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HERMITIAN SURFACES AND A TWISTOR SPACE OF ALGEBRAIC DIMENSION 2

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ABSTRACT. We study anti-self-dual hermitian surfaces with odd first Betti number. We show that the twistor space of a Hopf surface M has algebraic dimension 2, and we prove existence of half-conformally-flat metrics on the connected sum of copies of M and $\mathbb{CP}_2$. Finally we emphasize some differences with the case b_1 even.

1. INTRODUCTION

In this work we continue our study of complex surfaces M with anti-self-dual hermitian metric h and twistor space Z, by considering the case in which M is compact and $b_1(M)$ is odd. For topological reasons then, M cannot admit a Kähler metric, however, it was shown in [B₁] and [P₂] that the universal cover $\widetilde{M}$ has a Kähler metric of zero scalar curvature which is globally conformal to the lifted metric $\tilde{h}$. That is, (M, h) is locally conformal to a Kähler metric of zero scalar curvature, in particular (M, h) is *l.c.k.* in the notation of I. Vaisman.

Examples. All known examples of compact anti-self-dual hermitian surfaces (*a.s.d.h.s.*) of non-Kähler type (i.e. $b_1(M)$ odd), are the following:

(1) A Hopf surface M with its standard conformally flat metric. This is the quotient of $\mathbb{C}^2_* = \mathbb{C}^2 \setminus \{0\}$ by the group of conformal isometries generated by the map $(z_1, z_2) \mapsto (az_1, bz_2)$ where a and b are complex numbers satisfying $|a| = |b| \neq 0, 1$. M is diffeomorphic to $S^1 \times S^3$ and the metric $\tilde{h} = (dz \otimes d\overline{z}) / \|z\|^2$ of $\mathbb{C}^2$ descends to a conformally flat hermitian metric on M.

(2) Recently LeBrun [L] has explicitly constructed anti-self-dual hermitian metrics on the blow up of a Hopf surface at any number of points lying on a divisor and on some complex deformations of these surfaces. These metrics are not conformally flat because the signature of the manifold is nonzero.

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If we now look at the twistor fibration $t : Z \to M$ over M, we have that Z is a compact complex 3-manifold, furthermore (see also [P_2]) the integrable almost complex structures J and $-J$ on M, define an effective divisor $X = \Sigma + \overline{\Sigma}$ of Z. If $\sigma : Z \to Z$ denotes the real structure of Z, then $\sigma(X) = \sigma(\Sigma+\overline{\Sigma}) = \overline{\Sigma}+\Sigma = X$. So that X is σ-invariant, and the associated holomorphic line bundle $[X]$ is called a real bundle.

As proved in [B_1], if (M, h) is a compact a.s.d.h.s. with odd first Betti number, there is a hermitian metric in the conformal class of h with scalar curvature R which is strictly positive almost everywhere. As the twistor construction and all the other properties of M that we will consider are conformally invariant, we will assume from now on that h has positive scalar curvature.

We proved in [P_2] that when M is compact and the first Betti number of M is odd, $[X] \cong K^{-1/2} \otimes F$. Here K denotes the canonical line bundle of Z, whose sections are the holomorphic 3-forms, and we recall that it always admits a holomorphic square root $K^{1/2}$. On the other hand F is a holomorphic line bundle on Z with zero Chern class but never trivial. More precisely, we can describe F in the following way:

Proposition 1.1. *The holomorphic line bundle F is the pull-back via t of a complex line bundle E on M with a flat connection which however is never hermitian.*

Proof. The first part of the statement is an immediate consequence of the so-called Ward correspondence [AHS, Theorem 5.2] which says that the reality of F implies that $F = t^*E$ where E is a complex line bundle on M with an anti-self-dual connection D. However, by the naturality of Chern classes $0 = c_1(F) = t^*c_1(E)$; but $t^* : H^*(M, \mathbb{R}) \to H^*(Z, \mathbb{R})$ is injective by the Leray-Hirsch theorem, and so $c_1(E) = 0$ as well. It follows that the de Rham class of the curvature α of D is 0, but then $\alpha = 0$ by Hodge theory, because being closed and anti-self-dual α is also harmonic.

