PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 3, November 1991

NONEXISTENCE OF GENERALIZED SCATTERING RAYS AND SINGULARITIES OF THE SCATTERING KERNEL FOR GENERIC DOMAINS IN $\mathbb{R}^3$

LUCHEZAR STOJANOV

(Communicated by Frederick R. Cohen)

ABSTRACT. It is proved for fixed unit vectors $\omega \neq \theta$ in $\mathbb{R}^3$ and generic bounded open domains $\mathfrak{D} \subset \mathbb{R}^3$ that there do not exist generalized (ω, θ)-rays in $\Omega = \mathbb{R}^3 \setminus \mathfrak{D}$ containing nontrivial geodesics on $\partial\Omega$. Consequently, for generic domains the sojourn times of reflecting (ω, θ)-rays completely describe the set of singularities of the scattering kernel $s(t, \theta, \omega)$.

1. INTRODUCTION

Let Ω be a closed domain in $\mathbb{R}^3$ with C^∞ smooth boundary $\partial\Omega$ and bounded complement $\mathfrak{D} = \mathbb{R}^3 \setminus \Omega$. For fixed $\omega, \theta \in S^2$ the *scattering kernel* $s(t, \theta, \omega) = s_\Omega(t, \theta, \omega)$ related to the wave equation in $\mathbb{R} \times \Omega$ with Dirichlet boundary conditions on $\mathbb{R} \times \partial\Omega$ is a distribution in $\mathscr{S}'(\mathbb{R}_t)$ (see [6] for the definition). It was suggested by Guillemin [3] that the analysis of the singularities of $s(t, \theta, \omega)$ is connected with the sojourn times of the (ω, θ)-rays in Ω.

Let $\gamma: \mathbb{R} \to \Omega$ be a *generalized geodesic* in Ω; i.e. $\gamma = \imath \circ \tilde{\gamma}$, where $\tilde{\gamma}: \mathbb{R} \to T^*(\mathbb{R} \times \Omega)$ is a generalized bicharacteristic of the wave operator $\Box = \partial_t^2 - \Delta$ (see [7] or [4, §24.3]) and $\imath: T^*(\mathbb{R} \times \Omega) \to \Omega$ is the canonical projection. If there exist real numbers $a < b$ such that $\dot{\gamma}(t) = \omega$ for $t \leq a$ and $\dot{\gamma}(t) = \theta$ for $t \geq b$, then γ (and sometimes $\operatorname{Im}\gamma$) is called a *(ω, θ)-ray in* Ω. Such a curve γ consists of linear segments in Ω (two of them are infinite straightline rays) and gliding segments (i.e. geodesics with respect to the standard metric) on $\partial\Omega$. If $\operatorname{Im}\gamma$ contains only a finite number of linear segments and does not contain gliding ones, then γ is called a *reflecting* (ω, θ)*-ray* in Ω, otherwise γ is called a *generlized* (ω, θ)*-ray*. By $\mathfrak{L}_{\omega,\theta} = \mathfrak{L}_{\omega,\theta}(\Omega)$ we denote *the set of all* (ω, θ)*-rays in* Ω.

Received by the editors January 8, 1990 and, in revised form, May 8, 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 58G25, 58F30; Secondary 58F17.

Key words and phrases. Generalized and reflecting (ω, θ)-rays, sojourn time, scattering kernel, generic.

This work was partially supported by the Bulgarian Committee of Sciences, Grant 52.

Fix an open ball B with radii $a > 0$ that contains $\mathfrak{D}$. For $\eta \in S^2$ let Z_η be the hyperplane in $\mathbb{R}^3$ tangent to B such that Z_η is orthogonal to η and the halfspace H_η, determined by Z_η and having η as an inward normal, contains B. For a (ω, θ)-ray γ in Ω denote by T'_γ the length of this part of γ that is contained in $H_\omega \cap H_{-\theta}$. Then $T_\gamma = T'_\gamma - 2a$ is called the *sojourn time* of γ (cf. Guillemin [3]). It is easy to see that the definition of T_γ does not depend on the choice of the ball B.

Under some assumptions on Ω it was established in [10] that

$$\text{sing supp}\, s(t, \theta, \omega) \subset \{-T_\gamma : \gamma \in \mathfrak{L}_{\omega, \theta}\}, \tag{1}$$

where $\mathfrak{L}_{\omega, \theta} = \mathfrak{L}_{\omega, \theta}(\Omega)$ is the set of all (ω, θ)-rays in Ω. Moreover, in [10] a formula was proved for the main singularity of $s(t, \theta, \omega)$ for t close to some $T \in \text{sing supp}\, s(t, \theta, \omega)$. Recently, the inclusion (1) was established in [1] under weaker assumptions, and it was also shown there that for generic Ω in $\mathbb{R}^n$, $T_\gamma \in \text{sing supp}\, s(t, \theta, \omega)$ for a class of reflecting (ω, θ)-rays γ in Ω. In [8, 9, 14] all singularities of $s(t, \theta, \omega)$ have been examined for special classes of obstacles $\mathfrak{D}$.

