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ON WEAK REVERSE INTEGRAL INEQUALITIES FOR MEAN OSCILLATIONS

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ABSTRACT. We prove that if f verifies a reverse Hölder inequality with exponent p, $1 < p < +\infty$, then $(Mf + f^{\sharp})^p$ is a A_1-weight of Muckenhoupt, where Mf is the Hardy-Littlewood maximal function and $f^{\sharp}$ the Fefferman–Stein maximal function.

INTRODUCTION

Reverse integral inequalities occur in many fields of analysis such as quasiconformal mappings, weighted Sobolev imbedding theorems, and regularity of solutions of partial differential equations (see [3, 4, 7, 14] and many others).

A classical result about functions verifying reverse integral inequalities or related inequalities is extraintegrability. A prototype of these results is the famous higher integrability result of Gehring about the derivatives of the components of a quasiconformal mapping, published in 1973 [12]. See also [3, 4, 16, 9, 14].

Probably the first considered and simplest reverse integral (mean) inequality is the following [15, 1972]:

$$(A_1) \qquad \fint_Q g\,dx \le D \operatorname{ess}\inf_Q g\,,$$

where the constant $D \ge 1$ is the same for all the parallel subcubes of a fixed cube $Q_0 \subset R^n$ and g is a function that is positive in Q_0. A function verifying (A_1) is called an A_1–weight of Muckenhoupt. In [15] it is proved that any A_1–weight belongs to $L^r(Q_0)$ for $1 \le r < s$ and s depending only on n and D.

In this paper we consider weak reverse integral inequalities in a weighted case (see [4, 5, 14], e.g.), that is reverse integral inequalities with different supports. Such inequalities are much more useful in the applications to partial differential equations than the reverse integral inequalities [4, 7].

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In particular we consider inequalities like

$$\text{(WR)} \qquad \left(\fint_{\sigma Q}\left|f-\fint_{\sigma Q} f\,d\mu\right|^p d\mu\right)^{1/p} \le K \fint_Q \left|f-\fint_Q f\,d\mu\right| d\mu,$$

where $K > 1$ is independent on the parallel subcube Q of Q_0, $0 < \sigma \le 1$, $p > 1$ and μ is a doubling measure with density (see §1 for more details).

We prove that if (WR) holds, then the function $(Mf + f^\sharp)^p$ is an A_1-weight. Here Mf is the Hardy–Littlewood local maximal function and $f^\sharp$ is the Fefferman–Stein local maximal function (see [1] e.g.). From this, using some rearrangement techniques, a weighted version of the Herz theorem, [7, 13, 16], and the Muckenhoupt lemma, we deduce a higher integrability result for any f verifying condition (WR).

We emphasize that the main step in obtaining higher integrability from (WR) is in stating that the Muckenhoupt lemma [15] holds for $(Mf + f^\sharp)^p$. This approach is substantially different (see also [5, 9, 10, 16, 17] for similar cases) from many others available in literature [3, 4, 6, 10, 12, 14, 17, 19].

1. Main result

Let Q_0 be a cube of R^n whose edges are parallel to the axes. All the subcubes of Q_0 that are considered, have the edges that are parallel to the axes.

Let w be a nonnegative function that is locally summable in R^n, and for any measurable subset E of R^n consider the measure

$$\mu(E) = \int_E w(x)\,dx.$$

For any $\delta > 0$ and any cube $Q \subseteq Q_0$ we set δQ for the cube concentric with Q and such that $|\delta Q| = \delta^n |Q|$, where $|E|$ indicates the Lebesgue measure of the measurable set E.

We say that the weight function w satisfies the strong doubling condition ([3, 16] e.g.) if the inequality

$$\text{(D)} \qquad \mu(\delta Q) \le d\mu(Q)$$

holds for some positive constant d depending only on δ and n (and independent on the cube $Q \subseteq Q_0$).

For any $f \in L^1_{\mathrm{loc}}(Q_0)$ and any cube $\overline{Q} \subseteq Q_0$ we consider the local maximal function of Hardy–Littlewood:

$$Mf(x) = \sup_{\substack{x\in Q\\ Q\subset\overline{Q}}} \fint_Q |f(x)|\,d\mu, \qquad x \in \overline{Q},$$

and the local maximal function of Fefferman–Stein:

$$f^\sharp(x) = \sup_{\substack{x\in Q\\ Q\subset\overline{Q}}} \fint_Q \left|f(x) - \fint_Q f\,d\mu\right| d\mu, \qquad x \in \overline{Q},$$

where for any $h \in L^1_{\mathrm{loc}}(Q_0)$, $E \subseteq Q_0$, we set $\fint_E h(x)d\mu = (1/\mu(E)) \int_E h(x)d\mu$. We say that a nonnegative function $h \in L^1_{\mathrm{loc}}(Q_0)$ is an A_1–weight of Muckenhoupt if [15]

$$(A_1) \qquad \fint_Q h(x)\,d\mu \le D \operatorname*{ess\,inf}_Q h(x)$$

for some constant $D > 1$ independent on the cube $Q \subset Q_0$.