It remains to see that D is not a hermitian connection. This is proved by contradiction: suppose it is and consider the following exact sequence of sheaves on Z given by restricting to X the sheaf of holomorphic functions $\mathscr{O}_Z$:

$$0 \to K^{1/2}F^{-1} \to \mathscr{O}_Z \to \mathscr{O}_X \to 0 .$$

As the scalar curvature of h is assumed to be positive, when D is hermitian one has $H^1(Z, K^{1/2}F^{-1}) = 0$ by a vanishing theorem of Hitchin [H_1, Corollary 3.8]. This leads to a contradiction because the restriction map $H^0(Z, \mathscr{O}_Z) \to H^0(X, \mathscr{O}_X)$ cannot be onto since X has twice as many connected components as Z. □

Notice that the above can only happen because M is non-Kähler.

Using the same techniques as before, we have:

Proposition 1.2. *The complex line bundle E on M is holomorphic and does not admit nonzero global holomorphic sections.*

Proof. Since $F = t^*E$ and $t_{|_\Sigma} : \Sigma \to M$ is a biholomorphism, we get $E \cong F_{|_\Sigma}$ showing that E is holomorphic; looking at the exact sequence given by restriction $0 \to K^{1/2} \to F \to F_{|_X} \to 0$ we have that $H^0(Z, F) = 0$ by [Pn_1, Lemma 2.1], while $H^1(Z, K^{\frac{1}{2}}) = 0$ by the above vanishing theorem of Hitchin. It follows that $H^0(X, F_{|_X}) = 0$ and therefore $H^0(M, E) = 0$. □

Finally one can also find E by knowing the normal bundle ν_Σ of Σ in Z:

Proposition 1.3. $E^2 \cong \nu_\Sigma \otimes K_\Sigma$, *where* K_Σ *is the canonical bundle of* Σ.

Proof. A direct consequence of the adjunction formulas: since $[X] \cong K_Z^{-1/2} \otimes F$ one has $\nu_X \cong (K_Z^{-1/2})_{|_X} \otimes F_{|_X}$. On the other hand $K_X \cong (K_Z \otimes [X])_{|_X} \cong (K_Z^{1/2})_{|_X} \otimes F_{|_X}$ so that $\nu_X \otimes K_X \cong F^2_{|_X}$. The assertion then follows because $X = \Sigma \amalg \overline{\Sigma}$. □

Remark 1.4. As we will show later, the normal bundle ν_Σ is trivial when M is the Hopf surface. It follows from the proposition then, that $\nu_\Sigma \cong \mathscr{O}_\Sigma$ if and only if M is the Hopf surface. In this case indeed the canonical bundle of M would have a square root and therefore M would be spin; this forces M to be a Hopf surface as shown in [B_1].

Another consequence along the same lines is that the Hopf surface is the only hyperhermitian compact surface which is not hyper-Kähler, a fact also proved in [B_2]. This is because when M is hyperhermitian the natural map $p_1 : Z \to \mathbb{CP}_1$ is holomorphic with Σ as a fiber, so the normal bundle must again be trivial.

Let now $\tau(M) = b_2^+(M) - b_2^-(M)$ be the signature of M, where $b_2^\pm$ denotes the dimension of $\{\omega \in \Lambda^2_\pm(M) \;:\; d\omega = 0\}$. A topological consequence of the fact that the scalar curvature R of M is positive is:

Proposition 1.5. *Any compact a.s.d.h.s* M *with odd first Betti number satisfies*:

(1) $b_1(M) = 1$.

(2) $c_1^2 = -\chi = \tau = -b_2^- \leq 0$ *and equality holds if and only if* M *is a Hopf surface.*

Proof. This was essentially proved in [B_1], because when $R > 0$, all plurigenera of M vanish and therefore $b_1 = 1$ by the Enriques-Kodaira classification. To prove the first half of the second part of the proposition, we have to show that $b_2^+(M) = 0$. But this holds for any compact a.s.d. 4-manifold of positive scalar curvature by [H_1, Corollary 3.8] again.

