WHEN ARE TOUCHPOINTS LIMITS FOR GENERALIZED PÓLYA URNS?

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Abstract. Hill, Lane, and Sudderth (1980) consider a Pólya-like urn scheme in which \( X_0, X_1, \ldots \) are the successive proportions of red balls in an urn to which at the \( n \)th stage a red ball is added with probability \( f(X_n) \) and a black ball is added with probability \( 1 - f(X_n) \). For continuous \( f \) they show that \( X_n \) converges almost surely to a random limit \( X \) which is a fixed point for \( f \) and ask whether the point \( p \) can be a limit if \( p \) is a touchpoint, i.e. \( p = f(p) \) but \( f(x) > x \) for \( x \neq p \) in a neighborhood of \( p \). The answer is that it depends on whether the limit of \( (f(x) - x)/(p - x) \) is greater or less than 1/2 as \( x \) approaches \( p \) from the side where \( (f(x) - x)/(p - x) \) is positive.

Hill, Lane, and Sudderth (1980), hereafter referred to as [HLS], consider the following urn scheme. Let \( f: [0, 1] \rightarrow [0, 1] \) be any function and let an urn begin with \( l \) balls of which a proportion \( X_{l-1} \in (0, 1) \) are red and the remainder black. Add a new ball to the urn, whose color is red with probability \( f(X_{l-1}) \) and black otherwise. Let \( X_l \) be the new proportion of red balls and iterate the procedure, producing a sequence of proportions \( X_{l-1}, X_l, X_{l+1}, \ldots \). In the case where \( f \) is continuous, they show that \( X_n \) converges almost surely to some random variable \( X \). Furthermore, \( f(X) = X \) almost surely [HLS, Theorem 2.1 and Corollary 3.1]. Categorize points \( p \in (0, 1) \) for which \( p = f(p) \) by calling them upcrossings if \( (y - p)(f(y) - y) \) is positive for all \( y \) in some neighborhood of \( p \), and downcrossings if \( (y - p)(f(y) - y) \) is negative for all \( y \) in some neighborhood of \( p \). The terminology comes from the way the graph \( y = f(x) \) crosses the graph \( y = x \). The next results of [HLS] are that \( \text{prob}(X_n \rightarrow p) > 0 \) if \( p \) is a downcrossing and \( f \) maps \( (0, 1) \) into itself, while \( \text{prob}(X_n \rightarrow p) = 0 \) if \( p \) is an upcrossing. The only other kind of isolated point, \( p \), in the set \( \{x: x = f(x)\} \) is a touchpoint where \( f(y) > y \) for all \( y \neq p \) in a neighborhood of \( p \), or else \( f(y) < y \) for all \( y \neq p \) in a neighborhood of \( p \). They ask whether touchpoints can be in the support of the limiting random variable \( X \).
This note answers their question both ways for continuous $f$, giving a condition on $f$ near $p$ implying $\text{prob}(X_n \to p) > 0$ and another condition that implies $\text{prob}(X_n \to p) = 0$. These conditions almost meet, in the sense that they cover all cases where $(f(x) - x)/(p - x)$ has a limit as $x \to p$ except for the case where the limit is equal to 1/2. By symmetry between red and black balls, there is no loss of generality in considering only touchpoints of the first kind, where $f(y) > y$ for $y \neq p$ in a neighborhood of $p$. Therefore, the proofs will be given only for the touchpoints of the first kind. Furthermore, whether $X_n$ converges to $p$ with positive probability depends only on the germ of $f$ at $p$ [HLS, Lemma 4.1], so the arguments below will assume without loss of generality that $f(y) > y$ for all $y \neq p, 1$, as well as assuming that $f$ maps $(0, 1)$ into itself.