For $X = \partial\Omega$ denote by $C^\infty(X, \mathbb{R}^3)$ the space of all C^∞ maps of X into $\mathbb{R}^3$ endowed with the Withney C^∞ topology (cf. [2]), and by $\mathbf{C}(X) = C^\infty_{\text{emb}}(X, \mathbb{R}^3)$ its open subset consisting of all C^∞ embeddings of X into $\mathbb{R}^3$. Then $\mathbf{C}(X)$ is a Baire topological space, so every *residual subset* (i.e. a countable intersection of open dense subsets), is dense in it. Given $f \in \mathbf{C}(X)$ we denote by Ω_f the unbounded closed domain in $\mathbb{R}^3$ with $\partial\Omega_f = f(X)$.

The main result in this paper is the following:

Theorem 1.1. *Let $\theta \neq \omega$ be fixed unit vectors in $\mathbb{R}^3$. Then there exists a residual subset $\mathscr{R}$ of $\mathbf{C}(X)$ such that for each $f \in \mathscr{R}$ there are no generalized (ω, θ)-rays in Ω_f, and*

$$\text{sing supp}\, s_{\Omega_f}(t, \theta, \omega) = \{-T_\gamma : \gamma \in \mathfrak{L}_{\omega, \theta}(\Omega_f)\}. \tag{2}$$

Moreover, for $f \in \mathscr{R}$ and $\gamma \in \mathfrak{L}_{\omega, \theta}(\Omega_f)$ the main singularity of $s_{\Omega_f}(t, \theta, \omega)$ for t near $-T_\gamma$ is given by the same formula as that in [10, 1] (see [1, Theorem 2]).

2. Degenerate (ω, θ)-rays

Let Ω be as in the introduction, $X = \partial\Omega$, and $\omega \in S^2$.

A curve γ in Ω is called a *degenerate ω-ray* if it has the form $\gamma = \bigcup_{i=0}^{k-1} l_i \subset \Omega$ and the following conditions are satisfied:

(i) l_0 is the infinite linear segment starting at x_1 having direction $-\omega$;

(ii) for every $i = 1, \dots, k-1$, l_i is a linear segment $[x_i, x_{i+1}]$, $x_i \in X$ for $i = 1, \dots, k$;

(iii) if $k \geq 2$ then for any $i = 1, \dots, k-2$, the segments l_i and l_{i+1} satisfy the *reflection law* at x_{i+1} with respect to X; i.e., l_i and l_{i+1} make equal acute angles with the *interior* (with respect to Ω) *unit normal vector* $\nu(x_{i+1})$ to X at x_{i+1} and l_i, l_{i+1}, $\nu(x_{i+1})$ lie in a common plane; and

(iv) l_{k-1} is tangent to X at x_k, determining an asymptotic direction for X at x_k (cf. [15] for the definition of asymptotic direction).

The points $x_1, \dots, x_k$ are called *vertices* of γ. If every segment of γ is not tangent to X, then γ is called *ordinary*. The *defect* of such a ray γ is defined by $d(\gamma) = k - s$, where $s = \operatorname{card}\{x_1, \dots, x_k\}$.

Note that if the curvature of X does not vanish of infinite order, then for every generalized (ω, θ)-ray γ in Ω there exist a degenerate ω-ray γ_1 and a degenerate $(-\theta)$-ray γ_2 with $\gamma_i \subset \operatorname{Im}\gamma$, $i = 1, 2$ (cf. [7]).

For a set A and an integer $s \geq 2$ we use the notation:

$$A^{(s)} = \{(a_1, \dots, a_s) \in A^s : a_i \neq a_j \text{ for } i \neq j\}.$$

Lemma 2.1. *There exists a residual subset* $\mathfrak{R}(\omega)$ *of* $\mathbf{C}(X)$ *such that for* $f \in \mathfrak{R}(\omega)$ *if* γ *is a degenerate* ω*-ray in* Ω_f, *then* $d(\gamma) = 0$.

To prove the assertion one can use some arguments from the proof of Lemma 2.2 as well as a combinatorial classification of the degenerate ω-rays, similar to that for periodic reflecting rays used in §4 of [11]. Since the modifications are rather standard, we omit the details.

For an integer $k \geq 1$ and $\omega \in S^2$, denote by $\mathscr{D}(\omega; k)$ the set of those $f \in \mathbf{C}(X)$ such that the set of all $y = (y_1, \dots, y_k) \in f(X)^{(k)}$ for which $y_1, \dots, y_k$ are the successive vertices of a degenerate ω-ray on $f(X)$ is a discrete subset of $f(X)^{(k)}$.

Lemma 2.2. *The set* $\mathscr{D}(\omega; k)$ *contains a residual subset of* $\mathbf{C}(X)$.

