A CHARACTERIZATION OF SPECTRAL OPERATORS

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Abstract. A characterization of spectral operators due to N. Dunford is simplified. Especially, his complicated Condition (D) is replaced by a very simple one.

Dunford [2; 3; p. 2147] showed that an operator $T$ defined on a weakly complete Banach space $Y$ is spectral if and only if it satisfies the following conditions (i)–(iv).

(i) Dunford’s analyticity condition (A). $T$ has the single-valued extension property; that is, for every $x \in Y$ there exists a smallest closed set $\sigma_T(x)$, called the local spectrum of $T$ at $x$, and a unique analytic function $f : C \setminus \sigma_T(x) \to Y$, called the local resolvent of $T$ at $x$, such that $(z - T)f(z) = x$.

(ii) Dunford’s boundedness condition (B). There exists a bound $K$ such that $\|x\| \leq K\|x + y\|$ whenever $\sigma_T(x) \cap \sigma_T(y) = \emptyset$.

(iii) Dunford’s closure condition (C). For every closed set $F \subset C$, the set $Y_T(F) := \{x : \sigma_T(x) \subset F\}$ is closed.

(iv) Dunford’s decomposability condition (D). Let $\mathcal{S}(T)$ denote the collection of all $\delta \subset C$ for which $Y_T(\delta) + Y_T(C \setminus \delta)$ is dense in $Y$. Assuming $T$ satisfies condition (B), it follows that $Y$ is the direct sum of $Y_T(\delta)$ and $Y_T(C \setminus \delta)$ and the corresponding projections $E(\delta)$ and $E(C \setminus \delta)$ are bounded by $K$. Let $\mathcal{S}_2(T)$ denote the collection of all $\delta \in \mathcal{S}(T)$ such that for every $\epsilon > 0$ there exist $x_1, x_2 \in Y$ with $\|x_1 + x_2 - x\| < \epsilon$, $\sigma_T(x_1) \subset \delta \cap \sigma_T(x)$, and $\sigma_T(x_2) \subset \sigma_T(x) \setminus \delta$. We assume that for every $\lambda \in C$ and $r > 0$ there exist $\delta, \mu_n, \gamma_n \in \mathcal{S}_2(T)$ ($n = 1, 2, \ldots$) such that $\lambda \in \delta^0$, $\text{diam}(\delta) \leq r$, $\mu_n \subset \delta$, $\gamma_n \subset C \setminus \delta$, $\mu_n$ closed, $\gamma_n$ closed ($n = 1, 2, \ldots$), and $x = \lim[E(\mu_n)x + E(\gamma_n)x]$.

The purpose of this paper is to simplify the above criterion. For an arbitrary operator $T$, and a subset $\delta$ of $C$, we define $S_T(\delta)$ to be the set of all $x \in Y$ such that $(z - T)f(z) \equiv x$ for some analytic $Y$-valued function $f$ defined on an open set containing $C \setminus \delta$.

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Theorem. A (bounded linear) operator $T$ on a weakly complete Banach space $Y$ is spectral if and only if $T$ satisfies the following conditions.

(a) There exists $K > 0$ such that $\|x\| \leq K\|x + y\|$ whenever $x \in S_T(\delta)$, $y \in S_T(C \setminus \delta)$, and $\delta = \overline{\delta} \subset C$.

(b) $S_T(\Delta)$ is closed and $S_T(\Delta) + S_T(C \setminus \Delta)$ is dense in $Y$ whenever $\Delta$ is a closed (vertical or horizontal) half plane.