Let $\mathcal{F}_n$ be the $\sigma$-algebra generated by $\{X_i: i \leq n\}$, and let $\mathcal{F}_\tau$ be defined similarly for any stopping time $\tau$. The key to the proof of both conditions will be the decomposition of the submartingale $\{X_n, \mathcal{F}_n\}$ into a martingale and an increasing process. Write $X_{n+1} = X_n + A_n + Y_n$, where

$$A_n = \mathbb{E}(X_{n+1} - X_n | \mathcal{F}_n)$$

is $\mathcal{F}_n$-measurable and $Y_n = X_{n+1} - X_n - A_n$, so $\mathbb{E}(Y_n | \mathcal{F}_n) = 0$. Then calculate the following conditional probabilities given $\mathcal{F}_n$:

$$X_{n+1} = \begin{cases} \frac{nX_{n+1}}{n+1} = X_n + \frac{1-X_n}{n+1} & \text{with probability } f(X_n), \\ \frac{nX_n}{n+1} = X_n - \frac{X_n}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

This gives $A_n = (f(X_n) - X_n)/(n+1)$, which is nonnegative by assumption; and hence

$$Y_n = \begin{cases} \frac{1-f(X_n)}{n+1} & \text{with probability } f(X_n), \\ \frac{-f(X_n)}{n+1} & \text{with probability } 1 - f(X_n). \end{cases}$$

Also, $Y_n$ is a mean zero random variable given $\mathcal{F}_n$, with the conditional distribution of $Y_n$ given $\mathcal{F}_n$ satisfying $\min((f(X_n), 1 - f(X_n)))^2/(n+1)^{-2} = \inf Y_n^2 \leq \mathbb{E}(Y_n^2 | \mathcal{F}_n) \leq \sup Y_n^2 \leq (n+1)^{-2}$, where the inf is over $\omega$ in the $\mathcal{F}_n$-measurable set for which $X_n$ has the given value. Defining

$$Z_{n,m} = \sum_{i=n}^{m-1} Y_i$$

yields for each fixed $n$ a martingale $\{Z_{n,m}, \mathcal{F}_m\}$ with an $L^2$-bound $\mathbb{E}Z_{n,\infty}^2 \leq \sum_{i=n}^{\infty} (i+1)^{-2} \leq 1/n$. If $f$ is bounded away from 0 and 1 near $p$, then a lower $L^2$-bound is gotten by stopping the process $X_n$ when it exits an interval on which $\min(f(X_n), 1 - f(X_n)) > b$. If $\tau$ is any stopping time bounded above by the exit time of the interval, then the above lower bound on $\mathbb{E}Y_m^2$ gives

$$\mathbb{E}(Z_{n,\infty}^2 | \mathcal{F}_n) \geq \mathbb{E}(Z_{n,\tau}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n)b^2(n+1)^{-1}. \quad (1)$$
The idea will be that if \( f(x) - x < (p - x)/2 \), then the increasing part \( A \) pushes \( X \) toward \( p \) so slowly that by the time \( X \) gets close to \( p \), the increments of \( Z \) are very very small, and \( Z \) cannot push \( X \) above \( p \). So, in fact, one gets convergence to \( p \) from below. On the other hand, if \( f(x) - x > (p - x)/2 \), then the increasing part pushes \( X \) toward \( p \) fast enough so that the increments of \( Z \) are big enough compared with \( p - X \), so that, eventually, the addition of \( Z \) puts \( X \) over \( p \). A result along the lines of Pemantle [P1, P2] then implies that \( X_n \) cannot converge to \( p \).

**Remark.** It will be shown that convergence to a touchpoint near which \( f(x) > x \) is always from the left. Thus the behavior of the function to the right of the touchpoint is irrelevant.

**Theorem 1.** Let \( f \) be continuous in a neighborhood of a touchpoint \( p \) and suppose that \( f \) maps \((0, 1)\) into itself. Further suppose that \( x < f(x) \leq x + k(p - x) \) for some \( k < 1/2 \) and all \( x \) in some left neighborhood, \((p - \epsilon, p)\), of \( p \). Then \( \text{prob}(X_n \rightarrow p) > 0 \). [Similarly, if \( x > f(x) \geq x - k(x - p) \) for some \( k < 1/2 \) and all \( x \) in a right neighborhood, \((p, p + \epsilon)\) of \( p \), then also \( \text{prob}(X_n \rightarrow p) > 0 \).

**Corollary 2.** If \( f \) is differentiable at a touchpoint \( p \) and continuous in a neighborhood of \( p \), then \( \text{prob}(X_n \rightarrow p) > 0 \) under the same nontriviality assumption \( f((0,1)) \subset (0,1) \).