We prove the following:

Lemma 1.1. *Let f be a nonnegative function belonging to $L^1_{\mathrm{loc}}(Q_0)$, $0 < \sigma \le 1$. We suppose that w verifies condition* (D) and that

$$\text{(WR)} \qquad \left(\fint_{\sigma Q} \left|f - \fint_{\sigma Q} f d\mu\right|^p d\mu\right)^{1/p} \le K \fint_Q \left|f - \fint_Q f d\mu\right| d\mu,$$

where $p > 1$, with K positive constant and independent on the cube $Q \subseteq Q_0$. Then the function $(Mf + f^\sharp)^p$ is an A_1-weight of Muckenhoupt.

Proof. Fix $Q \le Q_0$ such that $Q/\sigma \subset Q_0$, $Q' \subset \sigma^2 Q$. We pick five cubes in this way to stay in Q_0. For a fixed $z \in Q'$, let $Q \subset \overline{Q}$ be such that $z \in Q$. Suppose $0 < \sigma \le \frac{1}{2}$, if $\frac{1}{2} < \sigma \le 1$ we proceed similarly. From the proof it will follow that the two cases $Q \subseteq Q'/\sigma$, $Q \nsubseteq Q'/\sigma$ must be treated separately. If $Q \subseteq Q'/\sigma$ then

$$(1.1) \qquad \begin{aligned} \fint_Q \left|f(x) - \fint_Q f d\mu\right| d\mu &= \fint_Q \left|f\chi_{Q'/\sigma} - \fint_Q f\chi_{Q'/\sigma} d\mu\right| d\mu \\ &\le (f\chi_{Q'/\sigma})^\sharp(z), \end{aligned}$$

where χ_E is the characteristic function of E. In the same way

(1.2)

$$\begin{aligned} \fint_Q f(x)d\mu &= \fint_Q (f\chi_{Q'/\sigma})(x)d\mu \\ &\le \fint_Q \left|f\chi_{Q'/\sigma} - \left(\fint_{Q'/\sigma} f\chi_{Q'/\sigma} d\mu\right)\chi_{Q'/\sigma}\right| d\mu + \fint_{Q'/\sigma} f\chi_{Q'/\sigma}(x)d\mu \\ &\le M\left(\left(f - \fint_{Q'/\sigma} f\chi_{Q'/\sigma} d\mu\right)\chi_{Q'/\sigma}\right)(z) + \operatorname{ess\,inf}_{Q'/\sigma} M(f\chi_{Q'/\sigma})(z). \end{aligned}$$

If $Q \nsubseteq Q'/\sigma$ it is easy to see that there exists a cube $\widetilde{Q}$ such that $\widetilde{Q} \supseteq Q \cup Q'$, $\widetilde{Q} \subset Q_0$, $|\widetilde{Q}| \le 3^n|Q|$. Using condition (D) the existence of a positive constant d independent on $Q \subset Q_0$ such that $\mu(\widetilde{Q}) \le d\mu(Q)$ is clear.

We then have

$$\begin{aligned} \fint_Q \left|f(x) - \fint_Q f d\mu\right| d\mu &\le d \fint_{\widetilde{Q}} \left|f\chi_Q - \fint_{\widetilde{Q}\cap Q} f\chi_Q d\mu\right| d\mu \\ &\le d \sup_{z\in C}{}' \fint_C \left|f\chi_Q - \fint_{C\cap Q} f\chi_Q d\mu\right| d\mu, \end{aligned}$$

where $\sup'$ is extended to all cubes C containing z and such that $C \cap Q$ is a cube. We deduce

$$\begin{aligned}\fint_Q \left| f(x) - \fint_Q f\,d\mu \right| d\mu &\le d \sup_{z\in C}{}' \fint_{C\cap Q} \left| f\chi_Q - \fint_{C\cap Q} f\chi_Q\,d\mu \right| d\mu \\ &\quad + d \sup_{z\in C}{}' \fint_{C\cap Q} f\chi_Q\,d\mu \\ &\le d f^\sharp(z) + d Mf(z) \\ &\le d \inf_{z\in Q} f^\sharp(z) + d \inf_{z\in Q} Mf(z),\end{aligned}$$

