]. Similarly, we obtain that $\omega^{-1}(t)$ has property [K] for any Whitney map $\omega$ for $C(X)$.

(3.4) Problem. If $X$ has property [K], is it true that $X \times I$ has property [K]?

It is known that there is a continuum $X$ such that $X$ has property [K], but $X \times X$ does not have property [K] (see [14, (4.7)]).

References


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