NONREALIZABILITY OF SUBALGEBRAS OF $\mathfrak{A}^*$

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Abstract. At the prime two, the dual of the Steenrod algebra is a polynomial algebra in generators $\xi_n$, $n \geq 1$. The Eilenberg-Mac Lane spectrum $K(\mathbb{Z}_2)$ has homology $Z_2[\xi_n | n \geq 1]$, the Brown-Peterson spectrum $BP$ has homology $Z_2[\xi_n^2 | n \geq 1]$, and the symplectic Thom spectrum $MSp$ has homology $Z_2[\xi_n^4 | n \geq 1] \otimes \Xi$. In this paper, we show that there is no spectrum $B_k$ with $H_*B_k = Z_2[\xi_n^k | n \geq 1]$ for $k \geq 2$.

In this paper, all spectra are localized at the prime two, and all coefficients are $\mathbb{Z}_2$. All our spectra $E$ have units $\mu: S \to E$, and all our ring spectra have a homotopy unit, are homotopy associative, and are homotopy commutative. Let $\mathfrak{A}^* = Z_2[\xi_n | n \geq 1]$ denote the dual of the Steenrod algebra. Recall [2, 7] that as algebras and $\mathfrak{A}^*$-comodules, $H_*KZ_2 = Z_2[\xi_n | n \geq 1]$, $H_*BP = Z_2[\xi_n^2 | n \geq 1]$, and $H_*MSp = Z_2[\xi_n^4 | n \geq 1] \otimes \Xi$, where $\Xi = Z_2[V_m | m \neq 2^q - 1]$. The $V_m$ are $\mathfrak{A}^*$-primitive elements of degree $4m$. In this paper we show that for $k \geq 2$ there is no ring spectrum $B_k$ such that $H_*B_k = Z_2[\xi_n^{2k} | n \geq 1]$ as algebras and $\mathfrak{A}^*$-comodules. For $k \geq 4$, we prove the stronger result that there is no spectrum $B_k$ such that $H_*B_k = Z_2[\xi_n^{2k} | n \geq 1] \otimes \Xi_k$ as $\mathfrak{A}^*$-comodules, where $\Xi_k$ is a set of $\mathfrak{A}^*$-primitive elements. Of course, $MSp$ is an example of a $B_2$. We cannot determine whether any $B_3$ exist. If spectra of the type $B_k$, $k \geq 3$, had existed, they would have defined generalized Adams spectral sequences which would have been efficient methods for computing $\pi_*S$. (For example, see [6] for a description of the $MSp$-Adams-Novikov spectral sequence for $\pi_*S$.)

Assume that $B_k$ exists with $k \geq 2$. Consider the Adams spectral sequence:

$$E_2^{n,I} = \text{Ext}_*^I(H^*B_k, Z_2)_n \Rightarrow \pi_*B_k.$$  

Note that since $B_k$ may not be a ring spectrum, the Adams spectral sequence (A) may not have a multiplicative structure. However, $H_*B_k = Z_2[\xi_n^k | n \geq 1]$ is

