ON THE WEIGHTED ESTIMATE OF THE SOLUTION ASSOCIATED WITH THE SCHRODINGER EQUATION

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(Communicated by J. Marshall Ash)

Abstract. Let \( u(x, t) \) be the solution of the Schrödinger equation with initial data \( f \) in the Sobolev space \( H^{-1+a/2}(\mathbb{R}^n) \) with \( a > 1 \). This paper shows that the weighted inequality
\[
\int_{\mathbb{R}^n} \int_{\mathbb{R}} |u(x, t)|^q dt (1 + |x|)^{-a} \, dx < C \|f\|_{H^{-1+a/2}(\mathbb{R}^n)}
\]
is false. Another improved weighted inequality is proved for the general case.

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Let \( f \) belong to the Schwartz space \( \mathcal{S}(\mathbb{R}^n) \) and set
\[
(1.1) \quad u(x, t) = \int_{\mathbb{R}^n} e^{-ix \cdot \varepsilon} e^{it|\varepsilon|^2} \hat{f}(\varepsilon) \, d\varepsilon, \quad x \in \mathbb{R}^n, \; t \in \mathbb{R}.
\]
Here \( \hat{f} \) denotes the Fourier transform of \( f \), defined by
\[
\hat{f}(\varepsilon) = \int_{\mathbb{R}^n} e^{-ix \cdot \varepsilon} f(x) \, dx.
\]
It is well known that \( u(x, t) \) is the solution of the Schrödinger equation with the initial data \( f \):
\[
\Delta u = i\partial u/\partial t, \quad t > 0, \; u(x, 0) = f(x).
\]

For \( s \in \mathbb{R} \) we also introduce Sobolev spaces \( H^s(\mathbb{R}^n) \) by setting
\[
H^s(\mathbb{R}^n) = \left\{ f + \mathcal{S}'(\mathbb{R}^n) : \|f\|_{H^s(\mathbb{R}^n)} = \left( \int_{\mathbb{R}^n} (1 + |x|^2)^{s/2} |\hat{f}(x)|^2 \, dx \right)^{1/2} < \infty \right\}.
\]

In [V] the following result of maximal operator \( u^*(x) = \sup_{|t| > 0} |u(x, t)| \) was established for functions in the Sobolev space \( H^{2a-1+a/2}(\mathbb{R}^n) \).

Theorem A [V, Theorem 2]. Let \( f \) be in \( H^s(\mathbb{R}^n) \) with \( s > a/2 \) and \( a > 1 \). Then
\[
(1.2) \quad \left( \int |u^*(x)|^2 \frac{dx}{(1 + |x|)^a} \right)^{1/2} \leq C \|f\|_{H^s(\mathbb{R}^n)}.
\]
In a crucial way, the proof of this theorem uses the following classical Sobolev inequalities which states that the $H^\gamma (\mathbb{R})$ with $\gamma > 1/2$ is embedded in $L^\infty (\mathbb{R})$ and the following:

**Theorem B** [V, Theorem 3]. If $\alpha \geq 0$ and $a > 1$, then

\[
(1.3) \quad \left( \int_{\mathbb{R}^n} \int_{\mathbb{R}} |\frac{\partial^\alpha u(x, t)}{\partial t^\alpha}|^2 \, dt \frac{dx}{(1 + |x|)^\alpha} \right)^{1/2} \leq C \|f\|_{H^{2\alpha-1+\alpha/2}(\mathbb{R}^n)}.
\]

But the proof of Theorem B is slightly in error with $\alpha = 0$, thus placing the validity of Theorem B in doubt when $\alpha = 0$. The purpose of this note is to show by counterexample, that estimates (1.3) cannot be expected to hold true for $\alpha = 0$.

