A NOTE ON CONTINUOUS MAPPINGS
AND THE PROPERTY OF J. L. KELLEY

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Abstract. In this paper, it is proved that if \( X \) is a continuum and \( \omega \) is any
Whitney map for \( C(X) \), then the following are equivalent:

1. \( X \) has property \([K]\).
2. There exists a (continuous) mapping \( F: X \times I \times [0, \omega(X)] \rightarrow C(X) \)
such that \( F(x) \times I \times \{t\} = \{A \in \omega^{-1}(t) \mid x \in A\} \) for each \( x \in X \) and
time \( t \in [0, \omega(X)] \), where \( I = [0, 1] \).
3. For each \( t \in [0, \omega(X)] \), there is an onto map \( f: X \times I \rightarrow \omega^{-1}(t) \) such
that \( f(x) \times I = \{A \in \omega^{-1}(t) \mid x \in A\} \) for each \( x \in X \). Some corollaries are
obtained also.

0. Introduction

By a continuum we mean a compact connected metric space. For a continuum
\( X \), \( C(X) \) denotes the hyperspace of all nonempty subcontinua of \( X \), with the
topology induced by the Hausdorff metric \( d_H \). Then the hyperspace \( C(X) \) is
a continuum, and in fact, is pathwise connected. In [6, 2.7. Theorem], Kelley
proved that \( C(X) \) is the continuous image of the Cantor fan, i.e., the cone over
the Cantor set.

In this paper, we prove the following: Let \( X \) be a continuum and \( \omega \) be any
Whitney map for \( C(X) \). Then the following are equivalent:

1. \( X \) has property \([K]\).
2. There exists a map \( F: X \times I \times [0, \omega(X)] \rightarrow C(X) \) such that
\( F(x) \times I \times \{t\} = \omega^{-1}(t) \)
for each \( x \in X \), where \( \omega^{-1}(t) = \{A \in \omega^{-1}(t) \mid x \in A\} \).
3. For each \( t \in [0, \omega(X)] \), there is an onto map \( f: X \times I \rightarrow \omega^{-1}(t) \) such
that \( f(x) \times I = \omega^{-1}(t) \) for each \( x \in X \).

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Note that if \( F \) is a map that satisfies (2) as above, then \( F(\{x\} \times I \times \{0\}) = \{x\} \) for each \( x \in X \) and \( F|X \times I \times \{t\} \to \omega^{-1}(t) \) is surjective. It is known that the hyperspace \( C(X) \) is contractible if and only if there is a map \( G: X \times [0, \omega(X)] \to C(X) \) such that \( G(x, t) \in \omega^{-1}(t) \) for each \( x \in X \) and \( t \in [0, \omega(X)] \), where \( \omega \) is any Whitney map for \( C(X) \) (see [6]).

To prove the above theorem, we use a selection theorem of D. W. Curtis [2, Theorem 2.2] and an idea of S. Ferry [4, the proof of 3.1]. As a corollary, the property of being weakly chainable (or uniformly pathwise connected) is a Whitney property for the class of continua that have property [K]. The first is a partial answer to Roger's problem [11, pp. 384, 112]. Also, the following problem is considered: Is it true that the property [K] is a Whitney property?

We refer the reader to Nadler's monograph [13] for hyperspace theory.

1. Preliminaries

Let \( X \) be a continuum. A map \( \omega: C(X) \to [0, \infty) \) is said to be a Whitney map for \( C(X) \) provided that \( \omega \) satisfies the following conditions:

1. \( \omega(\{x\}) = 0 \) for each \( x \in X \) and
2. if \( A, B \in C(X) \), \( A \subset B \), and \( A \neq B \), then \( \omega(A) < \omega(B) \).

In [15] Whitney proved that there always exists a Whitney map on any continuum. Then \( \omega^{-1}(t) \) \((0 \leq t < \omega(X))\) is a continuum and it is called a Whitney continuum. Let \( X \) be a continuum with metric \( d \). Then \( X \) has property [K] [6] if for every \( \varepsilon > 0 \) there exists \( \delta > 0 \) such that, for every pair of points \( x, y \) of \( X \) with \( d(x, y) < \delta \) and every subcontinuum \( A \) containing \( x \), there exists a subcontinuum \( B \) containing \( y \) with \( d_H(A, B) < \varepsilon \). In [14] Wardle proved that a continuum \( X \) has property [K] if and only if the set-valued function \( \alpha: X \to C(X) \), where \( \alpha(x) = \{A \in C(X)|x \in A\} \), is continuous. We may assume that \( X \) is naturally contained in \( C(X) \).