Proof. To prove the assertion it is sufficient to establish that $\mathscr{D}(\omega; k) \cap C^\infty_{\mathrm{emb}}(X, H_\omega)$ contains a residual subset of $C^\infty_{\mathrm{emb}}(X, H_\omega)$.

We proceed as in [11–13]. Let $\pi: \mathbb{R}^3 \to Z = Z_\omega$ be the orthogonal projection. Denote by U_k the set of those $y = (y_1, \dots, y_k) \in (\mathbb{R}^3)^{(k)}$ such that for every $i = 1, \dots, k-1$, y_i does not belong to the segment $[y_{i-1}, y_{i+1}]$, where by definition $y_0 = \pi(y_1)$. Define $F: U_k \to \mathbb{R}$ by

$$F(y) = \sum_{i=0}^{k-1} \|y_i - y_{i+1}\|.$$

If $y_1, \dots, y_k$ are the successive reflection points of a degenerate ω-ray γ in Ω_f with $d(\gamma) = 0$, then $y = (y_1, \dots, y_k) \in U_k$ and $F(y)$ is the length of $\gamma \cap H_\omega$. Moreover, for $y' = (y_1, \dots, y_{k-1})$ we have $\operatorname{grad}_{y'} F(y) = 0$, $\langle y_k - y_{k-1}, \nu(y_k)\rangle = 0$, and $w = y_k - y_{k-1}$ is an asymptotic direction for Y at y_k. The last condition can be expressed analytically as follows: Let $r: V \to Y$ be a chart, where V is an open subset of $\mathbb{R}^2$ and $r(V)$ is an open neighborhood of y_k in Y. Then $w = \lambda(\partial r/\partial u_1)(u) + \mu(\partial r/\partial u_2)(u)$ for some $\lambda, \mu \in \mathbb{R}$, where $r(u) = y_k$, $u = (u_1, u_2)$. Let L, M, N be the coefficients of the second fundamental form of Y at y_k; that is, $L(u) = \langle(\partial^2 r/\partial u_1^2)(u), \nu(y_k)\rangle$, $M(u) = \langle(\partial^2 r/\partial u_1 \partial u_2)(u), \nu(y_k)\rangle$, $N(u) = \langle(\partial^2 r/\partial u_2^2)(u), \nu(y_k)\rangle$. Then w is an asymptotic direction for Y at y_k iff (cf. [15])

$$L\lambda^2 + 2M\lambda\mu + N\mu^2 = 0. \tag{3}$$

It is easy to check that $\lambda = \langle w, (G\partial r/\partial u_1 - F\partial r/\partial u_2)/(EG - F^2)\rangle$ and $\mu = \langle w, (E\partial r/\partial u_2 - F\partial r/\partial u_1)/(EG-F^2)\rangle$, where $E(u) = \|(\partial r/\partial u_1)(u)\|^2$, $F(u) = \langle(\partial r/\partial u_1)(u), (\partial r/\partial u_2(u))\rangle$, and $G(u) = \|(\partial r/\partial u_2)(u)\|^2$ are the coefficients of the first fundamental form. Therefore (3) is equivalent to

$$\begin{aligned} &L\langle w, G\partial r/\partial u_1 - F\partial r/\partial u_2\rangle^2 \\ &\quad + 2M\langle w, G\partial r/\partial u_1 - F\partial r/\partial u_2\rangle\langle w, E\partial r/\partial u_2 - F\partial r/\partial u_1\rangle \\ &\quad + N\langle w, E\partial r/\partial u_2 - F\partial r/\partial u_1\rangle^2 = 0. \end{aligned}$$

Let $J_k^2(X, \mathbb{R}^3)$ be the *k-fold bundle of 2-jets* (cf. [2]). Given $f \in C^\infty(X, \mathbb{R}^3)$, the map $j_k^2 f\colon X^{(k)} \to J_k^2(X, \mathbb{R}^3)$ is defined by $j_k^2 f(x_1, \dots, x_k) = (j^2 f(x_1), \dots, j^2 f(x_k))$. Here $j^2 f(x) \in J^2(X, \mathbb{R}^3)$ is the 2-jet determined by f at $x \in X$. Denote by M the set of those $\tau = (j^2 f_1(x_1), \dots, j^2 f_k(x_k)) \in J_k^2(X, \mathbb{R}^3)$ such that $(x_1, \dots, x_k) \in X^{(k)}$; $(f_1(x_1), \dots, f_k(x_k)) \in U_k$; rank $df_i(x_i) = 2$ for every $i = 1, \dots, k$; and $f_i(x_i) - f_{i+1}(x_{i+1})$ is not tangent to $f_i(X)$ at $f_i(x_i)$ for all $i = 1, \dots, k-1$, and ω is not tangent to $f_1(X)$ at $f_1(x_1)$. Then M is open in $J_k^2(X, \mathbb{R}^3)$. Finally, define the singularity set Σ as the set of those $\tau \in M$ such that $\operatorname{grad}_{x'} F \circ (f_1 \times \cdots \times f_k)(x) = 0$, $\langle f_k(x_k) - f_{k-1}(x_{k-1}), \nu\rangle = 0$, and $f_k(x_k) - f_{k-1}(x_{k-1})$ is an asymptotic direction for $f_k(X)$ at $f_k(x_k)$, where ν is a nonzero normal vector to $f_k(X)$ at $f_k(x_k)$.