Proof. We first prove the sufficiency of the conditions. Assume the equation $(z - T)f(z) = 0$ has a nonzero analytic solution $f$. Let $N(\lambda; r)$ be an arbitrary open disc lying entirely in the domain of $f$. Since $(z - T)(z - \lambda)^{-1}f(\lambda) \equiv f(\lambda)$ for $z \neq \lambda$, it follows that $f(\lambda) \in S_T(\{\lambda\})$. Define $g(z) = (z - \lambda)^{-1}[f(z) - f(\lambda)]$ for $z \in N(\lambda; r) \setminus \{\lambda\}$, and $g(\lambda) = f'(\lambda)$ [1, p. 2]. Then $(z - T)g(z) \equiv -f(\lambda)$ for $z \in N(\lambda; r)$ and thus $-f(\lambda) \in S_T(C \setminus N(\lambda; r))$. In view of condition (B), $\|f(\lambda)\| \leq K\|f(\lambda) - f(\lambda)\| = 0$. Since $\lambda$ is arbitrary, $f \equiv 0$. Thus $T$ satisfies condition (A) and $S_T(\cap F_\alpha) = \cap S_T(F_\alpha)$ for any family $\{F_\alpha\}$ of subsets of $C$. Also in view of (a), $T$ satisfies condition (B).

Now, let $\delta \subset C$ be such that $Y = S_T(\delta) \oplus S_T(C \setminus \delta)$. Define $E(\delta)$ to be the projection onto $S_T(\delta)$ that is parallel to $S_T(C \setminus \delta)$. Because $(z - T)E(\delta)f(z) \equiv E(\delta)x$ whenever $f$ is the local resolvent of $T$ at $x$, it follows that $\sigma_T(E(\delta)x) \subset \sigma_T(x) \cap \delta$ provided that $S_T(\delta)$ is closed. Moreover, $\|E(\delta)\| \leq K$, $E(\delta_1)E(\delta_2) = E(\delta_1)$ if $\delta_1 \subset \delta_2$, and $E(\delta_1)E(\delta_2) = 0$ if $\delta_1 \cap \delta_2 = \emptyset$. In particular, $E(\Delta)$ and $E(C \setminus \Delta)$ are defined for any closed (vertical or horizontal) half plane $\Delta$.

We claim $E(R)$ can be defined for any semiclosed rectangle $R$ of the form $(a, b] \times (c, d]$. Let $x \in Y$. Let $\Delta$ be as in (b) and assume $\{\Delta_n\}$ is a strictly decreasing sequence of closed half planes converging to $\Delta$. Because $u_n = E(C \setminus \Delta)E(\Delta_n)x$ is bounded, every subsequence of $\{u_n\}$ has a weakly convergent subsequence. Let $u$ be the limit of any weakly convergent subsequence of $\{u_n\}$. Since $u_n \in S_T(\Delta_n) \cap S_T(C \setminus \Delta)$ for all $n \geq N$, it follows that $u \in S_T(\Delta_N) \cap S_T(C \setminus \Delta)$ and hence $u \in S_T(\Delta) \cap S_T(C \setminus \Delta) = \{0\}$. Thus $v_n = E(C \setminus \Delta)E(C \setminus \Delta_n)x$ converges weakly to $E(C \setminus \Delta)x$. Therefore, a sequence $w_n$ of finite convex combinations of $v_n$ converging strongly to $E(C \setminus \Delta)x$. Since $\sigma_T(v_n) \subset \sigma_T(x) \setminus \Delta$, $\sigma_T(w_n) \subset \sigma_T(x) \setminus \Delta$ for all $n$. Summing up, we have shown that for every $x \in Y$, $\Delta$ as in (b), and $\varepsilon > 0$, there exists $w \in Y$ such that $\|E(C \setminus \Delta)x - w\| < \varepsilon$ and $\sigma_T(w) \subset \sigma_T(x) \setminus \Delta$. Thus by consecutive applications of this result to the half planes $\{z: Re z \leq a\}$, $\{z: Re z \leq b\}$, $\{z: Im z \leq c\}$, and $\{z: Im z \leq d\}$, we obtain vectors $x_1$ and $x_2 \in Y$ such that $\sigma_T(x_1) \subset \sigma_T(x) \setminus R$, $\sigma_T(x_2) \subset S_T(R)$, and $\|x - x_1 - x_2\| < \varepsilon$. Thus $S_T(R) + S_T(C \setminus R)$ is dense in $Y$ and hence $Y = S_T(R) \oplus S_T(C \setminus R)$.