Let \( (Y, d) \) be a metric space, and for each positive integer \( n \) let \( P_n = \{(t_i) \in I^n|\sum_{i=1}^n t_i = 1\} \). A convex structure on \( (Y, d) \) [2,2.1] is a sequence of subsets \( M_n \subset Y^n \) and maps \( k_n: M_n \times P_n \to Y \) satisfying the following conditions:

\begin{enumerate}
\item[(C1)] \( k_n(y, \ldots, y; t_1, \ldots, t_n) = y; \)
\item[(C2)] \( k_n(y_1, y_2, \ldots, y_n; t_1, t_2, \ldots, t_{i-1}, 0, t_{i+1}, \ldots, t_n) = k_{n-1}(y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n; t_1, \ldots, t_{i-1}, t_{i+1}, \ldots, t_n); \)
\item[(C3)] for each \( \varepsilon > 0 \) there is \( \delta > 0 \) such that for every \( n \) and \( (t_i) \in P_n \),
\( d(k_n((y_i); (t_i)), k_n((y'_i); (t_i)) < \varepsilon \) if \( d(y_i, y'_i) < \delta \) for each \( i \).
\end{enumerate}

A subset \( C \) of \( Y \) is convex if for each \( n \), \( C^n \subset M_n \) and \( k_n(C^n \times P_n) \subset C \).

In [2, 2.2. Theorem] Curtis proved the following selection theorem:

(1.1) \textbf{Theorem} (D. W. Curtis). \textit{Let} \( X \) \textit{be paracompact}, \((Y, d)\) \textit{a metric space with a convex structure}, and \( \Phi: X \to Y \) \textit{a lower semicontinuous set-valued
function, with each \( \Phi(x) \) a complete, convex subset of \( Y \). Then \( \Phi \) admits a continuous selection \( s: X \to Y \).

2. Result

In this section, we prove the following main result of this paper:

(2.1) **Theorem.** Let \( X \) be a continuum and \( \omega \) be a Whitney map for \( C(X) \). Then the following are equivalent:

1. \( X \) has property \([K]\).
2. There exists \( F: X \times I \times [0, \omega(X)] \to C(X) \) such that \( F(\{X\} \times I \times \{t\}) = \omega_X^{-1}(t) \) for each \( x \in X \) and \( t \in [0, \omega(X)] \), where \( \omega_X^{-1}(t) = \{A \in \omega^{-1}(t) | x \in A\} \).
3. For each \( t \in [0, \omega(X)] \), there is an onto map \( f: X \times I \to \omega^{-1}(t) \) such that \( f(\{x\} \times I) = \omega_X^{-1}(t) \) for each \( x \in X \).

To prove (2.1) we need the following:

(2.2) **Theorem** cf. [7, (2.3)]. Let \( X \) be a continuum and \( \omega \) be any Whitney map for \( C(X) \). Then for any \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( A, B \in C(X) \), \( |\omega(A) - \omega(B)| < \delta \) and \( B \subset U(A, \delta) \), then \( d_H(A, B) < \varepsilon \), where \( U(A, \delta) \) denotes the \( \delta \)-neighborhood of \( A \) in \( X \).

A map \( f: X \to B \) between metric spaces is said to be strongly regular if \( f \) is proper and if for each \( b \in B \) and \( \varepsilon > 0 \) there is \( \delta > 0 \) such that if \( d(b, b') < \delta \), then there are maps \( g_{bb'}: f^{-1}(b) \to f^{-1}(b') \) and \( g_{b'b}: f^{-1}(b') \to f^{-1}(b) \) and homotopies \( h_t: f^{-1}(b) \to f^{-1}(b) \) and \( k_t: f^{-1}(b') \to f^{-1}(b') \) such that

1. \( d(g_{bb'}(x), x) < \varepsilon \) and \( d(h_t(x), x) < \varepsilon \) for all \( x \in f^{-1}(b) \) and \( 0 \leq t \leq 1 \),
2. \( d(g_{b'b}(x), x) < \varepsilon \) and \( d(k_t(x), x) < \varepsilon \) for all \( x \in f^{-1}(b') \) and \( 0 \leq t \leq 1 \),
3. \( h_0 = g_{bb'} \cdot g_{bb'} \) and \( h_1 = \text{id} \),
4. \( k_0 = g_{b'b} \cdot g_{b'b} \) and \( k_1 = \text{id} \).

