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QUOTIENTS OF SOLUTIONS OF LINEAR ALGEBRAIC DIFFERENTIAL EQUATIONS

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ABSTRACT. An example is given of two entire functions f, g that satisfy linear differential equations with polynomial coefficients, whose quotient $h = f/g$ is entire, yet h satisfies no such differential equation.

On the same day, Alfred van der Poorten and Alexander Eremenko independently asked me the same question that stems from the theorem of Ritt that if f and g are two simple exponential polynomials whose quotient h is entire, then h must also be a simple exponential polynomial, which means that $h(x) = \sum_{k=1}^{n} a_k e^{\lambda_k x}$, where a_k and λ_k are complex constants. They asked me whether the result remains true if "simple exponential polynomial" is replaced throughout by "entire solution of a linear differential equation with polynomial coefficients." Our theorem shows that the answer is no, using a "cousin" of a function shown to me by Walter Hayman for another purpose.

Theorem. *Let*

$$f(x) = e^{2\pi i x^2} - 1, \tag{1}$$

$$g(x) = e^{2\pi i x} - 1. \tag{2}$$

Then $h(x) = f(x)/g(x)$ is entire, but does not satisfy any linear differential equation with polynomial coefficients, even though f and g do.

Remark. The equations that f and g satisfy can be chosen to be *monic*, a condition that Eremenko asked for, in addition. This is trivial for g. As for f, observe that

$$f'(x) = (4\pi i x)e^{2\pi i x^2}, \tag{3}$$

$$f''(x) = (-8\pi^2 x^2 + 4\pi i)e^{2\pi i x^2}, \tag{4}$$

$$f'''(x) = -(16\pi^3 i x^3 + 24\pi^2 x)e^{2\pi i x^2}. \tag{5}$$

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Now it is evident that

$$f'''(x) + axf''(x) + bf'(x) = 0 \tag{6}$$

for suitable constants a and b.

Proof of the theorem. First, h is entire because the denominator vanishes exactly at the integers, and the numerator vanishes at each integer. The idea behind the rest of the proof is that if $h(x)$ were to satisfy a linear differential equation with polynomial coefficients, then so would

$$\tilde{h}(x) = (e^{2iex^2} - 1)/(e^{2iex} - 1), \tag{7}$$

which is obtained by replacing π in h by e. But $\tilde{h}(x)$ has infinitely many poles, since most of the zeros of the denominator are not cancelled by zeros of the numerator. But a solution of a linear differential equation with polynomial coefficients can have singularities in the finite complex plane only at the zeros of the leading coefficient.

Now for the details. Suppose

$$p_n(x)h^{(n)}(x) + p_{n-1}(x)h^{(n-1)}(x) + \cdots + p_0(x)h(x) = 0, \tag{8}$$

say as formal power series, where the p_j are polynomials. Let φ be an automorphism of $\mathbb{C}$, with $\varphi(\pi) = e$, and $\varphi(i) = i$, so that φ fixes $\mathbb{Q}(i)$, the Gaussian rationals. Note that the $h^{(j)}(x)$ are rational functions of π, x, $e^{2\pi ix}$, $e^{2\pi ix^2}$ with coefficients in $\mathbb{Q}(i)$. Extend the action of φ to $\mathbb{C}((x))$, the field generated by the formal power series in x, via

$$\varphi\left(\sum a_n x^n\right) = \sum \varphi(a_n)x^n.$$

It is easy to see that $\varphi(x) = x$, $\varphi(\pi) = e$, $\varphi(e^{2\pi ix}) = e^{2eix}$, $\varphi(e^{2\pi ix^2}) = e^{2eix^2}$, and that φ commutes with differentiation. Therefore, applying φ to (8), we would get

$$\tilde{p}_n(x)\tilde{h}^{(n)}(x) + \tilde{p}_{n-1}(x)\tilde{h}^{(n-1)}(x) + \cdots + \tilde{p}_0(x)\tilde{h}(x) = 0, \tag{9}$$

where

$$\tilde{p}_k(x) = \varphi(p_k(x)), \qquad k = 0, 1, \ldots, n. \tag{10}$$

As explained above, (9) is impossible. Hence (8) is impossible and the theorem is proved.

References

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