**Proof.** Since \( f(x) - x \) does not change sign at \( p \), the derivative of \( f(x) - x \) must be zero at \( p \) and Theorem 1 applies. \( \square \)

**Proof of Theorem 1.** Replacing \( f \) by a function agreeing with \( f \) on a neighborhood of \( p \), there is no loss of generality in assuming that \( f \) is continuous and that \( f(x) > x \) for all \( x \in [0, 1]\backslash\{p\} \). Thus it will suffice to prove that with positive probability there is an \( N \) for which \( n > N \) implies \( X_n < p \), since \( X_n \) converges to a fixed point of \( f \) [HLS, Corollary 3.1], which must then be \( p \).

Pick a \( k \) for which the hypothesis is satisfied and pick \( k_1 \) with \( k < k_1 < 1/2 \). Pick a constant \( \gamma \) just barely greater than 1 so that \( \gamma k_1 < 1/2 \). The function \( g(r) = re^{(r-n)/2k_1}r \) has value 1 at \( r = 1 \) and derivative \( g'(1) = 1 - 1/2k_1\gamma < 0 \), so there is an \( r \in (0, 1) \) for which \( g(r) > 1 \). Fix such an \( r \). Define

\[
T(n) = e^{n(1-r)/\gamma k_1}, \quad \text{so} \quad g(r^n) = r^n T(n)^{1/2} > 1.
\]

Choose \( M \) big enough so that \( \gamma r^M < \epsilon \) and define

\[
\tau_M = \inf\{j > T(M): X_{j-1} < p - r^M < X_j\}
\]

if such a \( j \) exists, and \( \tau_M = -\infty \) otherwise. By the nontriviality assumption that \( f \) maps \((0, 1)\) into itself, \( \text{prob}(\tau_M > T(M)) > 0 \). For each \( n \geq M \), define \( \tau_{n+1} = \inf\{j \geq \tau_n: X_j > p - r^{n+1}\} \). Note that if \( X_j \geq p \) for some \( j > T(M) \), then \( \tau_n \leq j \) for all \( n \geq M \). The theorem will be proved by showing that \( \text{prob}(\tau_n > T(n)) \) for all \( n \geq M > 0 \), which will imply that with
nonzero probability, $X_n$ is eventually less than $p$, proving the theorem. Begin
by assuming that $\tau_n > T(n)$ and calculate $\text{prob}(\tau_{n+1} > T(n+1)|\tau_n > T(N))$
as follows. Let $B$ be the event $\{\inf_{j > \tau_n} X_j \geq p - \gamma r^n\}$ and estimate

$$\text{prob}(B^c|\tau_n > T(N)) = \text{prob}\left(\inf_{j > \tau_n} X_j < p - \gamma r^n|\tau_n > T(N)\right)$$

$$\leq \text{prob}\left(\inf_{\tau_n < j < T(n+1)} X_j < -(\gamma - 1)r^n|\tau_n > T(N)\right)$$

$$\leq \mathbb{E}(Z_{\tau_n,\infty}^2|\tau_n > T(N))/(\gamma - 1)^2$$

$$\leq e^{-n(1-r)/k_1}(\gamma - 1)^{-2}r^{-n}$$

$$= (\gamma - 1)^{-2}[g(r)]^{-2n}.$$

Next, note that if $B$ holds, then

$$\sum_{T(n) < j < T(n+1)} A_j = \sum_{T(n) < j < T(n+1)} (f(X_j) - X_j)/(j + 1)$$

$$< (\ln[T(n + 1)] - \ln[T(n)])/(k \gamma r^n)$$

$$\leq (k \gamma r^n)/[(1 - r)/(k_1) + 1/T(n)]$$

$$= (k/k_1)(r^n - r^{n+1}) + k \gamma r^n/T(n).$$

But then if $B$ holds and $\tau_{n+1} = L \leq T(n + 1)$, it must be the case that