Finally, suppose that $b_2^- = 0$ also, then the metric h has to be conformally flat and this forces M to be a Hopf surface as shown in [B_1]. □

Remark 1.6. As in any l.c.k. manifold, there is a naturally defined flat line bundle L on M, which is never holomorphically trivial because h is not globally conformally Kähler [B_1]. Now since flat line bundles are given by representations of the fundamental group into $\mathbb{R}^+$, it follows from $b_1(M) = 1$

that L and E are defined by two representations which are multiples of each other.

2. A twistor space of algebraic dimension two and half-conformally flat metrics

Consider the complement of the origin in the complex plane $\mathbb{C}^2_*$ with coordinates $z = (z_1, z_2)$ and hermitian metric $\tilde{h} = ||z||^{-2}(dz_1 \otimes d\bar{z}_1 + dz_2 \otimes d\bar{z}_2)$. Fix a complex number λ with $|\lambda| \neq 0, 1$, the infinite cyclic group of holomorphic isometries $\Lambda := \{z \mapsto \lambda^n z : n \in \mathbb{Z}\} \subset GL(2, \mathbb{C})$ acts properly discontinuosly and without fixed points on $\mathbb{C}^2_*$. We can then consider the quotient manifold $H_\lambda := \mathbb{C}^2_*/\Lambda$ with the induced metric h. H_λ is a complex surface homeomorphic to $S^1 \times S^3$, called a Hopf surface. As $(\mathbb{C}^2_*, \tilde{h})$ and (H_λ, h) are both conformally flat, let W and Z be their respective twistor spaces.

To describe W, we think of $\mathbb{C}^2_*$ as $(S^4 \setminus \{0, \infty\})$. Since $\tilde{h}$ is also conformally flat, their twistor spaces coincide, and W is the open set $(\mathbb{CP}_3 \setminus \{L_0 \cup L_\infty\})$; where L_0 and L_∞ are the twistor lines above 0 and ∞ in S^4.

As Λ acts on $\mathbb{HP}_1 = S^4$ by conformal isometries: $q \mapsto \lambda^n q$, it also acts on $\mathbb{CP}_3$ by biholomorphisms, see [P_1]:

$$[Z_0, Z_1, Z_2, Z_3] \mapsto [Z_0, Z_1, \lambda^{-n} Z_2, \bar{\lambda}^{-n} Z_3]. \tag{2.1}$$

Λ acts freely on W, and $Z = W/\Lambda$ is the twistor space of H_λ.

Proposition 2.2. *If λ is real, Z admits a holomorphic fibration $p: Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$.*

Proof. Consider the holomorphic map $\tilde{p} : (\mathbb{CP}_3 \setminus \{L_0 \cup L_\infty\}) \to \mathbb{CP}_1 \times \mathbb{CP}_1$ given by $[Z_0, Z_1, Z_2, Z_3] \mapsto ([Z_0, Z_1], [Z_2, Z_3])$; is clear that $\tilde{p}$ commutes with the action of Λ, and so descends to $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$. Notice then that $\tilde{p}$ is a regular map, with fiber $\tilde{p}^{-1}([Z_0, Z_1], [Z_2, Z_3]) = [aZ_0, aZ_1, bZ_2, bZ_3] = [(a/b)Z_0, (a/b)Z_1, Z_2, Z_3] \cong (\mathbb{C} \setminus 0)$, also denoted by $\mathbb{C}_*$, where $a, b \in \mathbb{C}_*$. Passing to the quotient by Λ, we get the compact holomorphic fiber bundle $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$ with fiber the compact Riemann surface of genus 1, given by $\mathbb{C}_* / (z \sim \lambda z)$. □