Next, using some arguments from [11] or [12] (cf., for example, [11, proof of Lemma 7.1]) we establish that Σ is a smooth submanifold of M with $\operatorname{codim}\Sigma = 2k$. Then for $f \in C^\infty(X, \mathbb{R}^3)$, $j_k^2 f \pitchfork \Sigma$ implies that $\{x \in X^{(k)}\colon j_k^2 f(x) \in \Sigma\}$ is a discrete subset of $X^{(k)}$; i.e., $f \in \mathscr{D}(\omega; k)$. Consequently $\mathscr{D}(\omega; k)$ contains the residual subset:

$$\{f \in C^\infty(X, \mathbb{R}^3)\colon j_k^2 f \pitchfork \Sigma\} \cap C^\infty_{\text{emb}}(X, H_\omega) \cap \mathfrak{R}(\omega)$$

of $C^\infty_{\text{emb}}(X, H_\omega)$. This proves the assertion.

3. Nonexistence of generalized (ω, θ)-rays

In this section we prove that generic domains Ω do not admit generalized (ω, θ)-rays. To this end we combine Lemma 2.2 with a simple perturbation technique.

Lemma 3.1. *Let X be a smooth surface in $\mathbb{R}^3$ and $c\colon [a, b] \to X$ be a geodesic on $X (b > a)$. Let $c(t_0)$ be an arbitrary point on the geodesic $(a < t_0 < b)$ that is not a point of selfintersection, and U be an arbitrary neighborhood of $c(t_0)$*

in X *such that*

$$U \cap \operatorname{Im} c = \{c(t) : \alpha < t < \beta\} \tag{4}$$

for some α, $\beta \in (a, b)$. *Then there exists* $f \in \mathbf{C}(X)$ *arbitrarily close to* id *with respect to the* C^∞ *topology such that* $\operatorname{supp} f \subset U$, *and if* $\tilde{c}: [a, b] \to \widetilde{X}$ *is the geodesic on* $\widetilde{X} = f(X)$ *with* $\tilde{c}(t) = c(t)$ *for* $t \in [a, \alpha]$, *then*

$$\{\tilde{c}(t) : t \in (\alpha, \beta]\} \cap \{c(t) : t \in (\alpha, \beta]\} = \varnothing. \tag{5}$$

Proof. We may assume that U is small enough so that there exists coordinates x_0, x_1 in U given by a chart $r: V \to U \subset X$, where $V = (\alpha, \beta) \times (-\delta, \delta) \subset \mathbb{R}^2$ for some $\delta > 0$, $a < \alpha < t_0 < \beta < b$, such that the components g_{ij} of the standard metric g on X have the form:

$$g_{00}(x_0, x_1) = 1, \qquad g_{01}(x_0, x_1) = 0, \qquad g_{11}(x_0, x_1) = G(x_0, x_1) > 0$$

for $(x_0, x_1) \in V$. Moreover, we may assume

$$G(x_0, x_1) < 1 \quad \text{for all } (x_0, x_1) \in V. \tag{6}$$

Otherwise we can replace r by another chart, $\tilde{r}: V \to X$ given by $\tilde{r}(x_0, x_1) = r(x_0, \varepsilon x_1)$; then $\tilde{g}_{11}(x_0, x_1) = \varepsilon^2 g_{11}(x_0, x_1) < 1$ for $\varepsilon > 0$ sufficiently small. Moreover, (4) holds provided $t_0 - \alpha$, $\beta - t_0$, and δ are sufficiently small. Also note that $r(t, 0) = c(t)$ for $t \in (\alpha, \beta)$.

Take arbitrary C^∞ functions λ, $\mu: \mathbb{R} \to [0, 1]$ with

$$\operatorname{supp} \lambda = [\alpha, \beta], \qquad p = \mu(0) > 0, \qquad q = \mu'(0) > 0. \tag{7}$$

For $\varepsilon > 0$ small enough set $f_\varepsilon(y) = y$ for $y \in X \setminus U$ and $f_\varepsilon(y) = r(x) + \varepsilon\lambda(x_0)\mu(x_1)(\partial r/\partial x_0)(x)$ for $y = r(x)$, $x = (x_0, x_1) \in V$. Let $X_\varepsilon = f_\varepsilon(X)$. Then $\psi(x) = r(x) + \varepsilon\lambda(x_0)\mu(x_1)(\partial r/\partial x_0)(x)$ defines a chart, $\psi: V \to \psi(V) \subset X_\varepsilon$. Let $g(\varepsilon)$ be the standard metric on X_ε induced by $\mathbb{R}^3$; then for its components $g_{ij}(\varepsilon; x_0, x_1)$ we have:

$$\begin{aligned}
g_{00}(\varepsilon; x_0, x_1) &= 1 + 2\varepsilon\lambda'(x_0)\mu(x_1) + O(\varepsilon^2), \\
g_{01}(\varepsilon; x_0, x_1) &= \varepsilon\lambda(x_0)\mu'(x_1) + O(\varepsilon^2), \\
g_{11}(\varepsilon; x_0, x_1) &= G(x_0, x_1) + 2\varepsilon\lambda(x_0)\mu(x_1) \\
&\quad \cdot \langle (\partial r/\partial x_1)(x), (\partial^2 r/\partial x_0 \partial x_1)(x) \rangle + O(\varepsilon^2)
\end{aligned}$$

for ε close to 0.

Using (x_0, x_1) consider the canonical coordinates x_0, x_1, y_0, y_1 in $T^* X_\varepsilon$ and the Hamiltonian vectorfield generated by the Hamiltonian:

$$H(\varepsilon, x, y) = g_{00}(\varepsilon; x) y_0^2/2 + g_{01}(\varepsilon; x) y_0 y_1 + g_{11}(\varepsilon; x) y_1^2/2,$$

where $x = (x_0, x_1)$, $y = (y_0, y_1)$. Let $c(\varepsilon; t)$, $0 \le t$, be the geodesic on X_ε with $c(\varepsilon; t) = c(t)$ for each $t \in [0, \alpha]$, and $(x^{(\varepsilon)}(t), y^{(\varepsilon)}(t))$ be the corresponding integral curve in $T^* X_\varepsilon$. Writing the Hamiltonian equations for this curve, and then the corresponding variational equations for

$$X_i(t) = (d/d\varepsilon) x_i^{(\varepsilon)}(t)|_{\varepsilon=0}, \qquad Y_i(t) = (d/d\varepsilon) y_i^{(\varepsilon)}(t)|_{\varepsilon=0},$$

we get (cf. (7)):

$$\begin{aligned}
\dot{X}_0(t) &= Y_0(t) + 2p\lambda'(t)\\
\dot{X}_1(t) &= G(t,0)Y_1(t) + q\lambda(t)\\
\dot{Y}_0(t) &= -p\lambda''(t)\\
\dot{Y}_1(t) &= -q\lambda'(t)\\
X_0(\alpha) = X_1(\alpha) &= Y_0(\alpha) = Y_1(\alpha) = 0
\end{aligned}$$

for $(\alpha \le t \le \beta)$. Consequently, $Y_1(t) = -q\lambda(t)$ and $\dot{X}_1(t) = q\lambda(t)(1-G(t,0))$. Hence (6) yields $\dot{X}_1(t) > 0$ for every $t \in (\alpha, \beta)$, and therefore, $X_1(t) > 0$ for each $t \in (\alpha, \beta]$. This means that $(d/d\varepsilon)x_1^{(\varepsilon)}(t) > 0$ for $t \in (\alpha, \beta]$, provided $\varepsilon > 0$ is sufficiently small. Fix such an ε. Then $x_1^{(\varepsilon)}(t)$ is positive for $t \in (\alpha, \beta]$, and for $f = f_\varepsilon$, $\tilde{X} = X_\varepsilon$, and $\tilde{c}(t) = c(\varepsilon, t)$ we have (5). This proves the assertion.

Fix two unit vectors $\omega \ne \theta$ in $\mathbb{R}^3$.

Theorem 3.2. *There exists a residual subset $\mathscr{V}$ of $\mathbf{C}(X)$ such that for every $f \in \mathscr{V}$ there are no generalized (ω, θ)-rays in Ω_f.*

Proof. We are going to construct by induction a decreasing sequence $\mathscr{V}_1 \supset \mathscr{V}_2 \supset \cdots \supset \mathscr{V}_k \supset \cdots$ of residual subsets of $\mathbf{C}(X)$ such that for any k and any $f \in \mathscr{V}_k$ there are no generalized (ω, θ)-rays in Ω_f with not more than k vertices.

It follows by [5] that there exists a residual subset $\mathscr{K}$ of $\mathbf{C}(X)$ such that whenever $f \in \mathscr{K}$, for every $y \in f(X)$ the curvature of $f(X)$ at y does not vanish of third order with respect to any direction tangent to $f(X)$ at y. Then for $f \in \mathscr{K}$, if $\gamma: \mathbb{R} \to \Omega_f$ is a (ω, θ)-ray, then $\operatorname{Im}\gamma = \bigcup_{i=0}^k l_i$, where l_0 and l_k are infinite segments, starting at x_1 and x_k, respectively, with directions $-\omega$ and θ, and for any $i = 1, \dots, k-1$, l_i is either a linear segment $[x_i, x_{i+1}]$ in Ω_f or a geodesic $x_i x_{i+1}$ on $\partial\Omega_f = f(X)$, and $x_i = \partial\Omega_f$ for every $i = 1, \dots, k$. Moreover, two successive linear segments of γ satisfy the reflection law at their common end, and if a linear and a gliding segments of γ are successive, then the linear segment is tangent to the gliding one, determining an asymptotic direction for $\partial\Omega_f$ at their common end (cf. [7]).