$$Z_{\tau_n, L} = X_L - X_{\tau_n} - \sum_{j=\tau_n}^{L-1} A_j$$

$$\geq X_L - X_{\tau_n} - \sum_{T(n) < j < T(n+1)} A_j$$

$$\geq r^n - r^{n+1} - \xi_n - (k/k_1)(r^n - r^{n+1}) - k \gamma r^n/T(n)$$

$$= r^n(1 - r)(1 - (k/k_1)) - \xi_n - k \gamma r^n/T(n)$$

$$= r^n(1 - r)(1 - (k/k_1)) - \tilde{\xi}_n.$$

The term $\xi_n$ comes from the fact that $X_{\tau_n}$ may overshoot the stopping point $p - r^n$, and $\tilde{\xi}_n$ denotes the sum of $\xi_n$ and the $k \gamma r^n/T(n)$ term. Then $\xi_n$ is bounded by $X_{\tau_n} - X_{\tau_n - 1} < \tau_n^{-1} < T(n)^{-1}$ by assumption. Since $T(n)^{-1}$ is of order less than $r^{2n}$, the $\tilde{\xi}_n$ contribution vanishes asymptotically in the sense that

$$\frac{r^n(1 - r)(1 - (k/k_1)) - \tilde{\xi}_n}{r^n(1 - r)(1 - (k/k_1))} \to 1.$$
Now $E(Z_{\tau_n,\infty}^2 \mid \tau_n > T(N)) < T(n)^{-1}$, so

$$\text{prob}(\tau_{n+1} \leq T(n+1) \mid \tau_n > T(N)) \leq \text{prob}(\mathcal{B}^c \mid \tau_n > T(N)) + \text{prob} \left( \mathcal{B} \text{ and } \sup_{t \leq N} Z_{\tau_n,\tau_t \geq n} \leq n(1-r) \left( 1 - \frac{k}{k_1} \right) - \frac{\varepsilon_n}{2} \mid \tau_n > T(N) \right)$$

$$\leq (1-\gamma)^{-2} [g(r)]^{-2n} + T(n)^{-1} \left[ n(1-r) \left( 1 - \frac{k}{k_1} \right) - \frac{\varepsilon_n}{2} \right]^2$$

$$\leq (1-\gamma)^{-2} [g(r)]^{-2n} + \left[ (1-r)(1 - (k/k_1)) \right]^{-2} [g(r)]^{-2n} \times \left[ \frac{n(1-r)(1 - (k/k_1)) - \frac{\varepsilon_n}{2}}{n(1-r) \left( 1 - \frac{k}{k_1} \right)} \right]^2.$$

Because the last term of the numerator vanishes asymptotically, the sum of these probabilities converges. Then $\text{prob}(\tau_n > T(n) \text{ for all } n > M) = \text{prob}(\tau_M > T(M)) \prod_{n \geq M} (1 - \text{prob}(\tau_{n+1} \leq T(n+1) \mid \tau_n > T(N))) > 0$ since each factor is positive and $\sum \text{prob}(\tau_{n+1} \leq T(n) \mid \tau_n > T(N))$ is finite. In this case, $X_n$ must converge to $p$ from below. $\square$

**Theorem 3.** Suppose that $f(x) \geq x + k(p-x)$ for some $k > 1/2$ and all $x$ in some left neighborhood, $(p-\varepsilon, p)$, of $p$. Then $\text{prob}(X_n \rightarrow p) = 0$. [Similarly, if $f(x) \leq x - k(x - p)$ for some $k > 1/2$ and all $x$ in a right neighborhood, $(p, p+\varepsilon)$ of $p$, then also $\text{prob}(X_n \rightarrow p) = 0$.]

**Remark.** No continuity assumptions are needed this time.

**Proof.** Again there is no loss of generality in assuming that $f(x) \geq x$ for all $x$; similarly, assume $f(x) \geq \min(1, x + k|p-x|)$ on $[0, p]$. Furthermore, Lemma 2.2 of [HLS] says that replacing $f$ by a pointwise smaller function gives a process which can be defined on the same probability space so as always to be smaller. Thus replacing $f$ by the minimum of $1$ and $x + k|p-x|$ on $[0, p]$ and by $x$ on $[p, 1]$ gives a process which converges to $p$ whenever the original process does, so it suffices to prove the theorem for this choice of $f$. The importance of assuming this lies only in getting $f$ bounded away from 0 and 1 near $p$ (without assuming continuity) so that there will be a lower $L^2$-bound on $Z$.