We now want to analyze the fibration p more closely: let us denote by F_1 and F_2 the two factors in the base space $\mathbb{CP}_1 \times \mathbb{CP}_1$, and start by considering the fibration $\tilde{p} : W \to F_1 \times F_2$. Let then $\pi_i : F_1 \times F_2 \to F_i$ be the canonical projections and $\tilde{p}_i$ the compositions $\pi_i \circ \tilde{p}$, $i = 1, 2$. As $u \in F_1$ varies, $\tilde{p}_1^{-1}(u)$ describes the set of all hyperplanes in $\mathbb{CP}_3$ passing through the line L_∞, and we notice that each of these hyperplanes meets the line L_0 in exactly one point. It follows that $\tilde{p}_1^{-1}(u) \cong \mathbb{C}^2_*$ for each $u \in F_1$. Similarly $\tilde{p}_2^{-1}(v) \cong \mathbb{C}^2_*$. It is also useful to notice that, as every twistor line L meets any hyperplane through L_0 or L_∞ in exactly one point, we have that $\tilde{p}_1^{-1}(u) \cap L = \{1\text{pt.}\}$ and $\tilde{p}_2^{-1}(v) \cap L = \{1\text{pt.}\}$, for each twistor line L , $u \in F_1$, $v \in F_2$.

Passing to the quotient by the action of Λ we consider now the fibration $p : Z \to F_1 \times F_2$. Let $p_i = \pi_i \circ p$, $i = 1, 2$; we look at the fibers: if $u \in F_1$ we set $S_u := p_1^{-1}(u) \cong \mathbb{C}_*^2/\,(z \sim \lambda^{-1} z)$ and if $v \in F_2$, $S^v := p_2^{-1}(v) \cong \mathbb{C}_*^2/\,(z \sim \lambda z)$. From the previous discussion we then get the following:

Proposition 2.3. *When λ is real, the twistor space Z is foliated by two families of Hopf surfaces $\{S_u\}_{u \in \mathbb{CP}_1}$ and $\{S^v\}_{v \in \mathbb{CP}_1}$, each leaf being biholomorphic to H_λ. Furthermore every two leaves S_u and S^v intersect in the elliptic curve $p^{-1}(u, v)$, and each twistor line L intersects any of S_u and S^v in exactly one point.*

Notice that when λ is an arbitrary complex number, the map (2.1) commutes with the antiholomorphic involution

$$\sigma : [Z_0, Z_1, Z_2, Z_3] \mapsto [-\overline{Z}_1, \overline{Z}_0, -\overline{Z}_3, \overline{Z}_2]$$

of $\mathbb{CP}_3$, and therefore Z inherits the real structure of the twistor space $\mathbb{CP}_3$.

Furthermore the map $p_1 : Z \to \mathbb{CP}_1$ is still well defined for $\lambda \in \mathbb{C}$. This is just the fact proved in [B_2] that the twistor space of a hyperhermitian surface fibers holomorphically over $\mathbb{CP}_1$.

It is also clear from the above description that Z is diffeomorphic to the product of spheres $S^1 \times S^3 \times S^2$.

If V is an arbitrary complex manifold, we will denote by Θ_V the sheaf of sections of its holomorphic tangent bundle. In what follows we let M be the Hopf surface H_λ with λ *real* and Z be its twistor space, using the fibration we described before we now compute the cohomology of Θ_Z and derive some consequences.

It is well known that

$$H^i(M, \Theta_M) \cong \begin{cases} \mathbb{C}^4 & \text{for } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

In fact this result can be shown using the same techniques we use in the proof of the following:

Proposition 2.4.

$$H^i(Z, \Theta_Z) \cong \begin{cases} \mathbb{C}^7 & \textit{for } i = 0, 1, \\ 0 & \textit{otherwise.} \end{cases}$$

Proof. From the fiber bundle map $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$, we get an exact sequence

$$0 \to \mathscr{O}_Z \to \Theta_Z \to p^* \Theta_{\mathbb{CP}_1 \times \mathbb{CP}_1} \to 0 \tag{2.5}$$

because the never zero holomorphic tangent vector field

$$a Z_0 \frac{\partial}{\partial Z_0} + a Z_1 \frac{\partial}{\partial Z_1} + b Z_2 \frac{\partial}{\partial Z_2} + b Z_3 \frac{\partial}{\partial Z_3}$$

belongs to $\operatorname{Ker} p_*$, for any $a, b \in \mathbb{C}$, with $(a, b) \neq 0$.