Denote by $\mathscr{V}_1$ the set of those $f \in \mathscr{K}$ such that there are no degenerate (ω, θ)-rays in Ω_f with one vertex. The verification that $\mathscr{V}_1$ is open and dense in $\mathbf{C}(X)$ uses some arguments very similar to those below, so we omit the details here.

Let $k > 1$, and suppose we have already constructed the sets $\mathscr{V}_1 \supset \cdots \supset \mathscr{V}_{k-1}$ so that they have the desired properties. Next, we construct $\mathscr{V}_k$.

A function $\mathscr{H}: \{1, 2, \dots, k\} \to \{0, 1\}$ is called a *k-design* if $\mathscr{H} \not\equiv 0$, $\mathscr{H}(0) = \mathscr{H}(k) = 0$, and $\mathscr{H}(i)\mathscr{H}(i+1) = 0$ for each $i = 1, 2, \dots, k-2$. If γ is a generalized (ω, θ)-ray with $\operatorname{Im}\gamma = \bigcup_{i=0}^k l_i$ and k vertices, and for each

$i = 1, \dots, k-1$, $\mathscr{H}(i) = 0$ holds iff l_i is a linear segment, then γ is called a *ray with design* $\mathscr{H}$.

Fix a k-design $\mathscr{H}$ and set

$$q = \max\{i : 1 \le i \le k-1,\ \mathscr{H}(i) = 1\},$$

$$p = \min\{i : 1 \le i \le k-1,\ \mathscr{H}(i) = 1\}.$$

We now use the sets $\mathscr{D}(\omega; m)$ defined in §2. Also consider the set $\mathscr{T}_k$ of all $f \in \mathbf{C}(X)$ such that there are only finitely many reflecting (ω, θ)-rays in $\mathbf{\Omega}_f$ with not more than k vertices and all of them are ordinary. By Theorem 5.1 in [13], $\mathscr{T}_k$ contains a residual subset of $\mathbf{C}(X)$. Then by Lemma 2.2 the set:

$$\mathscr{W} = \mathscr{V}_{k-1} \cap \mathscr{D}(\omega, p) \cap \mathscr{D}(-\theta, q) \cap \mathscr{T}_k \tag{8}$$

also contains a residual subset of $\mathbf{C}(X)$. Fix an arbitrary $r \in \mathbb{N}$ and denote by $\mathscr{V}(k; r; \mathscr{H})$ *the set of those* $f \in \mathscr{W}$ *such that there are no generalized* (ω, θ)*-rays in* $\mathbf{\Omega}_f$ *with design* $\mathscr{H}$ *and sojourn times* $\le r$.

First, we show that $\mathscr{V}(k; r; \mathscr{H})$ in dense in $\mathscr{W}$. To this end we assume $\mathrm{id} \in W$, and then we have to prove that there exists $f \in \mathscr{V}(k; r; \mathscr{H})$ arbitrarily close to id with respect to the C^∞ topology. Observe that there are only finitely many generalized (ω, θ)-rays in $\mathbf{\Omega}_f$ with design $\mathscr{H}$ and sojourn times $\le r$. Indeed, assume there exists an infinite sequence $\{\gamma_m\}$ of distinct generalized (ω, θ)-rays $\gamma_m : \mathbb{R} \to \mathbf{\Omega}$ with design $\mathscr{H}$ and sojourn times $\le r$. Let $x_i^{(m)} = \gamma(t_i^{(m)})$, $0 = t_1^{(m)} < t_2^{(m)} < \cdots < t_k^{(m)}$, be the successive vertices and $l_i^{(m)}$ $(i = 0, 1, \dots, k)$ be the successive segments of γ_m. We may assume that there exist $\lim_m x_i^{(m)} = x_i$ for $i = 1, \dots, k$ and $\lim_m |l_i^{(m)}|$ ($|l|$ denotes the *length* of the segment l) for $i = 0, 1, \dots, k$. Then a standard continuity argument shows that for every $t \in \mathbb{R}$ there exists $\lim_m \gamma_m(t) = \gamma(t)$, and γ is a (ω, θ)-ray in $\mathbf{\Omega}$ (cf. [4]). Moreover, $l_i = \lim_m l_i^{(m)}$ are successive segments of γ (some of them may consists of only one point so they can be cancelled) with endpoint $x_1, \dots, x_k$ and $\operatorname{Im}\gamma = \bigcup_{i=0}^k l_i$. Since $\theta \ne \omega$ and $\mathrm{id} \in \mathscr{W} \subset \mathscr{V}_1$, the case $x_1 = \cdots = x_k$ is impossible.