**Theorem 1.** The inequality in Theorem B with $\alpha = 0$, i.e.

\[
(1.4) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}} |u(x, t)|^2 \, dt \frac{dx}{(1 + |x|)^a} \leq C \|f\|^2_{H^{-1+a/2}(\mathbb{R}^n)}
\]

does not hold for some $f \in H^{-1+a/2}(\mathbb{R}^n)$. In fact, for $n \geq 2$ there exists an $f_0 \in H^{-1+a/2}(\mathbb{R}^n)$ so that

\[
(1.5) \quad \int_{\mathbb{R}^n} \int_{\mathbb{R}} |u(x, t)|^2 \, dt \frac{dx}{(1 + |x|)^a} = \infty.
\]

For $n = 1$, the corresponding inequality

\[
(1.6) \quad \left\| \left( \int_{-\infty}^{\infty} |u(x, t)|^2 \, dt \right)^{1/2} \right\|_{L^\infty(\mathbb{R})} \leq C \|f\|_{H^{-1}(\mathbb{R})}
\]

is also false with some $f \in H^{-1}(\mathbb{R})$. Indeed, there is also an $f_0 \in H^{-1}(\mathbb{R})$ so that (1.6) fails to be true.

It may be interesting to determine the source of the error in Theorem B. The proof of Theorem B makes use of the following lemma from [V]:

**Lemma.** Let $g$ be in $L^2(S^{n-1})$. Then if $a > 1$, \( a > 1 \),

\[
(1.7) \quad \left( \int_{\mathbb{R}^n} \left| \int_{S^{n-1}} g(\varepsilon) e^{ix \cdot \varepsilon} d\sigma(\varepsilon) \right|^2 \frac{dx}{(1 + |x|)^a} \right)^{1/2} \leq C \left( \int_{S^{n-1}} |g(\varepsilon)|^2 d\sigma(\varepsilon) \right)^{1/2},
\]

$S^{n-1}$ being the unit sphere in $\mathbb{R}^n$ and $d\sigma(\varepsilon)$ the Lebesgue measure on $S^{n-1}$.

This lemma was proved in [V] only for $n = 2$ which is heavily dependent on geometry. The second purpose of this note is to give a proof in the general case. In fact, the following stronger result can be established.

**Theorem 2.** Let $g$ be in $L^2(S^{n-1})$. Then if $a > 1$, we have

\[
(1.8) \quad \left( \int_{\mathbb{R}^n} \left| \int_{S^{n-1}} g(\varepsilon) e^{ix \cdot \varepsilon} d\sigma(\varepsilon) \right|^2 \frac{dx}{|x|^a} \right)^{1/2} \leq C_a \left( \int_{S^{n-1}} |g(\varepsilon)|^2 d\sigma(\varepsilon) \right)^{1/2}.
\]
Theorem 2 leads to the conclusion that the estimate from line 24 to line 25 is wrong for $\alpha = 0$ in the proof of Theorem B. (See [V, p. 875].) Thus, it would not seem proper to prove Theorem A by using (1.3) for $\alpha = 0$.

We shall first give a proof of Theorem 1 in §2. The proof of Theorem 2 is postponed to §3. The constants $C$ need not be the same at each occurrence.

2. PROOF OF THEOREM 1

First suppose that $n \geq 2$. Let $f_0(x) = f_0(|x|)$ be a radial function that belongs to $L^2(\mathbb{R}^2)$ and its Fourier transform

$$\hat{f}_0(|x|) = |x|^{-\sigma-n/2+1}(1 + |x|)^{-\beta},$$

where

$$a/2 < \sigma < 1$$

and

$$n < \beta.$$  

Then it is not difficult to verify that $f_0 \in L^2(\mathbb{R}^n)$ and $f_0 \in H^{-1+a/2}(\mathbb{R}^n)$, since

$$\int_{\mathbb{R}^n} |f_0(x)|^2 dx = \int_{\mathbb{R}^n} |\hat{f}_0(x)|^2 dx = \int_{\mathbb{R}^n} |x|^{-2\sigma-n+2}(1 + |x|)^{-2\beta} dx < +\infty$$

and

$$\int_{\mathbb{R}^n} (1 + |x|^2)^{-1+a/2} |\hat{f}_0(x)|^2 dx$$

$$= \int_{\mathbb{R}^n} (1 + |x|^2)^{-1+a/2} |x|^{-2\sigma-n+2}(1 + |x|)^{-2\beta} dx < +\infty$$

given conditions (2.1) and (2.2).