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a sub-Hopf algebra of \( \mathfrak{A}^* \). Let \( T(k) \) denote the truncated polynomial algebra \( Z_2[\xi_n|n \geq 1]/(\xi_n^{2^k}|n \geq 1) \). By the change of rings theorem of A. Liulevicius [7]
\[
\text{Ext}_{\mathfrak{A}}(H^*B_k, Z_2) \cong \text{Cotor}_{\mathfrak{A}^*}(H_*B_k, Z_2)
\]
\[
= \text{Cotor}_{\mathfrak{A}^*}(Z_2[\xi_n^{2^k}|n \geq 1], Z_2) \cong \text{Cotor}_{T(k)}(Z_2, Z_2).
\]
We use the May spectral sequence [8] to study \( \text{Cotor}_{T(k)}(Z_2, Z_2) \):
\[
E^0T(k) \text{ is the associated graded algebra of } T(k) \text{ induced by the coproduct filtration. Let } \mathfrak{e}^*T(k) \text{ denote the cobar construction of } T(k). \text{ The following lemma describes a DGA algebra } M\mathfrak{E}_1 \text{ whose homology is } M\mathfrak{E}_2.
\]
**Lemma 1.** Let \( h_{nj} = [\xi_n^{2^j}] \in \mathfrak{e}^*T(k) = \mathfrak{e}^{2^n+2^j-2^j-1}T(k) \). Then
\[
M\mathfrak{E}_1 = Z_2[h_{nj}|n \geq 1 \text{ and } k > j > 0].
\]
Moreover, \( d_1(h_{nj}) = \sum_{t=1}^{k-j-1} h_{n-t,j+t}h_{tj} \) for \( 0 \leq j < k-1 \), and \( d_1(h_{n,k-1}) = 0 \).
\[\text{Proof.} \]
\[
E(S), P(S), \Gamma(S) \text{ denote the exterior algebra, polynomial algebra, and divided polynomial algebra, respectively, on the set } S, \text{ and let } V(\mathfrak{L}) \text{ denote the universal enveloping algebra of the restricted Lie algebra } \mathfrak{L} \text{ [10]. Let } \xi_{nj} = \{\xi_n^{2^j}\} \text{ in } E^0T(k). \text{ Then } E^0T(k) = E(\xi_{nj}|n \geq 1, k > j \geq 0) \text{ with } \tilde{\Delta}(\xi_{nj}) = \sum_{t=1}^{k-j-1} \xi_{n-t,j+t} \otimes \xi_{tj}. \text{ Thus, } (E^0T(k))^* = V(\mathfrak{L}), \text{ where } \mathfrak{L} \text{ is the restricted Lie algebra with } Z_2\text{-basis } \{\xi_n^*|n \geq 1, k > j \geq 0\}, \text{ zero restriction, and Lie bracket } [\xi_{mi}^*, \xi_{nj}^*] \text{ equal to } \xi_{m+n,i}^*, \xi_{m+n,j}^*, 0 \text{ if } m+i = j, n+j = i, m+i \neq j \text{ and } n+j \neq i, \text{ respectively. By } [9, \text{Remark 10}], \text{ there is a differential on the } Z_2\text{-coalgebra } X = \Gamma(s\mathfrak{L}) \otimes V(\mathfrak{L}) \text{ making } X \text{ a free } (E^0T(k))^*\text{-resolution of } Z_2. \text{ Thus,}
\[
M\mathfrak{E}_2 = H_*(\text{Hom}_{V(\mathfrak{L})}(X, Z_2))
\]
\[
\cong H_*(\text{Hom}_{Z_2}(\Gamma(s\mathfrak{L}), Z_2)) \cong H_*(Z_2[h_{nj}|n \geq 1, k > j \geq 0]),
\]
where \( h_{nj} = (s\xi_n^*)^* \text{ is represented by } [\xi_n^{2^j}] \text{ in the cobar construction. Clearly } d_1, \text{ being induced by } \tilde{\Delta}(\xi_{nj}), \text{ is given by } d_1(h_{nj}) = \sum_{t=1}^{k-j-1} h_{n-t,j+t}h_{tj} \text{ for } 0 \leq j < k-1. \text{ The situation is analogous to that of } [8, \text{Chapter 2, } \S3] \text{ and } [5, \text{ }\S1]. \quad \square
\]
**Lemma 2.** For \( k \geq 4 \), there is a nonzero element \( h_0h_{k-1}^2 \in E_2^{2^k-2,3}. \)
\[\text{Proof.} \]
\[
M\mathfrak{E}_1 = Z_2{h_{20}h_{k-1,1}, h_{11}h_0}, \text{ and } d_1(h_{20}h_{k-1,1}), d_1(h_{11}h_0) \text{ contains } h_{20}h_{21}, h_{k-3,3}, h_{11}h_{30}, h_{k-3,3}, \text{ respectively, as a nonzero summand because } k \geq 4. \text{ Thus,}
\]
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$M^{E_2^{k-1,2}}_{\infty} = 0$, $h_0^2h_{k-1}$ is nonzero in $M^{E_{\infty}^{k-2,3}}_{\infty}$ and defines a nonzero element $h_0^2h_{k-1}$ of $E_2^{k-2,3}$. □