If every l_i is nontrivial, i.e. it does not consist of one point, then γ would be a generalized (ω, θ)-ray with design $\mathscr{H}$. Then $\delta_m = \bigcup_{i=0}^{p-1} l_i^{(m)}$ for any $m \in \mathbb{N}$ and $\delta = \bigcup_{i=0}^{p-1} l_i$ are degenerate ω-rays in $\mathbf{\Omega}$ with $\delta_m \xrightarrow[m]{} \delta$, which is a contradiction with $\mathrm{id} \in \mathscr{D}(\omega, p)$. Therefore l_i vanishes for at least one i. It then follows that γ is a reflecting (ω, θ)-ray in $\mathbf{\Omega}$, otherwise we would get a contradiction with $\mathrm{id} \in \mathscr{V}_{k-1}$. Clearly, γ has at most $k-1$ reflection points. Moreover, applying some arguments similar to those in §4 of [13], we see that some segment of γ is tangent to X, which is a contradiction with $\mathrm{id} \in \mathscr{T}_k$.

Hence there exist only finitely many generalized (ω, θ)-rays $\gamma, \gamma_2, \dots, \gamma_n$ in $\mathbf{\Omega}$ with design $\mathscr{H}$ and sojourn times $\le r$. Let $\operatorname{Im}\gamma = \bigcup_{j=0}^k l_j$. Then l_j is a linear segment iff $\mathscr{H}(j) = 0$. Let $x_i = \gamma(t_i)$ be the successive vertices of γ,

$0 = t_1 < t_2 < \cdots < t_k$. Then

$$l_q = \{\gamma(t) \colon t_q \le t \le t_{q+1}\}$$

is a geodesic on X and $l_{q+1}, \dots, l_{k-1}$ are linear segments. There is no $a \in [t_q, t_{q+1})$ such that

$$\{\gamma(t) \colon a \le t \le t_{q+1}\} \subset l_s \tag{9}$$

for some $s < q$. Indeed, if such a and s exist, then there would be two distinct generalized geodesics in Ω passing through x_{q+1} in direction $\overrightarrow{x_{q+1}x_{q+2}}$, which is a contradiction with id $\in \mathscr{W} \subset \mathscr{K}$ (cf. [7] or [4]). Hence for every $s = 1, \dots, q-1$ there exists $a \in [t_q, t_{q+1})$ so that (9) does not hold. Consequently, there is $t_0 \in (t_q, t_{q+1})$ such that $\gamma(t_0)$ is not a point of selfintersection of γ. Moreover, applying the same argument, and eventually replacing V by a smaller neighborhood of x_k, we see that t_0 can be chosen so that $\gamma(t_0) \notin (\bigcup_{i=2}^n \operatorname{Im}\gamma_i) \cup \overline{V}$. Next, choose a small coordinate neighborhood U of $\gamma(t_0)$ in X with

$$U \cap \left(\left(\bigcup_{i=2}^{n} \operatorname{Im}\gamma_i \right) \cup \overline{V} \cup \bigcup_{\substack{j=0 \\ j \neq q}}^{k} l_j \right) = \varnothing$$

and such that (4) holds for $c(t) = \gamma(t)$, $a = t_q$, $b = t_{q+1}$, and some α, β. By Lemma 3.1 there exists $f \in \mathbf{C}(X)$ arbitrarily close to id such that $\operatorname{supp} f \subset U$ and (5) holds for $\tilde{X} = f(X)$ and the geodesic $\tilde{c} \colon [a, b] \to \tilde{X}$ with $\tilde{c}(t) = c(t)$ for $t \in [a, \alpha]$. Since id $\in \mathscr{D}(\omega; p)$ it is easily seen that if f is sufficiently close to id, then the only generalized (ω, θ)-rays in Ω_f with design $\mathscr{H}$ and sojourn times $\le r$ are $\gamma_2, \dots, \gamma_n$ and eventually, a ray δ with $\delta(0) = x_1$ and $\dot{\delta}(0) = \omega$. Assume that for any choice of f there exists such a generalized (ω, θ)-ray $\delta = \delta_f$. Then clearly $\delta(t) = \gamma(t)$ for all $t \le \alpha$. Let $z_1 = x_1, z_2, \dots, z_k$ be the successive vertices of δ. Observe that for f sufficiently close to id the last vertex z_k of δ belongs to V. Otherwise we would find a sequence $f_m \to$ id such that the last vertex of $\delta_m = \delta_{f_m}$ is not contained in V and $\lim_m \delta_m(t) = \delta(t)$ exists for all $t \in \mathbb{R}$; then δ would be a generalized (ω, θ)-ray in Ω with design $\mathscr{H}$ and sojourn time $\le r$ different from $\gamma, \gamma_2, \dots, \gamma_n$: a contradiction. Hence $z_k \in V$ for f sufficiently close to id, which is a contradiction with the choice of V. Thus for f sufficiently close to id there are only $n-1$ generalized (ω, θ)-rays with design $\mathscr{H}$ and sojourn times $\le r$ in Ω. Moreover, a simple argument shows that for special construction of f considered above (if U is sufficiently small and f is sufficiently close to id) we have $f \in \mathscr{W}$.