Let $R_0 = (-a, a] \times (-a, a]$ be a semiclosed square containing $\sigma(T)$. For each $n \in \mathbb{N}$, let $\{D_{nk}: k = 1, 2, \ldots, n\}$ be the family of disjoint semiclosed squares of diameter $a\sqrt{2}/2^n$ covering $R_0$. It is clear that $E$ is a bounded additive set function on the Boolean algebra generated by $\{D_{nk}: k = 1, 2, \ldots, 4^n; n = 1, 2, \ldots\}$. Define $S_n = \sum_{k=1}^{4^n} z_{nk}E(D_{nk})$, where $z_{nk}$ is
the centre of \( D_{nk} \). For \( m > n \), we have
\[
\|S_m - S_n\| = \left\| \sum_{k=1}^{4^n} c_{mk} E(D_{mk}) \right\| \leq 4a\sqrt{2}/2^n,
\]
where \( c_{mk} \) is the difference between the centre of \( D_{mk} \) and the centre of \( D_{nj} \) such that \( D_{mk} \subset D_{nj} \) [3; p. 2181]. Hence \( \{S_n\} \) converges to a scalar-type spectral operator \( A \) [3; p. 2192]. Obviously, \( AT = TA \). We claim \( T - A \) is quasinilpotent. Let \( T_{nk} \) and \( A_{nk} \) be the restrictions of \( T \) and \( A \) to \( E(D_{nk})Y = \overline{S_T(D_{nk})} \) for all possible pairs \((n, k)\). Since \( S_T(D_{nk}) \) is the intersection of four spectral manifolds corresponding to closed half planes, it follows that \( S_T(D_{nk}) \) is closed and hence \( \sigma(T|S_T(D_{nk})) \subset \sigma(T|S_T(D_{nk})) \subset D_{nk} \) [1, p. 23]. Also, let \( S_{mnk} = S_m|\overline{S_T(D_{nk})} \) and observe that \( \|(z - S_{mnk})^{-1}\| \leq K/\text{dist}(z, D_{nk}) \).

Letting \( m \to \infty \) it yields \( \|(z - A_{nk})^{-1}\| \leq K/\text{dist}(z, D_{nk}) \) for all possible \((n, k)\).

Thus \( \sigma(A_{nk}) \subset \overline{D_{nk}} \) and hence \( \sigma(T_{nk} - A_{nk}) \subset \{\alpha - \beta: \alpha \in D_{nk}, \beta \in D_{nk}\} \) [8, Chapter 0]. Therefore, \( \sigma(T_{nk} - A_{nk}) \) lies in a disc of radius \( a\sqrt{2}/2^{n-1} \) for all possible \((n, k)\). It follows that
\[
\sigma(T - A) \subset \bigcup_{k=1}^{4^n} \sigma(T_{nk} - A_{nk}) \subset \{\lambda: |\lambda| < a\sqrt{2}/2^{n-1}\}
\]
for all \( n \), and thus \( \sigma(T - A) = \{0\} \). Therefore, \( T - A \) is quasinilpotent and consequently, \( T \) is spectral.

The necessity of the conditions is an immediate consequence of Dunford’s characterization. \( \square \)

Remarks. (1) We have shown that if \( T \) satisfies condition (a) of the theorem, then it necessarily satisfies condition (A).

(2) A sort of strong regularity of \( E \) depicted by \( x = \lim[E(\mu_n) + E(\nu_n)]x \) required in Dunford’s condition (D) is not needed in our criterion. We at most require \( S_T(\Delta) + S_T(\mathbb{C} \setminus \Delta) \) to be dense in \( Y \). In other words, we need only the closed (vertical or horizontal) half planes to belong to \( \mathcal{S}_T \).