On the other hand, with a simple change of variable, by (1.1) we get the following representation of $u(x, t)$ in polar coordinates

$$u(x, t) = \frac{1}{2} \int_0^\infty e^{ist} s^{(n-2)/2} \int_{s^{-1}}^{s} \hat{f}_0(s^{1/2} e) e^{is^{1/2} x^s} d\sigma(e) ds.$$  

Using Plancherel’s theorem in the $t$ variable, it follows that

$$\int_{\mathbb{R}^n} \int_{\mathbb{R}} |u(x, t)|^2 dt \frac{dx}{(1 + |x|)^a}$$

$$= \frac{1}{4} \int_{\mathbb{R}^n} \int_0^\infty s^{(n-2)/2} \int_{s^{-1}}^{s} \hat{f}_0(s^{1/2} e) e^{is^{1/2} x^s} d\sigma(e) \left| \int_{s^{-1}}^{s} \hat{f}_0(s^{1/2} e) e^{is^{1/2} x^s} d\sigma(e) \right|^2 ds \frac{dx}{(1 + |x|)^a}$$

$$= \frac{1}{4} \int_{\mathbb{R}^n} \int_0^\infty s^{(n-2)/2} \int_{s^{-1}}^{s} e^{is^{1/2} x^s} d\sigma(e) \left| \int_{s^{-1}}^{s} e^{is^{1/2} x^s} d\sigma(e) \right|^2 ds \frac{dx}{(1 + |x|)^a}$$

$$= \frac{(2\pi)^n}{4} \int_{\mathbb{R}^n} \int_0^\infty s^{n-2} |\hat{f}_0(s^{1/2})|^2 \left| \frac{f_{(n-2)/2}(s^{1/2}|x|)}{|x|^{n-2}} \right|^2 ds \frac{dx}{(1 + |x|)^a}.$$
In the last equality we used the fact that [SW, p. 154]

$$\int_{S^{n-1}} e^{ix\varepsilon} d\sigma(\varepsilon) = (2\pi)^{n/2} J_{(n-2)/2}(\|x\|)\|x\|^{(n-2)/2},$$

where $J_k(x)$ is Bessel’s function of order $k$. Using the expression for $f_0, we get from (2.3) that

$$\int_{S^{n-1}} |u(x, t)|^2 dt \int_{S^{n-1}} |x|^a$$

$$= c_n \int_0^{\infty} s^{n/2-2} |\hat{f}_0(s^{1/2})|^2 \int_0^{\infty} |J_{(n-2)/2}(\gamma)|^2 \frac{\gamma d\gamma}{(1 + \gamma/\sqrt{s})^a} ds$$

$$\geq c_n \int_1^{\infty} s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2 \int_1^{\infty} |J_{(n-2)/2}(\gamma)|^2 \gamma^{-(a-1)} d\gamma ds$$

$$= c_n \int_1^{\infty} |J_{(n-2)/2}(\gamma)|^2 \gamma^{-(a-1)} d\gamma \cdot \int_0^{1} s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2, ds = +\infty,$$

since the last integral

$$\int_0^{1} s^{n/2-2+a/2} |\hat{f}_0(s^{1/2})|^2 ds = \int_0^{1} s^{-1+a/2-\sigma} (1 + s^{1/2})^{-2\beta} ds = +\infty$$
given condition (2.1). Thus the proof of the first part of Theorem 1 is complete.

For $n = 1$ the corresponding inequality (1.6) is false with the following function $f_0 \in H^1(\mathbb{R})$:

$$\hat{f}_0(x) = |x|^{-\sigma} (1 + |x|)^{-\beta} \quad (0 < \sigma < 1/2, \, \sigma + \beta > 1/2)$$
can also be easily verified. We omit the detail.

