GEODESICS AND BOUNDED HARMONIC FUNCTIONS
ON INFINITE PLANAR GRAPHS

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Abstract. It is shown there that an infinite connected planar graph with a uniform upper bound on vertex degree and rapidly decreasing Green's function (relative to the simple random walk) has infinitely many pairwise finitely-intersecting geodesic rays starting at each vertex. We then demonstrate the existence of nonconstant bounded harmonic functions on the graph.

Let $g$ be an infinite, simple, connected, planar graph. $g$ also denotes the vertex set of the graph. If two vertices $x$ and $y$ are connected by an edge, we write $xEy$. For a vertex $x$, the degree of $x$ is $d(x) \equiv |\{y \in g: yEx\}|$, and we assume:

$$(1) \quad \delta \equiv \sup_{x \in g} d(x) < \infty.$$ 

A finite [infinite] walk $\gamma$ is a sequence $(\gamma(0), \ldots, \gamma(n))$ [(\gamma(0), \gamma(1), \ldots)] of elements of $g$ such that $\gamma(k)E\gamma(k + 1)$ for all $0 \leq k \leq n - 1$ [for all $k \geq 0$]. We say that $\gamma$ starts at $\gamma(0)$ and, in the first case, ends at $\gamma(n)$ and has length $n$. Since $g$ is connected, we may define a metric:

$$d(x, y) \equiv \inf\{n: n is the length of a finite walk from x to y\}.$$ 

A path is a walk whose vertices are distinct. A geodesic $\gamma$ is a path such that $d(\gamma(m), \gamma(n)) = |m - n|$ for all possible $m$ and $n$. For $x \in g$, $\Gamma(x, n)$ is the set of geodesics that have length $n$ and start at $x$; $\Gamma(x)$ is the set of geodesics that have infinite length and start at $x$.

The following propositions are useful; the first is easy to prove by a diagonal type argument.

**Proposition 1.** For all $x \in g$, $\Gamma(x) \neq \emptyset$.

**Proposition 2.** Given $x, y \in g$ and $\gamma \in \Gamma(x)$, there exists a $\gamma' \in \Gamma(y)$ such that $\gamma$ and $\gamma'$ eventually coincide.

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Proof. Let \( x, y \in g, \gamma \in \Gamma(x) \). By the triangle inequality, \(|d(y, \gamma(n)) - n| = |d(y, \gamma(n)) - d(x, \gamma(n))| \leq d(x, y)\) and, since \( d(y, \gamma(n)) - n \) is nonincreasing, \( a = \lim_{n \to \infty} [d(y, \gamma(n)) - n] = d(y, \gamma(N)) - N \) for some \( N \). Define a path \( \gamma' \) where \((\gamma'(0), \ldots, \gamma'(d(y, \gamma(N))))\) is a finite geodesic from \( y \) to \( \gamma(N) \) and, for \( k \geq d(y, \gamma(N)) \), \( \gamma'(k) = \gamma(k - a) \). Then \( \gamma' \in \Gamma(y) \). \( \square \)

Consider the transition probabilities for a Markov chain defined by:

\[
p(x, y) \equiv \begin{cases} 1/d(x) & \text{if } y \in x, \\ 0 & \text{otherwise.} \end{cases}
\]

We denote this chain by \( X(0), X(1), \ldots \). We let \( P^x(\cdot) \equiv P(\cdot | X(0) = x) \) and \( E^x(\cdot) \) be the associated expectation operator. Hence, \( p(x, y) = P(X(1) = y | X(0) = x) = P^x(X(1) = y) \). \( X(\cdot) \) is called the simple random walk on \( g \).

Let \( p^{(n)}(x, y) \) be the \( n \)-fold convolution of \( p \) with itself, and define Green's function as \( G(x, y) = \sum_{n \geq 0} p^{(n)}(x, y) \). Probabilistically, \( p^{(n)}(x, y) = P^x(X(n) = y) \) and \( G(x, y) = E^x(\sum_{n \geq 0} \chi_{\{y\}}(X(n))) \) is the average number of times that the random walk, starting at \( x \), hits \( y \). It is easy to see that the random walk is transient if and only if \( G \) exists (see [2]; his proof for the case when \( g \) is a tree applies to our case without change). By the strong Markov property,

\[
(2) \quad G(x, y) = P^x(\exists n \geq 0: X(n) = y)G(y, y).
\]

We assume that Green's function is rapidly decreasing in the sense that

\[
(3) \quad \sum_{n \geq 0} n \cdot \sup\{G(x, y): x, y \in g, d(x, y) = n\} < \infty.
\]

Remark. It is known that the Cheeger condition

\[
\exists c > 0: \forall \text{ finite } K \subset g: \#\{\text{edges from } K \to K^c\}/|K| \geq c
\]

implies \( G(x, y) \leq ce^{d(x, y)} \) (for some \( c \) and \( \varepsilon \))—see [1] or [4]. Hence the Cheeger condition implies condition (3).