In order to use (2.5) we now look at the direct image sheaves and we have:

$$p_{*i}\mathscr{O}_Z \cong \begin{cases} \mathscr{O}_{\mathbb{CP}_1\times\mathbb{CP}_1} & \text{for } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Recalling that the fiber E over p is a connected elliptic curve, this is clear when $i \neq 1$. On the other hand, for any small Stein open subset $U \subset \mathbb{CP}_1 \times \mathbb{CP}_1$, we have

$$\begin{aligned}(p_{*1}\mathscr{O}_Z)(U) &:= H^1(p^{-1}(U), \mathscr{O}_Z) = H^1(U \times E, \mathscr{O}_Z) \\ &\cong (H^0(U, \mathscr{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}) \otimes H^1(E, \mathscr{O}_E)) \oplus (H^1(U, \mathscr{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}) \otimes H^0(E, \mathscr{O}_E)) \\ &= H^0(U, \mathscr{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}) \otimes H^1(E, \mathscr{O}_E)\end{aligned}$$

by applying Serre duality on the fiber,

$$\begin{aligned}&\cong \mathscr{O}_U \otimes (H^0(E, \Omega^1_E))^* = (\mathscr{O}_U \otimes (H^0(E, \mathscr{O}_E))^* \\ &:= \mathscr{O}^*_{\mathbb{CP}_1\times\mathbb{CP}_1}(U) \cong \mathscr{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(U).\end{aligned}$$

The projection formula then allows us to immediately get the other direct image sheaves:

$$p_{*i}(p^*\Theta_{\mathbb{CP}_1\times\mathbb{CP}_1}) \cong \Theta_{\mathbb{CP}_1\times\mathbb{CP}_1} \otimes p_{*i}\mathscr{O}_Z \cong \begin{cases} \Theta_{\mathbb{CP}_1\times\mathbb{CP}_1} & \text{for } i = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

Using these results the final ingredient we need to conclude the proof is that, by a theorem of Leray, there are two spectral sequences

$$E_\infty^{p,q} \Rightarrow H^{p+q}(Z, \mathscr{O}_Z) \quad \text{and} \quad F_\infty^{p,q} \Rightarrow H^{p+q}(Z, p^*\Theta_{\mathbb{CP}_1\times\mathbb{CP}_1})$$

with

$$E_2^{p,q} = H^p(\mathbb{CP}_1 \times \mathbb{CP}_1, p_{*q}\mathscr{O}_Z) = \begin{cases} \mathbb{C} & \text{if } p = 0 \text{ and } q = 0, 1, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$F_2^{p,q} = H^p(\mathbb{CP}_1 \times \mathbb{CP}_1, p_{*q}\Theta_Z) = \begin{cases} \mathbb{C}^6 & \text{if } p = 0 \text{ and } q = 0, 1, \\ 0 & \text{otherwise.} \end{cases}$$

This says that $E_2^{p,q} = E_\infty^{p,q}$ and $F_2^{p,q} = F_\infty^{p,q}$. The result finally follows by using the exact sequence (2.5) and noticing that the global holomorphic vector fields on Z are:

$$H^0(Z, \Theta_Z) \cong \left\{ \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix} : A, \in M(2, \mathbb{C}) \right\} \Big/ kI \cong \mathbb{C}^7. \quad \square$$

The algebraic dimension $a(V)$ of a complex manifold V is defined to be the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions on V, minus 1. So that, for example, $a(V) = 0$ means that the only meromorphic functions on V are the constants. When V is compact, one has the following theorem of Moishézon:

$a(V) \leq dim_{\mathbb{C}} V$ with equality if and only if V can be blown up to an algebraic manifold.

When the equality is satisfied, V is called a Moishézon space and by the theorem it has Hodge symmetries: $h^p(V, \Omega^q) = h^q(V, \Omega^p)$.