(2.3) **Lemma.** Let \( \Lambda(X) \subset C(C(X)) \) denote the space of maximal order arcs in \( C(X) \), and let \( e: \Lambda(X) \to X \) be the map defined by \( e(\alpha) = \alpha(0) \) for \( \alpha \in \Lambda(X) \) (see [2]). If \( X \) has property \([K]\), then \( e \) is a strongly regular mapping.

**Proof.** Let \( \omega \) be a Whitney map for \( C(X) \). Also, let \( \varepsilon > 0 \) and \( x_0 \in X \). By (2.2) there exist open subsets \( U_1^*, U_2^*, \ldots, U_n^* \) of \( \Lambda(X) \) such that \( \bigcup U_i^* \supset e^{-1}(x_0) \) and \( \text{diam} U_i^* < \varepsilon/2 \) for each \( i \), where \( U_i^* \) is of the following form:

\[
U_i^* = \{\alpha \in \Lambda(X) | \alpha(t^i_j) \in V_j^i \text{ for some } 0 \leq t^i_0 < t^i_1 < \cdots < t^i_n \leq \omega(X) \text{ and open subsets } V_0^i, V_1^i, \ldots, V_n^i \text{ of } X\},
\]

where \( \alpha(t^i_j) \in \alpha \) with \( \omega(\alpha(t^i_j)) = t^i_j \).
Let $W^*_i = \{ \alpha \in \Lambda(X) | \alpha(t^j) \subset Cl V^j_i \text{ for each } j = 0, 1, \ldots, n \}$. Note that $W^*_i$ is closed in $\Lambda(X)$ and $U^*_i \subset W^*_i$. Note that if $V^j_i$ ($i = 1, 2, \ldots, \) is a neighborhood of $x_0$ such that $Cl V^j_{i+1} \subset V^j_i$, $\cap Cl V^j_i = \{x_0\}$, and $U^*_i = \{ \alpha \in \Lambda(X) \mid \alpha(0) \in V^j_i \}$, then $\cap W^*_i = e^{-1}(x_0)$.

We may assume that $\text{diam } W^*_i < \varepsilon$ and $\cap_{j \in J} U^*_i(x_0) = \emptyset$ if and only if $\cap_{j \in J} U^*_i(x_0) \neq \emptyset$, where $J$ is a subset of $\{0, 1, \ldots, n\}$ and $U^*_i(x_0) = e^{-1}(x_0) \cap U^*_i$. Since $e$ is an open map, there is a neighborhood $O$ of $x_0$ in $X$ such that if $x \in O$, then $e^{-1}(x) \subset \cup U^*_i$ and $e^{-1}(x) \cap \cap_{j \in J} U^*_i \neq \emptyset$ if and only if $\cap_{j \in J} U^*_i(x_0) \neq \emptyset$. For each $x \in O$, let $W^*_i(x) = e^{-1}(x) \cap W^*_i$.

