NONINTEGRABILITY OF SUPERHARMONIC FUNCTIONS

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ABSTRACT. In this article we prove the following: If $u$ is a nonzero superharmonic function on a proper subdomain $D$ in $\mathbb{R}^n$, then

$$\int_D |u(x)|^p \delta_D(x)^{np-n-2p} \, dx = \infty,$$

where $0 < p \leq 1$ and $\delta_D(x)$ denotes the distance between $x$ and the boundary of $D$.

Let $D$ be a domain in the euclidean space $\mathbb{R}^n$ ($n \geq 2$), with $D \neq \mathbb{R}^n$. We denote by $\delta_D(x)$ the distance between $x \in D$ and $\partial D$, the boundary of $D$. In contrast to [3], we obtain the following result:

Theorem. Let $0 < p \leq 1$. If a superharmonic function $u$ on $D$ in $\mathbb{R}^n$ satisfies

(\ast) \quad \int_D |u(x)|^p \delta_D(x)^{np-n-2p} \, dx < \infty,

then $u$ vanishes identically.

It is obvious that in the condition (\ast) the exponent $np - n - 2p$ cannot be replaced by any larger number (e.g., see the remark below). Incidentally, we note that if $D$ is a bounded $C^1$-domain, then $\int_D s(x)^p \delta_D(x)^{np-n-p} \, dx = \infty$ for any nonzero subharmonic function $s \geq 0$ on $D$ (cf. [4]).

Now, denote by $B(x, r)$ the open ball of radius $r > 0$ centered at $x \in \mathbb{R}^n$ and by $G_D$ the Green function of $D$ (in the sense of [2, p. 84]) if it exists. Positive constants depending only on $\alpha$, $\beta$, ... are denoted by $c(\alpha, \beta, \ldots)$ and are not necessarily the same on any two occurrences.

Lemma. If a nonnegative subharmonic function $s$ on $D$ satisfies

$$\int_D s(x)^p \delta_D(x)^{np-n-2p} \, dx < \infty$$

for some $0 < p \leq 1$, then $s(x) = o(\delta_D(x)^{2-n})$ as $\delta_D(x) \to 0$. In particular, if $n = 2$, then $s$ is bounded and $s(x) \to 0$ as $x \to \partial D$.

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Proof. For \( x \in D \), let \( B = B(x, \delta_D(x)) \). By the same argument as in [1, p. 172], we obtain

\[
s(x)^p \leq c(n, p)\delta_D(x)^{-n} \int_B s(y)^p \, dy
\]

\[
\leq c(n, p)\delta_D(x)^{p(2-n)} \int_B s(y)^p \delta_D(y)^{np-n-2p} \, dy,
\]

since \( \delta_p(y) \leq 2\delta_D(x) \) for \( y \in B \). From this, the assertions of the lemma follow immediately.

Proof of the theorem. Define \( s(x) = -\min\{u(x), 0\} \). First we assume that \( D \) has the Green function. Since \( \int_D s(x)^p \delta_D(x)^{np-n-2p} \, dx < \infty \),

\( v_{s,p}(\cdot) := \int_D G_D(\cdot, y)s(y)^p \delta_D(y)^{np-n-2p} \, dy \)

is a potential. For any \( x \in D \), set \( t = \delta_D(x)/2 \), \( B = B(x, 2t) \), and \( B_0 = B(x, t) \). Then, for \( y \in B_0 \), \( \delta_D(y) \leq 3t/2 \) and \( s(y)^{p-1} \geq c(n, p, s)\delta_D(y)^{(n-2)(1-p)} \) by the lemma. Hence the sub-mean-value property of \( s \) gives

\[
v_{s,p}(x) \geq \int_{B_0} G_B(x, y)s(y)^p \delta_D(y)^{np-n-2p} \, dy
\]

\[
\geq c(n) \int_{B_0} |x - y|^{2-n}s(y)[s(y)^{p-1}\delta_D(y)^{np-n-2p}] \, dy
\]

\[
\geq c(n, p, s) \int_0^t \int_{|\xi| = 1} r^{2-n}s(x + r\xi)t^{-2}r^{n-1} \, dr \, d\sigma(\xi)
\]

\[
\geq c(n, p, s)s(x).
\]

Since every subharmonic function dominated by a potential is nonpositive, we deduce that \( s \equiv 0 \); that is, \( u \geq 0 \) on \( D \). On the other hand, the same argument as above leads to \( v_{1,1} = \int_D G_D(\cdot, y)\delta_D(y)^{-2} \, dy \equiv \infty \), and hence any potential \( v = \int_D G_D(\cdot, z)\, d\lambda(z) \) with \( \lambda \neq 0 \) satisfies

\[
\int_D v(y)\delta_D(y)^{-2} \, dy = \int_D \int_D G_D(y, z)\delta_D(y)^{-2} \, d\lambda(z) \, dy
\]

\[
= \int_D \int_D G_D(z, y)\delta_D(y)^{-2} \, dy \, d\lambda(z)
\]

\[
= \int_D v_{1,1}(z) \, d\lambda(z) = \infty.
\]

Since every positive superharmonic function has a nonzero potential as its minorant, we conclude that \( u \equiv 0 \) under (*) by remarking that \( u^p \) is superharmonic and \(-2 \geq np - n - 2p\).

Next we consider the case that \( D \) does not have the Green function. Then \( n = 2 \) (cf. [2, Theorem 5.12]). On account of the lemma, \( s \) is bounded and \( s(x) \to 0 \) as \( x \to \partial D \). For \( \varepsilon > 0 \), set \( s_\varepsilon = \max\{s - \varepsilon, 0\} \) and choose a ball \( B_\varepsilon = B(x_\varepsilon, r_\varepsilon) \) in \( D \) such that \( s_\varepsilon = 0 \) on \( B_\varepsilon \). For \( \eta > 0 \), we also set

\[
s_{\varepsilon, \eta}(x) = \begin{cases} 
\max\{s_\varepsilon(x) - \eta \log(|x - x_\varepsilon|/r_\varepsilon), 0\}, & x \in D \setminus B_\varepsilon \\
0, & x \in B_\varepsilon.
\end{cases}
\]
Then $0 < s_{\epsilon, \eta} \leq s_{\epsilon} \leq s$, and $s_{\epsilon, \eta}$ is subharmonic on $D$. Since $s_{\epsilon, \eta}(x) \to 0$ as $x \to \partial D$ or as $x$ tends to infinity, the maximum principle implies that $s_{\epsilon, \eta} \equiv 0$, so that $s(x) \leq \epsilon + \eta \log(|x - x_{\epsilon}|/r_{\epsilon})$. Letting $\eta \downarrow 0$ and then $\epsilon \downarrow 0$, we deduce that $s \equiv 0$. Thus $u$ is nonnegative, so that $u$ is constant. The conclusion $u \equiv 0$ follows easily. This completes the proof.

Remark. Let $D = B(0, 2)\setminus\{0\}$ and $B = B(0, 1)$ and define $u(x) = -G_{B}(x, 0)$ on $B \cap D$ and $u(x) = 0$ on $D \setminus B$. Then $u$ is superharmonic on $D$ and satisfies
\[
\int_{D} |u(x)|^{p} \delta_{D}(x) \| \theta_{n-p-n-2p+\epsilon} \, dx \leq \infty
\]
for any $\epsilon > 0$.

REFERENCES


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