As a corollary of the last proof we then have:

Theorem 2.6. *The algebraic dimension of the twistor space Z of the Hopf surface M considered above, is equal to 2.*

Proof. The holomorphic fibration $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$, shows that $a(Z) \geq 2$. On the other hand $a(Z) \leq 2$ because in the course of the previous proof we have shown that $h^1(Z, \mathcal{O}) = 1$, while all global holomorphic 1-forms are zero on any twistor space [H_2]. This shows that Z does not admit Hodge symmetries and therefore cannot be Moishézon. □

Notice that in fact $h^1(Z, \mathcal{O}) = b_1(M) = 1$ for any compact a.s.d.h.s. M of non-Kähler type, so that by the results of [Pn_3] the twistor space of a compact *a.s.d.h.s.* can never be Moishézon.

To conclude this section let $A\#B$ indicate the connected sum of the two manifolds A and B. As a result of some new techniques of S. Donaldson and R. Friedman, another corollary of the last proposition is the following:

Theorem 2.7. *For any natural numbers p and q the manifold*

$$\left(\mathop{\#}_{i=1}^{p} M\right)\left(\mathop{\#}_{j=1}^{q} \mathbb{CP}_2\right)$$

admits self-dual metrics, while $(\#_{i=1}^p M)(\#_{j=1}^q \overline{\mathbb{CP}}_2)$ admits anti-self-dual metrics.

Proof. This is a direct consequence of [DF, Theorem 6.1 and Proposition 6.2]. In fact we have shown that $H^2(Z, \Theta) = 0$ for the Hopf surface, while it is well known that the twistor space of $\mathbb{CP}_2$ is the flag manifold $F_{1,2} \subset \mathbb{CP}_2 \times \mathbb{CP}_2$ and that $H^2(F_{1,2}, \Theta) = 0$. □

The fact that $(S^1 \times S^3)\#\mathbb{CP}_2$ admits self-dual metrics has also been proved in [DF] and [F] using different methods.

3. Structural differences

We start this section by noticing that the twistor space of the Hopf surface above, is the first example of a compact twistor space with algebraic dimension 2, [Pn_2, Pn_3, DF]. Furthermore it is the twistor space of a hermitian anti-self-dual surface. In this section we want to consider some holomorphic properties of this example which we think are useful to understand the difference in character between compact a.s.d.h.s. of Kähler and non-Kähler type.

To do this we first introduce some notation: N will denote a compact a.s.d.h.s. with *even* first Betti number and twistor space W, while $Y \subset W$

will denote the natural real divisor defined by the complex structure of N. We will compare the holomorphic properties of N and W with the ones of the Hopf surface M and its twistor space Z.

Let then $p : Z \to \mathbb{CP}_1 \times \mathbb{CP}_1$ be the holomorphic map described in §2 and recall that all holomorphic line bundles on $\mathbb{CP}_1 \times \mathbb{CP}_1$ are of the form $\mathcal{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(m, n)$, for some $m, n \in \mathbb{Z}$; where $\mathcal{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(m, 0)$ and $\mathcal{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(0, m)$ denote the pull-back of the holomorphic line bundle of Chern class m on the first and second $\mathbb{CP}_1$-factor, respectively. By $\mathcal{O}_Z(m, n)$ we will then denote the pull-back bundle $p^*(\mathcal{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(m, n))$.

Using this notation and the previous discussion one has, for example, that $[X] = \mathcal{O}_Z(2, 0)$ while $K_Z^{-1/2} = \mathcal{O}_Z(1, 1)$ and $F = \mathcal{O}_Z(1, -1)$. In fact one can also check that the Chern class of $\mathcal{O}_Z(1, -1)$ vanishes by intersecting with any twistor line.

Furthermore the normal bundle $\nu_\Sigma \cong (\mathcal{O}_Z(2, 0))_{|_\Sigma}$ is trivial, and $E := (\mathcal{O}_Z(1, -1))_{|_\Sigma} \cong K_M^{1/2}$. As it is easy to see that the complex line bundle $\mathbb{C} \otimes L$ of §1 is isomorphic to K_M when M is the Hopf surface, we also have $L \cong E^2$ in this case.