3. Proof of Theorem 2

Let us develop $g(\varepsilon) \in L^2(S^{n-1})$ into a series of spherical harmonics

$$g(\varepsilon) \sim \sum_{k=0}^{\infty} a_k Y_k(\varepsilon) \quad (\varepsilon \in S^{n-1}),$$

where $Y_k(\varepsilon)$ is a spherical function of order $k$, i.e., the value on $S^{n-1}$ of a homogeneous polynomial $P(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ satisfying Laplace’s equation $\Delta P = 0$. We may always normalize the $Y_k(\varepsilon)$ and assume that

$$\|Y_k\| = \left(\int_{S^{n-1}} |Y_k(\varepsilon)|^2 d\sigma(\varepsilon)/|S^{n-1}|\right)^{1/2} = 1,$$

$|S^{n-1}|$ being the Lebesgue measure of $S^{n-1}$. Thus the functions $Y_k(\varepsilon)$ form an orthonormal system on $S^{n-1}$ and Bessel’s inequality gives

$$\left(\sum_{k=0}^{\infty} |a_k|^2\right)^{1/2} \leq \left(\int_{S^{n-1}} |g(\varepsilon)|^2 d\sigma(\varepsilon)\right)^{1/2}.$$
Our first step will be to replace the function \( g(\varepsilon) \) in (1.8) by the development (3.1) and prove the equation

\[
\int_{S^{n-1}} g(\varepsilon) e^{ix \cdot \varepsilon} d\sigma(\varepsilon) = \sum_{k=0}^{\infty} a_k \int_{S^{n-1}} Y_k(\varepsilon) \cdot e^{ix \cdot \varepsilon} d\sigma(\varepsilon).
\]

Since \( g \in L^2(S^{n-1}) \) and the development (3.1) converges to \( g \) over \( S^{n-1} \), (3.3) follows in norm \( L^2 \) by Schwarz' inequality.

Now we invoke the formulas \([AH, p. 572]\)

\[
\int_{S^{n-1}} Y_k(\varepsilon) e^{ix \cdot \varepsilon} d\sigma(\varepsilon) = (2\pi)^{\frac{n+1}{2}} i^k Y_k(x') \cdot J_{k+\lambda}(|x|)/|x|^\lambda \quad (\lambda = (n - 2)/2)
\]

and (3.3), so we get the equality as follows

\[
\left| \int_{S^{n-1}} g(\varepsilon) e^{ix \cdot \varepsilon} d\sigma(\varepsilon) \right|^2 = c_k \sum_{k, l=0}^{\infty} (-1)^l i^{k+l} a_k \overline{a_l} Y_k(x') \overline{Y_l(x')} \cdot J_{k+\lambda}(|x|)/|x|^{2\lambda},
\]

where \( x' = x/|x| \in S^{n-1} \). Thus, the left-hand side of (1.8) is equal to

\[
c_n \int_0^\infty \int_{S^{n-1}} \sum_{k, l=0}^{\infty} (-1)^l i^{k+l} a_k \overline{a_l} Y_k(x') \overline{Y_l(x')} d\sigma(x') J_{k+\lambda}(|x|) J_{l+\lambda}(|x|) |x|^{-2\lambda} \ d\gamma
\]

The last integral was evaluated in \([L, 13.4.2(3)]:\)

\[
\int_0^\infty t^{-d} J_\mu(bt) J_\nu(bt) \ dt = \frac{(b/2)^{d-1} \Gamma(\lambda) \Gamma(\frac{\mu+\nu-d+1}{2})}{2\Gamma(\frac{\nu-\mu+d+1}{2}) \Gamma\left(\frac{\nu+\mu+d+1}{2}\right) \Gamma\left(\frac{\mu-\nu+d+1}{2}\right)}
\]

under the conditions

\[
\text{Re}(\mu + \nu + 1) > \text{Re}(d) > 0, \quad b > 0.
\]

In our case the last integral in (3.4) is

\[
\int_0^\infty |J_{k+\lambda}(\gamma)|^2 \gamma^{1-a} d\gamma = \frac{1}{2^{a-1}} \frac{\Gamma(\frac{a}{2})}{\Gamma(\frac{a+\lambda}{2})} \frac{\Gamma(k+\frac{a}{2} - \frac{\lambda}{2})}{\Gamma(k+\frac{a}{2} - 1 + \frac{\lambda}{2})} \leq c/(k+(n-a)/2)^{a-1} \leq c/k^{a-1} \quad (k = 1, 2, \ldots).
\]

Then Theorem 2 follows from (3.2) to (3.5).
REFERENCES


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