We show that $W^*_i(x)$ is convex. Consider $\alpha_1, \alpha_2, \ldots, \alpha_n \in e^{-1}(x)$ and $(t_1, t_2, \ldots, t_n) \in P^n$. Suppose first that each $t_i \geq 0$. Define $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) = \alpha$, where $\alpha = \{ \alpha(t_1) \cup \cdots \cup \alpha(t_n) \mid 0 \leq i \leq n \}$, and $\tau_i = t_i/(t_1 + \cdots + t_n)$. In the case that $t_i = 0$ for some $i$, $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n)$ is defined by the boundary condition (C2) (see the proof of [2, (4.1)]). Clearly, if $\alpha_1, \alpha_2, \ldots, \alpha_n \in W^*_i(x)$, then $k_n(\alpha_1, \alpha_2, \ldots, \alpha_n; t_1, t_2, \ldots, t_n) \in W^*_i(x)$ and $\{k_n\}$ satisfies all the conditions for a convex structure. Hence $W^*_i(x)$ is convex, in particular, an AR if $\cap_{j \in J} W^*_i(x) \neq \emptyset$ for a subset $J$ of $\{0, 1, \ldots, n\}$, then $\cap_{j \in J} W^*_i(x)$ is an AR. Note that $\cap_{j \in J} W^*_i(x) \neq \emptyset$ if and only if $\cap_{j \in J} W^*_i(x) \neq \emptyset$. Hence we can see that there is a map $f: e^{-1}(x) \to e^{-1}(x)$ such that $\cap W^*_i(x) \subset W^*_i(x)$ for each $i$. Also, there is a map $g: e^{-1}(x) \to e^{-1}(x)$ such that $\cap W^*_i(x) \subset W^*_i(x)$ for each $i$. Clearly, $f \circ g(W^*_i(x)) \subset W^*_i(x)$ and $g \circ f(W^*_i(x)) \subset W^*_i(x)$ for each $i$. We can easily see that $f$ and $g$ satisfy the desired conditions. Thus $e: \Lambda(X) \to X$ is a strongly regular mapping with AR fibers.

Proof of (2.1). First, we prove that (1) implies (2). We use Ferry’s idea as in the proof of [4, 3.1]. In [2] it was proved that $\Lambda(X)$ has a convex structure. By (2.3) $e$ is a strongly regular mapping with AR fibers. Note that $e^{-1}(x)$ is convex (see [2]). Since $\Lambda(X)$ is a compact metric space, it can be embedded in the Hilbert cube $Q = [0, 1]^{\infty}$. Let $F(Q, \Lambda(X))$ be the space of maps from $Q$ to $\Lambda(X)$ in the sup norm and let $H \subset F(Q, \Lambda(X))$ be the subspace $\{g \mid g$ retracts $Q$ onto some $e^{-1}(x)\}$. Note that $H$ is complete.

Define a convex structure on $H$ as follows: Suppose that $M_n \subset \Lambda(X)^n$ and $k_n: M_n \times P_n \to \Lambda(X)$ satisfy the conditions of convex structure on $\Lambda(X)$. By [2, 4.1] $e^{-1}(x)$ is a convex subset for each $x \in X$. Let $q: H \to X$ be the map defined by $q(r) = x$, if $r$ retracts $Q$ onto $e^{-1}(x)$. Let $M^*_n = \{(r_1, \ldots, r_n) \in H^n \mid q(r_1) = \cdots = q(r_n)\}$ and let $k_n^*: M^*_n \times P_n \to H$ be the map defined by $k_n^*(r_1, \ldots, r_n; t_1, \ldots, t_n)(z) = k_n(r_1(z), \ldots, r_n(z); t_1, \ldots, t_n)$ for each $z \in Q$. Clearly, $H$ has a convex structure. By [4, Step I, p. 376] $q$ is an open map. Since $q^{-1}(x)$ is convex in $H$, by (1.1) there is a section $s: X \to H$ of $q$, i.e., $q \circ s = \text{id}_X$. 

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Define a map \( G: X \times Q \to \Lambda(X) \) by \( G(x, z) = s(x)(z) \) for each \( x \in X \) and \( z \in Q \). Define a map \( G': X \times Q \times [0, \omega(X)] \to C(X) \) by \( G'(x, z, t) = G(x, z) \cap \omega^{-1}(t) \). Finally, choose an onto map \( h: I \to Q \) and then define \( F: X \times I \times [0, \omega(X)] \to C(X) \) by \( F(x, s, t) = G'(x, h(s), t) \). Clearly, \( F \) satisfies the desired conditions.

Clearly, (2) implies (3). Also, (3) implies (1) (see \([13, (16.14)]\)).

3. Some corollaries

In this section, we give some applications of (2.1). In \([10, 3]\) Lelek and Fearnley respectively defined the notion “weakly chainable,” and they proved that a continuum \( X \) is weakly chainable if and only if \( X \) is a continuous image of the pseudo-arc. Clearly, the product of two weakly chainable continua is weakly chainable. By (2.1) we have:

\[\text{(3.1) Corollary. The property of being weakly chainable is a Whitney property for the class of continua that have property \([K]\).}\]

(3.1) is a partial answer to Rogers problem \([11, \text{p. 384}]\).