In this way, by induction we construct $g \in \mathscr{V}(k; r; \mathscr{H})$ arbitrarily close to id. Hence $\mathscr{V}(k; r; \mathscr{H})$ is dense in $\mathscr{W}$. To prove that $\mathscr{V}(k; r; \mathscr{H})$ is open in $\mathscr{W}$ it is sufficient to establish that if $\{f_n\} \subset \mathscr{W} \setminus \mathscr{V}(k; r; \mathscr{H})$ and $f_n \to$ id $\in \mathscr{W}$, then X admits a generalized (ω, θ)-ray with design $\mathscr{H}$ and sojourn time $\leq r$. This follows easily using some arguments from above and we omit the details.

The set $\bigcap_{\mathscr{H},r} \mathscr{V}(k;r;\mathscr{H})$, where $r \in \mathbb{N}$ and $\mathscr{H}$ runs over the finite set of all k-designs, is a residual subset of $\mathscr{W}$; therefore it contains a residual subset $\mathscr{V}_k$ of $\mathbf{C}(X)$. Clearly $\mathscr{V}_k$ has the desired properties. This completes the construction of the sequence $\{\mathscr{V}_k\}$.

Finally, setting $\mathscr{V} = \bigcap_{k=1}^{\infty} \mathscr{V}_k$, we obtain a residual subset of $\mathbf{C}(X)$ such that for any $f \in \mathscr{V}$ there are no generalized (ω, θ)-rays in Ω_f.

Proof of Theorem 1.1. It follows by Theorem 2 in [1] that there exists a residual subset $\mathscr{A}$ of $\mathbf{C}(X)$ such that for any $f \in \mathscr{A}$ and any reflecting (ω, θ)-ray γ in Ω_f, if $T_\gamma \neq T_\delta$ for every generalized (ω, θ)-ray δ in Ω_f, then $-T_\gamma$ belongs to the left-hand side of (2). Set $\mathscr{R} = \mathscr{A} \cap \mathscr{K} \cap \mathscr{V}$, then $\mathscr{R}$ is a residual subset of $\mathbf{C}(X)$. Given $f \in \mathscr{R}$, by [10] or [1] we have that the left-hand side of (2) is contained in the right-hand side. Since $f \in \mathscr{V}$, there are no generalized (ω, θ)-rays in Ω_f, and the above remark implies that (2) holds.

References

1. F. Cardoso, V. Petkov, and L. Stojanov, *Singularities of the scattering kernel for generic obstacles*, Ann. Inst. Henri Poincaré, Phys. Théorique **53** (1990), 445–466.
2. M. Golubitsky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, New York, 1973.
3. V. Guillemin, *Sojourn time and asymptotic properties of the scattering matrix*, Publ. Res. Inst. Math. Sci. **12** (1977), 69–88.
4. L. Hörmander, *The analysis of linear partial differential operators*, vol. III, Springer-Verlag, Berlin, 1985.
5. E. Landis, *Tangential singularities*, Funct. Anal. Appl. **14** (1980), 25–34; English Transl. **14** (1980), 98–106.
6. A. Majda, *A representation formula for the scattering operator and the inverse problem for arbitrary bodies*, Comm. Pure Appl. Math. **30** (1977), 165–194.
7. R. Melrose and J. Sjöstrand, *Singularities in boundary value problems*. I, Comm. Pure Appl. Math. **31** (1978), 593–617.
8. S. Nakamura, *Singularities of the scattering kernel for several convex obstacles*, Hokkaido Math. J. **18** (1989), 487–496.
9. S. Nakamura and H. Soga, *Singularities of the scattering kernel for two balls*, J. Math. Soc. Japan **40** (1988), 205–220.
10. V. Petkov, *High frequency asymptotics of the scattering amplitude for non-convex bodies*, Comm. Partial Differential Equations **5** (1980), 293–329.
11. V. Petkov and L. Stojanov, *Periods of multiple reflecting geodesics and inverse spectral results*, Amer. J. Math. **109** (1987), 619–668.
12. ——, *Spectrum of the Poincare map for periodic reflecting rays in generic domains*, Math. Z. **194** (1987), 505–518.
13. ——, *On the number of periodic reflecting rays in generic domains*, Ergodic Theory Dynamical Systems **8** (1988), 81–91.

14. ——, *Singularities of the scattering kernel and scattering invariants for several strictly convex obstacles*, Trans. Amer. Math. Soc. **312** (1989), 203–235.

15. T. Wilmore, *Introduction to differential geometry*, Clarendon Press, Oxford, 1959.

INSTITUTE OF MATHEMATICS, BULGARIAN ACADEMY OF SCIENCES, P. O. BOX 373, 1090 SOFIA, BULGARIA

Current address: Technische Hochschule Darmstadt, FB Mathematik, Scheossgartenstrasse 7, D-6100 Darmstadt, Germany