Lemma 1. For any integer \( m \geq 0 \), there is an \( N(m) \geq 0 \) such that if \( A \) is the union of \( m \) geodesics and \( d(x, A) \geq N(m) \), then \( P^x(\exists n: X(n) \in A) < 1 \).

Proof. For any \( n \geq 0, x \in g \), let \( S(x, n) \) and \( B(x, n) \) be the metric sphere and ball respectively with centers \( x \) and radii \( n \). If \( \gamma \) is a geodesic, then \( |\gamma \cap S(x, n)| \leq |\gamma \cap B(x, n)| \leq 2n + 1 \). Hence \( |A \cap S(x, n)| \leq (2n + 1)m \) and
we get
\[ P^x(\exists n \geq 0: X(n) \in A) \leq \sum_{y \in A} P^x(\exists n \geq 0: X(n) = y) \]
\[ = \sum_{y \in A} \frac{G(x, y)}{G(y, y)} \quad \text{(by (2))} \]
\[ \leq \sum_{y \in A} G(x, y) \quad \text{(since } G(y, y) \geq 1) \]
\[ \leq \sum_{n \geq d(x, A)} |A \cap S(x, n)| \cdot \sup\{G(x, y): d(x, y) = n\} \]
\[ \leq m \sum_{n \geq d(x, A)} (2n + 1) \cdot \sup\{G(x, y): d(x, y) = n\}. \]

By (3), choose \( N(m) \) so that \( m \sum_{n \geq N(m)} (2n + 1) \cdot \sup\{G(x, y): d(x, y) = n\} < 1. \]

**Lemma 2.** For any \( K \subseteq \mathcal{G}, \) if \( \inf_{x \in \mathcal{G}} P^x(\limsup_{n \to \infty} (X(n) \in K)) < 1, \) then \( \sup_{x \in \mathcal{G}} d(x, K) = \infty. \)

**Proof.** By condition (1), for any \( y \in K \) and \( x \in \mathcal{G}, \)
\[ P^x(\exists n: X(n) \in K) \geq P^x(X(d(x, y)) = y) \geq \left( \frac{1}{\delta^2} \right)^{d(x, y)}. \]
Thus, if \( \sup_{x \in \mathcal{G}} d(x, K) < \infty, \) then \( \inf_{x \in \mathcal{G}} P^x(\exists n: X(n) \in K) > 0 \) and, therefore, \( \inf_{x \in \mathcal{G}} P^x(\limsup_{n \to \infty} (X(n) \in K)) = 1. \)

**Theorem 1.** For any \( x \in \mathcal{G}, \) there are infinitely many geodesic rays \( \gamma_1, \gamma_2, \ldots \) starting at \( x \) such that if \( i \neq j, \) then \( |\gamma_i \cap \gamma_j| < \infty. \)

**Proof.** We construct such a family inductively. There is always one geodesic ray starting at \( x \) (Proposition 1). Suppose \( \gamma_1, \gamma_2, \ldots, \gamma_m \in \Gamma(x) \) such that if \( i \neq j, \) then \( |\gamma_i \cap \gamma_j| < \infty. \) Let \( \partial A = \bigcup_{i=1}^{m} \gamma_i. \) By Proposition 2, it is enough to show that there exists a geodesic ray \( \gamma \) such that \( \gamma \cap \partial A = \emptyset. \) Therefore, by the diagonal method of Proposition 1, it is enough to show that there exists \( z \in \mathcal{G} \) such that for all \( k, \) there exists \( \gamma_k \in \Gamma(z, k) \) so that \( \gamma_k \cap \partial A = \emptyset. \)

Let \( A = \mathcal{G} \setminus \partial A \) and \( N = N(m + 2) \) where \( N(\cdot) \) is as in Lemma 1. As in the proof of Lemma 1, \( \sum_{y \in \partial A} G(x, y) < \infty \) and so
\[ P^x \left( \limsup_{k \to \infty} (X(k) \in \partial A) \right) = 0. \]

By Lemma 2, we can choose \( z \in A \) such that \( d(z, \partial A) \geq N. \)

Suppose that there exists \( n \) such that for all \( y \in \Gamma(z, n), \) \( y \cap \partial A \neq \emptyset. \) We show that this leads to a contradiction—we show that this implies the existence of two geodesic segments \( \gamma^*_l \) and \( \gamma^*_u \) such that:
(a) \( d(z, \gamma^*_l \cup \gamma^*_u) \geq N \) and
(b) every infinite path starting at \( z \) hits \( \gamma^*_l \cup \gamma^*_u \cup \partial A. \)