**Theorem 3.** Spectra $B_k$ and $B_k$ do not exist for $k \geq 4$.

**Proof.** Consider the unit map $\mu: S \to B_k$. Let the $h_k$ be represented by the $h_{1,k}$ in the May spectral sequences. By Adams [1], $d_2(h_k) = h_0^2h_{k-1}$ for $k \geq 4$ in the Adams spectral sequence for $\pi_*$:

$$E_2^{n,1} = \text{Ext}^1_\mathbb{Z}(\mathbb{Z}_2, \mathbb{Z}_2) \otimes \mathbb{Z}_2^{\infty}.$$

$\mu$ induces a map of spectral sequences $\mu_\ast$ between the Adams spectral sequences (C) and (A). Observe that $\mu_2$ is induced by the map of algebras $\mu_\ast: \mathbb{Z}_2 \to H^*B_k = \mathbb{Z}_2[e_m|m \geq 1]$, and thus $\mu_2$ is an algebra homomorphism. Clearly $\mu_2(h_i) = h_i$ for $0 \leq i < k - 1$, and $\mu_2(h_k) = 0$. Thus, $h_0^2h_{k-1}$ must be zero in $E_2$ of the Adams spectral sequence (A), contradicting Lemma 2. Therefore, $B_k$ cannot exist for $k \geq 4$. The $E_2$-term of the Adams spectral sequence for $\pi_*B_k$ equals $\text{Cotor}_T(Z_2, Z_2) \otimes \mathfrak{g}_k$. Thus, the above argument, with the Adams spectral sequence for $\pi_*B_k$ replacing the Adams spectral sequence (A), shows that $B_k$ cannot exist for $k \geq 4$. □

**Theorem 4.** A ring spectrum $B_2$ does not exist.

**Proof.** The $E_2$-term of the Adams spectral sequence (A) is $\text{Cotor}_T(Z_2, Z_2)$, which is computed in [5, §3], where $T(2)$ is called $B$. In the notation of [5, §3], $P(0, 1) \in E_2^{7,2}$ is a nonzero infinite cycle because $E_2^{8,0} = 0$. Let $\rho \in \pi_*B_3$ project to $P(0, 1)$. In $E_2^{14,3}$, $h_0^2P(0, 1)^2 = \Phi_0^2\Phi_1P(0, 1) \neq 0$. Note $E_2^{15,0} = E_2^{15,1} = 0$, $E_2^{15,2} = Z_2(P(0, 2))$, and $E_2^{15,3} = Z_2(h_0P(0, 2))$. Observe that $P(0, 2) \in \langle \Phi_0, h_0, \Phi_2 \rangle$ in $E_2$ by [5, Theorem 4.3(1)]. Now $\Phi_0$ and $\Phi_2$ are infinite cycles because there are no elements whose product with $h_0$ is 0 in degrees 0 and 12. $\langle \Phi_0, 2, \Phi_2 \rangle$ is defined because $\Phi_0, \Phi_2$ is the only nonzero element of degree one, thirteen, respectively. By [4, Theorem 8.1], $P(0, 2)$ is an infinite cycle. Thus, $h_0P(0, 1)^2$ is nonzero in $E_2^{14,5}$ and $2\rho^2 \neq 0$, contradicting the fact that $\pi_*B_2$ is a commutative graded ring. Therefore, $B_2$ cannot exist. □

Recall that $H_*\text{MSp} = \mathbb{Z}_2[e_n|n \geq 1] \otimes \mathfrak{g}$, where $\mathfrak{g}$ is a polynomial algebra of $\mathfrak{g}^*$-primitive elements.