The following argument is self-contained, but the reader may wish to look at Pemantle [P2, Lemmas 1 and 2] to see the template from which this proof was constructed.

**Lemma 4.** There are constants $a, c > 0$ and a neighborhood $N$ of $p$ such that for any $n$

$$\text{prob}(Z_{n,\infty} \geq cn^{-1/2} \text{ or } X_{n+j} \notin \mathcal{N} \text{ for some } j \in \mathcal{F}_n) > a.$$
Proof. Pick $b > 0$ and $\mathcal{N}$ a neighborhood of $p$ such that $f(\mathcal{N}) \subseteq [b, 1-b]$. Assume that $X_n \in \mathcal{N}$ or else the result is trivially true. For $k > 0$, let $\tau \leq \infty$ be the first time $X_j$ exits $\mathcal{N}$ or $Z_{n,j}$ exits $(-kn^{-1/2}, kn^{-1/2})$. Then equation (1) gives $\mathbb{E}(Z_{n,\tau}^2 | \mathcal{F}_n) \geq \text{prob}(\tau = \infty | \mathcal{F}_n)b^2(n+1)^{-1}$. On the other hand, $\mathbb{E}(Z_{n,\tau}^2 | \mathcal{F}_n) \leq \mathbb{E}(X_{\tau} - X_n)^2 \leq k^2/n$, since $Z$ is just the martingale part of $X$. Putting these together gives $\text{prob}(\tau = \infty | \mathcal{F}_n) \leq k^2(n+1)/b^2n$, and choosing $k$ small enough makes this at most $1/3$. Let

$$q = \text{prob}(\tau < \infty, X_t \notin \mathcal{N} | \mathcal{F}_n),$$

so that the conditional probability of $Z_{n,j}$ exiting $(-kn^{-1/2}, kn^{-1/2})$ given $\mathcal{F}_n$ is at least $2/3 - q$. Any martingale $\mathcal{M}$ started at zero that exits an interval $(-L, L)$ with probability at least $r$ and has increments bounded by $L/2$ satisfies $\text{prob}(\sup \mathcal{M} \geq L/2) \geq (3r-1)/4$; stopping $\mathcal{M}$ upon exiting $(-L, L/2)$ and letting $s = \text{prob}(\sup \mathcal{M} > L/2)$ gives $0 = \mathbb{E}\mathcal{M} \leq sL + (r-s)(-L) + (1-r)(L/2) = 2L(s - (3r-1)/4)$. Thus $Z_{n,j} \geq k/2\sqrt{n}$ for some $j$ with probability at least $(1 - 3q)/4$.

Now for any $j$, condition on the event $Z_{n,j} \geq k/2\sqrt{n}$; then the conditional probability of the event $Z_{n,\infty} < k/4\sqrt{n}$ can be bounded away from 1 using the following one-sided Tschebysheff estimate:

**Lemma 5.** If $\mathcal{M}$ has mean zero and $L < 0$, then

$$\text{prob}(\mathcal{M} \leq L) \leq \mathbb{E}\mathcal{M}^2/(\mathbb{E}\mathcal{M}^2 + L^2).$$

**Proof.** Write $w$ for $\text{prob}(\mathcal{M} \leq L)$. From

$$0 = \mathbb{E}\mathcal{M}^2 = w\mathbb{E}(\mathcal{M} | \mathcal{M} \leq L) + (1-w)\mathbb{E}(\mathcal{M} | \mathcal{M} > L)$$

and $\mathbb{E}(\mathcal{M} | M \leq L) \leq L$, it is immediate that

$$\mathbb{E}(\mathcal{M} | \mathcal{M} > L) \geq -L \frac{w}{1-w}.$$  

Then

$$\mathbb{E}\mathcal{M}^2 = w\mathbb{E}(\mathcal{M}^2 | M \leq L) + (1-w)\mathbb{E}(\mathcal{M}^2 | \mathcal{M} > L)$$

$$\geq wL^2 + (1-w)(\mathbb{E}(\mathcal{M}^2 | \mathcal{M} > L))^2$$

$$\geq wL^2 + (1-w)L^2(w^2/(1-w)^2)$$

$$= L^2 w/(1-w),$$

from which the desired conclusion follows. □

Apply this to the process $Z_{j,i}$ stopped at the entrance time $\tau$ of the interval $(-\infty, -k/4\sqrt{n})$ to get

