Abstract. Let $H$ be a Hamiltonian on a four-dimensional symplectic manifold. Suppose the system is completely integrable and on some nonsingular compact level surface $Q$ the integral is such that the connected components of the set of critical points form submanifolds. Then we prove that the topological entropy of the system restricted to $Q$ is zero. As a corollary we deduce the nonexistence of completely integrable geodesic flows by means of integrals as described above for compact surfaces with negative Euler characteristic.

1. Introduction

Let $M^4$ be a four-dimensional symplectic manifold and $H$ a Hamiltonian on $M$. Denote by $\text{sgrad} H$ the symplectic gradient of $H$. Let $Q$ be a nonsingular compact level surface of $H$. Suppose the system is completely integrable; that is, suppose there exists an additional function on $M$ which is independent of $H$ (almost everywhere) and is in involution with $H$ (such a function is called an integral). Restricting this integral to $Q$ gives a smooth function $f$. Denote by $h_{\text{top}}$ the topological entropy of the Hamiltonian flow restricted to $Q$. In this article we want to prove the following:

Theorem 1. Let $M^4$ be a smooth symplectic manifold and let $\text{sgrad} H$ be a Hamiltonian field on $M^4$. Suppose that the system is completely integrable and that on some nonsingular compact level surface $Q$ the integral $f$ verifies either of the following conditions:

(a) $f$ is real analytic.
(b) The connected components of the set of critical points of $f$ form submanifolds.

Then $h_{\text{top}} = 0$.

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Corollary 1. Let $N^2$ be a compact connected surface. Assume $N^2$ supports a geodesic flow that is completely integrable by means of an integral like the one in Theorem 1. Then $\chi(N^2) \geq 0$.

Remark 1. Corollary 1 was proved by Kozlov [K] in the case of an analytic integral by completely different methods. If we assume condition (b), the integral could even be of class $C^1$.

Observe that the class of functions considered in (b) includes the Bott integrals studied by Fomenko in [F].

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2. Proofs

We first state a result of Katok that we will use:

Theorem 2 ([Ka, Corollary (4.3)]). If $g$ is a $C^{1+\alpha}$ ($\alpha > 0$) diffeomorphism of a compact two-dimensional manifold and $h_{\text{top}}(g) > 0$, then $g$ has a hyperbolic periodic point with a transversal homoclinic point, and consequently there exists a $g$-invariant hyperbolic set $\Lambda$ such that the restriction of $g$ to $\Lambda$ is topologically conjugate to a subshift of finite type.

We note that Theorem 2 extends to flows without singularities on 3-manifolds. Theorem 1 follows from Theorem 2 and the following lemma.

Lemma 1. Under the hypothesis of Theorem 1 there are no transversal homoclinic orbits.

Proof. If the function is real analytic, Lemma 1 was proved by Moser in [M]. Therefore assume that condition (b) is verified. Denote by $\text{Crit}(f)$ the set of critical points of $f$. Since $f$ is an integral, the flow of $s\text{grad} H$ leaves $\text{Crit}(f)$ invariant. Condition (b) says that $\text{Crit}(f)$ is a disjoint union of circles and compact connected surfaces. These surfaces are tori and Klein bottles because $s\text{grad} H$ is never zero.

Suppose now that there is a transversal homoclinic orbit. Then we have the analogue for flows of the hyperbolic set $\Lambda$ in Theorem 2. We will also call if $\Lambda$ (for the properties of shifts and suspended horseshoes, we refer to [S]). We claim that there exists a surface $X^2$ in $\text{Crit}(f)$ such that $\Lambda \subset X^2$. To prove this, observe first that since $f$ is an integral it follows that if $\gamma$ is a hyperbolic closed orbit, then $\gamma \subset \text{Crit}(f)$. Otherwise the symplectic gradient of $f$ would generate a nonzero eigenvector with eigenvalue 1 for the Poincaré map of $\gamma$. This idea can be traced back to Poincaré (see [K] for details). But the hyperbolic closed orbits in $\Lambda$ are dense, and $\text{Crit}(f)$ is a closed set; therefore $\Lambda \subset \text{Crit}(f)$. Moreover, since the flow on $\Lambda$ is transitive (i.e., there is a dense orbit), we deduce the claim.

We now consider the flow of $s\text{grad} H$ restricted to $X^2$. Since $s\text{grad} H$ is never zero, for any closed orbit $\gamma$, $X^2 - \gamma$ is a cylinder or a Möbius band. By a Poincaré-Bendixson argument (see [PM, p. 34, Exercises 4 and 5]) we deduce
that sgrad $H$ has no nontrivial recurrent orbits, i.e., if $\omega(\gamma)$ denotes the limit set of the orbit $\gamma$ and $\gamma \subset \omega(\gamma)$, then $\omega(\gamma)$ is a closed orbit. But this is absurd because dense orbits in $\Lambda$ have nontrivial recurrence. The lemma is proved.

Remark 2. Note that the proof of Lemma 1 still works if we allow the surfaces to have boundaries. We also note that Moser in [M] proves that $\Lambda \subset \text{Crit}(f)$ with a different argument that needs only that $f$ be of class $C^1$.

Proof of the corollary. In [D], Dinaburg proved that if $\pi_1(N^2)$ has exponential growth, then $h_{\text{top}} > 0$. Hence, from Theorem 1, we get that $\pi_1(N^2)$ cannot have exponential growth, and therefore $\chi(N^2) \geq 0$.

References


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