By Lemma 1, condition (a) implies \( P^z(\exists j: X(j) \in \gamma^*_l \cup \gamma^*_u \cup \partial A) < 1 \) whereas condition (b) implies \( P^z(\exists j: X(j) \in \gamma^*_l \cup \gamma^*_u \cup \partial A) = 1. \)
For each \( y \in S(z, n) \), choose \( \gamma_y \in \Gamma(z, n) \). In addition, we choose these geodesics so that \( \bigcup \gamma_y \) is a tree. For any \( y \in S(z, n) \), let \( \gamma^*_y = (\gamma_y(\eta), \ldots, \gamma_y(n)) \) where \( \eta = \max\{j \leq n : \gamma_y(j) \in \partial A\} \). Note that for any \( t, u \in S(z, n) \), condition (a) holds. Let \( Z = \{y \in S(z, n) : \text{there exists an infinite path in } A, \text{ starting at } z, \text{ which last hits } B(z, n) \text{ at } y\} \). \( Z \) is nonempty by choice of \( N \) and \( z \). For \( Y \subset Z \), let \( C(Y) \) be the connected component of \( B(z, n) \setminus (\partial A \cup \bigcup_{y \in Y} \gamma^*_y) \) which contains \( z \).

We claim that \( C(Z) = C(\{t, u\}) \) for some \( t, u \in Z \). If so, then condition (b) holds for \( t \) and \( u \). To prove this claim, it is enough to show that if \( t, u, v \) are distinct elements of \( Z \), then \( C(\{t, u, v\}) = C(\{t', u'\}) \) for some \( t', u' \in \{t, u, v\} \).

Let \( t, u, v \) be distinct elements of \( Z \), and let \( \rho, \sigma, \tau \) be infinite paths in \( A \) starting at \( z \) which last hit \( B(z, n) \) at \( t, u, v \) respectively. Since \( \partial A \) is connected, \( \partial A \) is in one of the components of \( G \setminus (\rho \cup \sigma \cup \tau) \). By planarity, without loss of generality, any path from \( t \) to \( \partial A \) must hit \( \sigma \cup \tau \). Define \( \rho^*(j) = \rho(j + M + 1) \) where \( M = \max\{k : \rho(k) \in B(z, n)\} \). Then the complement of \( \partial A \cup \gamma^*_t \cup \rho^* \) contains two components, say \( B \) and \( C \), such that \( u \in B \), \( v \in C \), and, without loss of generality, \( z \in B \). Then, any path contained in \( A \) from \( z \) to \( v \) must hit either \( \gamma^*_t \) or \( \rho^* \). Since \( \rho^* \cap B(z, n) = \emptyset \), \( C(\{t, u, v\}) = C(\{t', u'\}) \). \( \Box \)

A function \( f : g \to \mathbb{R} \) is harmonic if and only if \( \sum_{y \in E(x)} f(y) = d(x) \cdot f(x) \) for all \( x \). In particular, since \( \liminf_{k \to \infty} (X(k) \in A) \) is invariant under the Markov shift, \( f(x) \equiv \mathbb{P}^x(\liminf_{k \to \infty} (X(k) \in A)) = \mathbb{P}_x f(x) \) and so \( f \) is bounded and harmonic. We use an idea similar to one Kendall uses in the case of Brownian motion on manifolds [3] to find a set \( A \) so that \( \mathbb{P}^x(\liminf_{k \to \infty} (X(k) \in A)) \) is nonconstant.

**Theorem 2.** There are nonconstant, bounded, harmonic functions on \( g \).

**Proof.** Let \( N = N(2) \) where \( N(\cdot) \) is as in Lemma 1. Fix \( x \in g \) and, by Theorem 1, choose \( 4N \) rays \( \gamma_1, \gamma_2, \ldots, \gamma_{4N} \in \Gamma(x) \) whose pairwise intersections are finite. Without loss of generality, these geodesics are numbered in a clockwise fashion (we may do this since \( g \) is planar). Let \( M \) be such that \( i \neq j \) implies \( (\gamma_i \cap \gamma_j) \setminus B(x, M) = \emptyset \). Let \( C = \gamma_1 \cup \gamma_{2N}, u = \gamma_N(M + N), v = \gamma_{3N}(M + N), \) and \( A \) and \( B \) be the connected components of \( g \setminus C \) containing \( u \) and \( v \) respectively. By Lemma 1, since \( d(u, C) \geq N \) and \( d(v, C) \geq N \),

\[
P^u \left( \liminf_{k \to \infty} (X(k) \in A) \right) \geq P^u(\forall j : X(j) \notin C) > 0
\]

and

\[
P^v \left( \limsup_{k \to \infty} (X(k) \in A) \right) \leq P^v(\exists j : X(j) \in C) < 1.
\]

By Lemma 2,

\[
\sup_{w \in g} d(w, A) = \infty.
\]
Since, for \( w \in B \),
\[
P^{w} \left( \liminf_{k \to \infty} (X(k) \in A) \right) \leq P^{w} (\exists j: X(j) \in C)
\]
\[
\leq 2c \sum_{n \geq d(w, A)} (2n + 1) \cdot \sup \{G(x, y): d(x, y) = n\}
\]
(as in the proof of Lemma 1), and since \( d(w, A) \) is unbounded,
\[
\inf_w P^w \left( \liminf_{n \to \infty} (X(n) \in A) \right) = 0
\]
and so is not constant. \( \square \)

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**References**


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