$$\text{prob}(Z_{n,\infty} \leq k/4\sqrt{n} | \mathcal{F}_j) \leq \text{prob}(Z_{j,\tau} \leq -k/4\sqrt{n} | \mathcal{F}_j) \leq \mathbb{E}Z_{j,\tau}^2/(\mathbb{E}Z_{j,\tau}^2 + k^2/16n) \leq \mathbb{E}Z_{n,\infty}^2/(\mathbb{E}Z_{n,\infty}^2 + k^2/16n) \leq 16/(k^2 + 16).$$
Combining this with the previous result shows that the conditional probability of $Z_{n,\infty} > k/4\sqrt{n}$ given $\mathcal{F}_n$ is at least $(1 - 3q)k^2/(64 + 4k^2)$. Recall that $q$ is the conditional probability of the process exiting $\mathcal{N}$ given $\mathcal{F}_n$, so that the probability we are trying to bound below is at least the maximum of $q$ and $(1 - 3q)k^2/(64 + 4k^2)$. For any value of $q$ the maximum is at least $k^2/(64 + 7k^2)$, thus the statement of the lemma is proved with $c = k/4$ and $a = k^2/(64 + 7k^2)$. \(\Box\)

Let $\tau$ be any finite stopping time. Conditioning on $\mathcal{F}_t$ then gives a stopping time version of the previous lemma:

\[
\begin{align*}
\text{prob}(Z_{\tau,\infty} > c\tau^{-1/2} \text{ or } X_{\tau+j} \notin \mathcal{N} \text{ for some } j|\mathcal{F}_t) > a.
\end{align*}
\]

A corollary of this is a sort of converse to the proof of Theorem 1, saying that if $X_n \to p$ then it does so from the left.

**Corollary 6.** Let $p$ be a touchpoint of the first kind, i.e. $f(y) > y$ for all $y \neq p$ in a neighborhood of $p$. Then the probability of the event that either $X_n > p$ finitely often or $X_n$ does not converge to $p$ is 1.

**Proof.** Suppose to the contrary that the probability that $X_n$ converges to $p$ and is greater than or equal to $p$ infinitely often is nonzero. Then there are $n, M$, and some event $\mathcal{B} \in \mathcal{F}_n$ such that $n < M$ and conditional on $\mathcal{B}$, the probability of $X_j$ converging to $p$ and being greater than $p$ some time before $M$ but never leaving $\mathcal{N}$ after time $n$ is at least $1 - a/3$. Define $\tau$ to be the minimum of $M$ and the least $j \geq n$ such that $X_j > p$. Then letting $\mathcal{E}$ be the event that $X_j$ converges to $p$ without leaving $\mathcal{N}$ after time $n$,

\[
\begin{align*}
\text{prob}(\mathcal{E}|\mathcal{B}, \tau < M) \geq 1 - a/3.
\end{align*}
\]

So

\[
\begin{align*}
\text{prob}(\mathcal{E}|\mathcal{B}, \tau < M) \geq 1 - a/3 - \text{prob}(\tau = M|\mathcal{B}) \geq 1 - 2a/3.
\end{align*}
\]

Now $\tau < M$ implies that $X_j > p$. But since $A_n$ is an increasing process, it follows that $X_j \to p$ and $X_t > p$ together imply $Z_{t,\infty} < 0$. Thus

\[
\begin{align*}
\text{prob}(Z_{t,\infty} < 0 \text{ and } X_{n+j} \in \mathcal{N} \text{ for all } j|\mathcal{B}, \tau < M) \geq 1 - 2a/3,
\end{align*}
\]

and hence

\[
\begin{align*}
\text{prob}(Z_{t,\infty} > c\tau^{-1/2} \text{ or } X_{t+j} \notin \mathcal{N} \text{ for some } j|\mathcal{B}, \tau > M) \leq 2a/3.
\end{align*}
\]

But this contradicts (2), since the events $\mathcal{B}$ and $\tau < M$ are both in $\mathcal{F}_t$. \(\Box\)