(3) Dunford’s requirement that \( S_T(F) \) be closed for every set \( F \) is reduced in our criterion to the requirement that \( S_T(\Delta) \) be closed for all closed half planes. The following example shows that \( T \) may fail to be spectral if, in Condition (b) of the theorem, \( \Delta \) is assumed only to be a closed right or upper half plane.

(4) Some of the ideas are taken from the proof of Theorem 1 of [7].

Example. Let \( V \) be a nonunitary contraction operator on a Hilbert space \( K \) with \( \sigma(B) = \{1\} \) [5, Problem 150]. Let \( \varphi \) be a conformal mapping from the unit disc onto \( \Delta_1 = \{\text{re}^{i\theta}: 0 \leq r \leq 1, 0 \leq \theta \leq \pi/4\} \) such that \( \varphi(1) = 0 \). Let \( A = \varphi(V) \). Then \( \sigma(A) = \{0\} \) and \( \Delta_1 \) is a spectral set for \( A \) [4, §1.1; 9, p. 143; 6, proof of Proposition 1]. Then the set \( \Delta_n = \{\text{re}^{i\theta}: 0 \leq r \leq 1, 0 \leq \theta \leq \pi/4n\} \) is a spectral set for \( A^{1/n} = \varphi_n(A) \), where \( \varphi_n(\text{re}^{i\theta}) = r^{1/n} e^{i\theta/n} \).
and \( \text{re}^{i\theta} \in \Delta_n \) \((n \in \mathbb{N})\). Let \( T = A \oplus A^{1/2} \oplus \cdots \). As it is shown in the proof of Proposition 1 of [6], \( \sigma(T) = [0, 1] \), \( \overline{S_T(\{0\})} = H \), and \( T \) satisfies the single-valued extension property where \( H = K \oplus K \oplus \cdots \). Thus \( S_T(\{0\}) \) is not closed [1; p. 23]. Hence, in the light of Dunford's criterion, \( T \) is not spectral. Let \( 0 \neq x \in H \) be arbitrary. We claim \( 0 \in \sigma_T(x) \). Let \( f \) be the local resolvent of \( T \) at \( x \) and assume, if possible, that \( 0 \notin \sigma_T(x) \). Let \( x_n \) be the projection of \( x \) in the \( n \)th copy of \( K \). Then \( \sigma_T(x_n) = \sigma_{A^{1/n}}(x_n) \subseteq \sigma_T(x) \). Since \( A^{1/n} \) is quasinilpotent, \( x_n = 0 \) \((n \in \mathbb{N})\). Thus \( x = 0 \), a contradiction. Thus condition (a) holds trivially. Moreover, if \( \delta \) is any Borel set, then \( S_T(\delta) + S_T(C \setminus \delta) \) is dense in \( H \) because it contains \( S_T(\{0\}) \). Now, let \( \Delta \) be any closed horizontal half plane. Then, either \([0, 1] \subseteq \Delta \), in which case \( S_T(\Delta) = S_T(\sigma(T)) = H \), or \( 0 \notin \Delta \), from which follows \( S_T(\Delta) = \{0\} \). Finally, assume \( \Delta \) is a closed right half plane. Then \( S_T(\Delta) = \{0\} \) if \( 0 \notin \Delta \), and \( S_T(\Delta) = H \) if \( 0 \in \Delta \).

\textit{Added in proof.} We are grateful to the referee who brought to our attention a preprint of Professor B. L. Wadhwa's talk [10], where he shows that \( T \) is spectral if and only if \( T \) satisfies the above Dunford's Conditions (A), (B), (C) and the following Condition (D'): \( S_T(\delta) + S_T(C \setminus \delta) \) is dense in \( Y \) for all closed sets \( \delta \). Thus the main result of our paper extends Wadhwa's result by omitting Condition (A) and restricting Conditions (C) and (D') to closed half planes.

\section*{References}


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