because $\widetilde{Q} \supset Q'$. Otherwise, if $z \notin Q$, there exists $z' \in Q$ such that

$$\fint_{C\cap Q} \left| f - \fint_{C\cap Q} f\,d\mu \right| d\mu \le f^\sharp(z'), \qquad \fint_{C\cap Q} f\,d\mu \le Mf(z')$$

and then

$$\fint_Q \left| f - \fint_Q f\,d\mu \right| d\mu \le d \inf_{z\in Q'} f^\sharp(z) + d \inf_{z\in Q'} Mf(z). \tag{1.3}$$

If $Q \not\subseteq Q'/\sigma$, obviously

$$\fint_Q f\,d\mu \le d \fint_{\widetilde{Q}} f\,d\mu \le d \inf_{z\in Q'} Mf(z). \tag{1.4}$$

Using Hardy–Littlewood maximal inequality [1] we have

$$\begin{aligned}\int_{Q'/\sigma} \left(M\left(\left(f\chi_{Q'/\sigma} - \fint_{Q'/\sigma} f\chi_{Q'/\sigma}\,d\mu \right)(x) \right) \right)^p d\mu \\ \le c_0 \int_{Q'/\sigma} \left| f\chi_{Q'/\sigma} - \fint_{Q'/\sigma} f\chi_{Q'/\sigma}\,d\mu \right|^p d\mu,\end{aligned} \tag{1.5}$$

where c_0 is a constant depending only on n and p.

From (1.1)–(1.4) we obtain

$$\begin{aligned}(f^\sharp(z) + Mf(z))^p \le c_1 \Bigg(&|(f\chi_{Q'/\sigma})^\sharp(z)|^p + d^p \inf_{z\in Q'} |Mf(z)|^p \\ &+ \left| M\left(\left(f\chi_{Q'/\sigma} - \fint_{Q'/\sigma} f\chi_{Q'/\sigma}\,d\mu \right) \chi_{Q'/\sigma} \right)(z) \right|^p \\ &+ d^p \inf_{z\in Q'} |f^\sharp(z)|^p \Bigg),\end{aligned}$$

where c_1 is a constant depending only on p. Dividing by $\mu(Q')$, integrating

over Q'/σ, and using (1.5) we obtain

$$(1.6)\quad \fint_{Q'} |f^\sharp(z) + Mf(z)|^p d\mu \le c_1 d_1 \fint_{Q'/\sigma} |f(\chi_{Q'/\sigma})^\sharp(z)|^p d\mu + c_1 d^p \inf_{z\in Q'} |Mf(z)|^p + c_1 d^p \inf_{z\in Q'} |f^\sharp(z)|^p + c_0 c_1 d_1 \fint_{Q'/\sigma} \left| f\chi_{Q'/\sigma} - \fint_{Q'/\sigma} f\chi_{Q'/\sigma} d\mu \right|^p d\mu,$$

where d_1 is the constant appearing in (D) for $\delta = 1/\sigma$.

On the other hand

$$(1.7)\quad \fint_{Q'/\sigma} |(f\chi_{Q'/\sigma})^\sharp(z)|^p d\mu$$
$$= \fint_{Q'/\sigma} \left| \left(f\chi_{Q'/\sigma} - \left(\fint_{Q'/\sigma} f\chi_{Q'/\sigma} d\mu \right) \chi_{Q'/\sigma} \right)^\sharp (z) \right|^p d\mu$$
$$\le 2 \fint_{Q'/\sigma} \left| M\left(f\chi_{Q'/\sigma} - \left(\fint_{Q'/\sigma} f\chi_{Q'/\sigma} d\mu \right) \chi_{Q'/\sigma} \right)(z) \right|^p d\mu$$
$$\le 2 \fint_{Q'/\sigma} |f\chi_{Q'/\sigma} - \fint_{Q'/\sigma} f\chi_{Q'\setminus\sigma} d\mu|^p d\mu.$$

Then from (1.6) and (1.7)

$$\fint_{Q'} |(f^\sharp + Mf)(z)|^p d\mu \le c_1 d^p \inf_{z\in Q'} |(f^\sharp + Mf)(z)|^p + c_2 \fint_{Q'/\sigma} \left| f(z) - \fint_{Q'/\sigma} f\, d\mu \right|^p d\mu,$$

where c_2 is a constant depending only on n, p, and w.