**Continuation of the proof of Theorem 3.** It remains to show that under the hypothesis of the theorem, the probability is zero that $X_n$ eventually resides in $(p - \varepsilon, p)$. If the probability were nonzero, then for any $\delta$ there would be an
event $\mathcal{B}$ in some $\mathcal{F}_M$ for which $\text{prob}(X_{M+j} \in (p-\varepsilon, p))$ for all $j \geq 0|\mathcal{B}) > 1 - \delta$. In fact, conditioning on $X_M$, $\mathcal{B}$ may be taken to determine $X_M$. So it suffices to show that the probability of the event $X_{M+j} \in (p-\varepsilon, p)$ for all $j \geq 0$ given $X_M$ is bounded away from 1. For what follows condition on $\mathcal{F}_M$ and on $X_M \in (p-\varepsilon, p)$. Also choose $M$ large enough so that for any $n > M$, $n^{-k/2k_l} < cn^{-1/2}$ where $c$ is chosen as in Lemma 4, and choose $\varepsilon$ small enough so that $(p-\varepsilon, p)$ is a subset of a neighborhood $\mathcal{N}$ to which Lemma 4 applies.

Begin by setting up constants and stopping times: pick a $k < 3/4$ for which the hypothesis of the theorem is satisfied and pick $k_1$ so that $k > k_1 > 1/2$. For $n \geq M$ define

$$V_n = (k/k_1) \ln(n) + 2 \ln(p - X_n) \quad \text{for } X_n < p \quad \text{and } -\infty \text{ otherwise.}$$

By assumption on $X_M$, $V_M > -\infty$. Let $\tau$ be the least $n \geq M$ such that $X_n \notin (p-\varepsilon, p)$ or $V_n < 0$. Observe that if $V_n > 0$ then $1/n < (p - X_n)^{2k_1/k} \leq (p - X_n)^{4/3}$, so $|X_{n+1} - X_n|$ is small compared to $p - X_n$, so $V_{\tau\wedge n}$ can never reach $-\infty$ and is in fact bounded below by $\min (-1, V_M)$. Now for $n < \tau$ calculate

$$E(\ln(p - X_{n+1})|\mathcal{F}_n) \leq \ln E(p - X_{n+1}|\mathcal{F}_n)$$

$$= \ln(p - X_n - A_n)$$

$$\leq \ln((p - X_n)(1 - k/(n + 1)))$$

$$= \ln(p - X_n) + \ln(1 - k/(n + 1));$$

so

$$E(V_{n+1}|\mathcal{F}_n) \leq V_n + (k/k_1)(\ln(n + 1) - \ln(n)) + 2 \ln(1 - k/(n + 1))$$

$$= V_n + (k/k_1)(n^{-1} + o(n^{-1})) - 2k(n^{-1} + o(n^{-1}))$$

$$= V_n - ((2 - 1/k_1)k + o(1))n^{-1} < V_n - Cn^{-1}$$

for large $n$ and some $C > 0$. So $V_{\tau\wedge n}$ is a supermartingale for large $n$, bounded below by $\min (-1, V_M)$, and hence converges almost surely. Clearly it cannot converge without stopping, since the increments of the expectation sum to $-\infty$, therefore the stopping time is reached almost surely.

In other words, conditional upon any event in any $\mathcal{F}_M$, the probability is 1 that for some $n > M$, either $X_n$ will leave $(p-\varepsilon, p)$ or $(k/k_1) \ln(n) < -2\ln(p - X_n)$. Let $\sigma \leq \infty$ be the least $n > M$ for which $(k/k_1) \ln(n) < -2\ln(p - X_n)$. We have just shown that the conditional probability of some $X_n$ leaving $(p-\varepsilon, p)$ given $\sigma = \infty$ is one. On the other hand, the conditional probability of some $X_{n+j}$ leaving $(p-\varepsilon, p)$ given $\sigma = n < \infty$ is at least $a$ by Lemma 4 since $X_{n+j} \notin N$ trivially implies $X_{n+j} \notin (p-\varepsilon, p)$, while $Z_{n,\infty} > cn^{1/2}$ implies $Z_{n,n+j} > cn^{1/2} > n^{-k/2k_l} > p - X_n$ for some $j$, which implies $X_{n+j} > p$. $\Box$
When are touchpoints limits for generalized Pólya urns?

References


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