In \([9]\) Kuperberg defined the notion “uniformly pathwise connected” and he proved that a continuum \( X \) is uniformly pathwise connected if and only if \( X \) is a continuous image of the Cantor fan. By \([9, (3.7)]\) we easily see that the product of two uniformly pathwise connected continua is uniformly pathwise connected. Hence we have:

\[\text{(3.2) Corollary. The property of being uniformly pathwise connected is a Whitney property for the class of continua that have property \([K]\).}\]

In \([12, \text{p. 558}; 14, \text{p. 295}]\), Nadler and Wardle respectively asked the following problem:

\[\text{(*) If a continuum } X \text{ has property \([K]\), is it true that } C(X) \text{ or } \omega^{-1}(t) \text{ has property \([K]\)?}\]

(See \([5]\) for a partial answer.) Here, we consider the following problem:

\[\text{(**) If a continuum } X \text{ has property \([K]\), is it true that } X \times I \text{ has property \([K]\)?}\]

Then we have:

\[\text{(3.3) Corollary. If problem (** has an affirmative answer, then problem (*) has an affirmative answer.}\]

\[\text{Proof. Suppose that (** has an affirmative answer. Then } X \times I^n \text{ has property \([K]\). Now we show that } X \times Q \text{ has property \([K]\). Let } \varepsilon > 0. \text{ Consider the projection } p_n: X \times Q \to X \times I^n. \text{ For some integer } n, p_n \text{ is a monotone and } \varepsilon/2-\text{mapping. Choose } \delta > 0 \text{ such that if } A \text{ is a subset of } X \times I^n \text{ with diam } A < \delta, \text{ then diam } p_n^{-1}(A) < \varepsilon/2. \text{ Then there is } \delta' > 0 \text{ such that if } w, w' \in X \times I^n, d(w, w') < \delta', A \in C(X \times I^n), \text{ and } w \in A, \text{ then there is } B \in C(X \times I^n) \text{ such that } w' \in B \text{ and } d_H(A, B) < \delta. \text{ Choose } \delta'' > 0 \text{ such that if } z, z' \in X \times Q \text{ and } d(z, z') < \delta'', \text{ then } d(p_n(z), p_n(z')) < \delta'. \text{ Let } z, z' \in X \times Q, d(z, z') < \delta'',}\]

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\[ z \in A \in C(X \times Q) \]. Then there is \( B' \in C(X \times I^n) \) such that \( p_n(z') \in B' \) and \( d_H(p_n(A), B') < \delta' \). Let \( B = p_n^{-1}(B') \). Then \( z' \in B \) and \( B \in C(X \times Q) \). We also see that
\[
d_H(A, B) = d_H(A, p_n^{-1}(B')) \\
\leq d_H(A, p_n^{-1}(p_n(A))) + d_H(p_n^{-1}(p_n(A)), p_n^{-1}(B')) \\
< \varepsilon/2 + \varepsilon/2 = \varepsilon.
\]

Hence \( X \times Q \) has property \([K]\). As in the proof of (2.1), \( \Lambda(X) \) can be embedded in the Hilbert cube \( Q \). Consider the embedding \( i: \Lambda(X) \to X \times Q \) defined by \( i(\alpha) = (\alpha(0), \alpha) \). Then the map \( G \) satisfies \( G \cdot i = \text{id} \) (see the proof of (2.1)). By [14, (2.9)], \( \Lambda(X) \) has property \([K]\). Then \( T: \Lambda(X) \times [0, \omega(X)] \to C(X) \), which is defined by \( T(\alpha, t) = \alpha \cap \omega^{-1}(t) \), is monotone. By [14, (4.3)], \( C(X) \) has property \([K]\). Similarly, we obtain that \( \omega^{-1}(t) \) has property \([K]\) for any Whitney map \( \omega \) for \( C(X) \).

(3.4) **Problem.** If \( X \) has property \([K]\), is it true that \( X \times I \) has property \([K]\)?

It is known that there is a continuum \( X \) such that \( X \times I \) has property \([K]\), but \( X \times X \) does not have property \([K]\) (see [14, (4.7)]).

**References**