**Lemma 5.** There is a map of spectra $f: B_3^{(24)} \to \text{MSp}$ such that

(a) $f|B_3^{(0)} = \mu: S \to \text{MSp}$;
(b) $f_\ast(\xi) \equiv \xi$ modulo the ideal spanned by $\mathfrak{g}$ for all $\xi \in H_*B_3$ with $n < 24$.

**Proof.** (a) We construct $f$ on $B_3^{(q)}$ by induction on $q \geq 0$. Let $f|B_3^{(0)} = \mu$. Note that $H_*B_3$ is nonzero only in degrees divisible by eight. Assume that $f$ has been defined on the $8(t - 1)$-skeleton of $B_3$, $1 \leq t \leq 3$. By [3, Lemma
VI.3.2], the obstruction to extending \( f \) to the \( 8t \)-skeleton of \( B_3 \) is an element of \( H^*(B_3; \pi_{8t-1} M\text{Sp}) \). However, the first nonzero element of \( \pi_r M\text{Sp} \) in a degree congruent to 3 mod 4 occurs when \( r = 31 \). Thus, we can extend \( f \) to the 24-skeleton of \( B_3 \).

(b) If \( \xi \in H^n B_3, n \leq 23 \), then \( (1 \otimes \epsilon)\psi f_\ast(\xi) = (1 \otimes \epsilon)(1 \otimes f_\ast)\psi(\xi) = (1 \otimes \epsilon)\psi(\xi) = \xi \otimes 1 \). Kernel \( (1 \otimes \epsilon)\psi \) equals the ideal spanned by the \( \mathfrak{A}^* \)-primitive elements of positive degree of \( H_\ast M\text{Sp} = \mathbb{Z}_2[\xi_n^r | n \geq 1] \otimes \mathfrak{S} \), which is \( \overline{\mathfrak{S}} \). Thus, \( f_\ast(\xi) \equiv \xi \) modulo the ideal spanned by \( \overline{\mathfrak{S}} \).

**Theorem 6.** A ring spectrum \( B_3 \) does not exist.

**Proof.** Observe that \( ME_1^{11,1} = Z_2(h_{22}) = ME_\infty^{11,1} \) because \( ME_1^{12,0} = 0 \). Thus, \( E_1^{11,1} = Z_2(R) \). A straightforward calculation shows that \( ME_2^{10,k} = 0 \) for \( k \geq 3, k \neq 6 \), and that \( ME_2^{10,6} = Z_2(h_{11}^4 h_{20}) \). Thus, the only possibility for a nonzero differential on \( R \) is \( d_5(R) = h_1^2 Q \), where \( h_1, Q \) is represented by \( h_{11}, h_{20} \), respectively, in the May spectral sequence. Let \( f: B_3^{(24)} \to M\text{Sp} \) denote the map of Lemma 5. Then \( f_\ast(h_1^2 Q) = \eta^2 q_0 \not\in \pi_{10} M\text{Sp} \). Thus, \( R \) is an infinite cycle. Let \( \lambda \in \pi_{11} B_3 \) project to \( R \). Note that \( 0 \neq h_{10} h_{22}^2 \in ME_1^{22,3} \) is an infinite cycle. Since \( ME_1^{23,k} = 0 \) for \( 0 \leq k \leq 2 \), \( 0 \neq h_0 R^2 \in E_2^{22,3} \) and \( E_2^{23,0} = E_2^{23,1} = 0 \). Therefore, \( h_0 R^2 \) is a nonbounding infinite cycle. Thus, \( 2\lambda^2 \neq 0 \), which contradicts the fact that \( \pi_\ast B_3 \) is a commutative graded ring. Therefore, \( B_3 \) cannot exist.

**Bibliography**


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