In general, given a holomorphic line bundle $G \to V$ over a compact complex manifold V, one can produce meromorphic functions on V by taking the quotient of two holomorphic sections of G. Then one can set the Kodaira dimension $k(V, G)$ of G to be $-\infty$ if no positive tensor power of G admits global holomorphic sections; otherwise $k(V, G) + 1$ equals the transcendence degree over $\mathbb{C}$ of the field of meromorphic functions generated by taking quotients of holomorphic sections of positive powers of G. For example $k(V, G) = 0$ if and only if the linear system $m|G|$ consists of exactly one divisor for some m positive, and is trivial otherwise. It is also clear that $k(V, G) \leq a(V)$. Finally, when K denotes the canonical line bundle of V, $k(V, K)$ is called the *Kodaira dimension* of V and is a bimeromorphic invariant.

Remark 3.1. When $b_1(M)$ is even, it was shown in [P_2] and [Pn_3] that

$$k(W, [Y]) = k(W, K^{-1}) = a(W) \leq 1 .$$

For the Hopf surface M instead,

$$k(Z, [X]) = 1 \quad \text{while } k(Z, K^{-1}) = a(Z) = 2 .$$

Proof. $k(Z, [X]) = k(Z, \mathcal{O}_Z(2, 0)) = k(\mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1}(2)) = 1$. While $k(Z, K^{-1}) = k(Z, \mathcal{O}_Z(2, 2) = k(\mathbb{CP}_1 \times \mathbb{CP}_1, \mathcal{O}_{\mathbb{CP}_1\times\mathbb{CP}_1}(2, 2)) = 2$. $\square$

Remark 3.2. The projection $p_1 : Z \to \mathbb{CP}_1$ on the first factor, comes from the hyperhermitian structure of the Hopf surface [B_2]. The same is true for the twistor space of a hyper-Kähler surface, i.e. a torus or a $K3$ surface; in the hyper-Kähler case however, the canonical bundle K of the twistor space is the the pull back of $\mathcal{O}_{\mathbb{CP}_1}(-4)$; we show this is not valid in the hyperhermitian case:

$$K_Z \cong \mathcal{O}_Z(-2, -2) \not\cong \mathcal{O}_Z(-4, 0) \cong p_1^*(\mathcal{O}_{\mathbb{CP}_1}(-4)).$$

Finally we want to spend a word on deformations.

Remark 3.3. When (N, h) is an a.s.d.h.s. with $b_1(N)$ even, the metric h can be assumed to be Kähler with zero scalar curvature, in particular h is an extremal Kähler metric in the sense of Calabi. It was shown in [C] then that any small deformation N_t of the complex structure of N admits such a Kähler metric, i.e. an a.s.d.h. metric h_t.

For the Hopf surface the situation is different. To show this we consider complex structures on $S^1 \times S^3$. These are gotten by taking the quotient of $\mathbb{C}^2_*$ by an infinite cyclic group Γ of biholomorphisms acting freely. Let $H = \mathbb{C}^2_* / \Gamma$ be such a complex surface and consider the question of finding an a.s.d.h. metric on H. Of course, since the signature $\tau(H) = 0$, any such metric has to be conformally flat and therefore its universal covering is $\mathbb{C}^2_*$ with the standard conformal structure. It easily follows from this then, that H is an a.s.d.h.s. if and only if Γ is generated by an element $\gamma : (z, w) \mapsto (az, bw)$ where $a, b \in \mathbb{C}$ satisfy $0 < |a| = |b| < 1$. This says that the moduli space of such metrics on $S^1 \times S^3$ is $S^1 \times S^1 \times (0, 1)$; but it also says that if we take a deformation $M_t = \mathbb{C}^2_* / \langle\gamma_t\rangle$ of the standard Hopf surface $M = M_0 = \mathbb{C}^2_* / \langle\gamma_0\rangle$, where $\gamma_0(z, w) = (az, aw)$, which is given by $\gamma_t = (az + tw, aw)$; the only element with a *a.s.d.h.* metric will be M_0. When $t \neq 0$ one has a complex structure on $S^1 \times S^3$ which does not admit any a.s.d.h. metric.

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