Now we use condition (WR) to obtain

$$\fint_{Q'} |(f^\sharp + Mf)(z)|^p d\mu \le c_1 d^p \inf_{z\in Q'} |(f^\sharp + Mf)(z)|^p + c_2 K \left(\fint_{Q'/\sigma^2} \left| f - \fint_{Q'/\sigma^2} f\, d\mu \right| d\mu \right)^p$$
$$\le c_1 d^p \inf_{z\in Q'} |(f^\sharp + Mf)(z)|^p + c_2 K \inf_{z\in Q'} |f^\sharp(z)|^p,$$

because $Q' \subset Q'/\sigma^2$. The lemma is proved.

We remark that the proof of Lemma 1.1 does not depend on any covering lemma.

2. Higher integrability result

In this section we deduce from Lemma 1.1 a higher integrability result for functions satisfying condition (WR).

We recall the following Muckenhoupt lemma [13, p. 213] on higher integrability from reverse mean value inequality.

Lemma 2.1. *Let $h(t)$ be a nonnegative and monotone decreasing function on the interval $0 \le x \le b$. If there exists a constant $D > 1$ such that*

$$\frac{1}{s}\int_0^s h(t)\,dt \le Dh(s)$$

for every $0 \le s \le b/20$, then

$$\int_0^b |h(t)|^r\,dt \le \frac{20^r b^{1-r} D}{D - r(D-1)}\left(\int_0^b h(t)\,dt\right)^r$$

for any $1 \le r \le D/(D-1)$.

Using Lemmas 1.1 and 2.1 we prove the following:

Theorem 2.1. *Let f be a nonnegative function belonging to $L^1_{\mathrm{loc}}(Q_0)$ and satisfying condition* (WR) *with w satisfying condition (D). Then f belongs to $L^r_{\mathrm{loc}}(Q_0)$, for any $p \le r < p + \varepsilon$, where ε is a positive constant depending on K, p, w, and n.*

Proof. Put $g(x) = (Mf + f^\sharp)^p(x)$ for any $x \in \overline{Q}$. From Lemma 1.1 we deduce that $g(x)$ is an A_1-weight. Then for any $x \in \overline{Q}$ we have

$$Mg(x) \le Dg(x) \tag{2.1}$$

with $D = cK$, c a positive constant depending on n, p, w.

We indicate by $h^*(t)$ the decreasing rearrangement of h with respect to the measure μ (see [5, 14, 15], e.g.), and we deduce from (2.1) the inequality

$$(Mg)^*(t) \le Dg^*(t)$$

for any $0 < t < \mu(\overline{Q})$. Using the weighted Herz theorem (see [5, 14] e.g.)

$$\frac{1}{t}\int_0^t g^*(s)\,ds \le C(Mg)^*(t)$$

and then

$$\frac{1}{t}\int_0^t g^*(s)\,ds \le CDg^*(t) \tag{2.2}$$

for any $0 < t < \mu(\overline{Q})$, where C depends only on n and the constant d appearing in condition (D).

From (2.2), using Lemma 2.1 we deduce

$$\int_0^{\mu(\overline{Q})} |g^*(t)|^r\,dt \le \frac{20^r \mu(\overline{Q})^{1-r} CD}{CD - r(CD-1)}\left(\int_0^{\mu(\overline{Q})} g^*(t)\,dt\right)^r$$

for any $1 \le r < CD/(CD-1)$, from which

$$\int_{\overline{Q}} (Mf + f^\sharp)^{pr}(x)\,d\mu \le \frac{20^r \mu(\overline{Q})^{1-r} CD}{CD - r(CD-1)}\left(\int_{\overline{Q}} (Mf + f^\sharp)^p(x)\,d\mu\right)^r,$$

for any $1 \leq r < CD/(CD-1)$. Then, using Hardy–Littlewood and Fefferman–Stein maximal inequalities, (see [2] e.g.), f belongs to $L^{pr}(\overline{Q})$ for $1 \leq r < CD/(CD-1)$ and the theorem is proved.

We note that condition (WR) is not immediately comparable with the reverse Hölder inequality [3, 5, 10, 12, 14, 16]

$$\left(\fint_{\sigma Q} |f|^p \, d\mu\right)^{1/p} \leq K \fint_Q |f| \, d\mu.$$

Moreover, in the significant cases $K > 1$, $0 < \varepsilon < 1$, condition (WR) and the Gurov–Reshetnyak condition [3, 9, 14]

$$\left(\fint_{\sigma Q} |f - \fint_{\sigma Q} f \, d\mu|^p \, d\mu\right)^{1/p} \leq \varepsilon \fint_{\sigma Q} |f| \, d\mu$$

are not related.

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