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SECTORIALNESS OF SECOND ORDER ELLIPTIC OPERATORS IN DIVERGENCE FORM

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ABSTRACT. A sectorial estimate is given to second order linear elliptic differential operators of divergence form. The estimate is a slight improvement of Pazy's. The obtained constant depends on p of the space $L^p(\Omega)$ $(1 < p < \infty)$ and does not depend on the operators themselves. The same constant has appeared in the sectorial estimate for second order linear ordinary differential operators due to Fattorini.

The result is in connection with Stein's estimate of the analytic semigroups generated by linear elliptic differential operators.

1. INTRODUCTION

Let Ω be a bounded domain in R^m with smooth boundary, and let $A(x, D)$ be the second order differential operator of divergence form

$$A(x, D)u = -\sum_{j,k=1}^{m} \frac{\partial}{\partial x_k}\left[a_{jk}(x)\frac{\partial u}{\partial x_j}\right].$$

We assume that the coefficients $a_{jk}(x) = a_{kj}(x)$ are real-valued and continuously differentiable on $\overline{\Omega}$, and that $A(x, D)$ is uniformly elliptic or degenerate elliptic, i.e., there is a constant $C_0 \geqq 0$ such that for all $\xi \in R^m$,

$$(1) \qquad \sum_{j,k=1}^{m} a_{jk}(x)\xi_j\xi_k \geqq C_0|\xi|^2 := C_0\sum_{j=1}^{m}\xi_j^2.$$

Let $1 < p < \infty$ and consider the linear operator in $L^p(\Omega)$:

$$(2) \qquad D(A_p) = W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega),$$

$$(3) \qquad (A_pu)(x) = A(x, D)u(x) \quad \text{for } u \in D(A_p),$$

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where $W^{k,p}(\Omega)$ is the usual Sobolev space and $W_0^{k,p}(\Omega)$ is the completion of $C_0^\infty(\Omega)$ with respect to the norm of $W^{k,p}(\Omega)$ $(k = 1, 2)$.

In particular, if $C_0 > 0$, then in his book [7] Pazy proved the following estimates: for $u \in D(A_p)$, $\operatorname{Re}(A_p u, |u|^{p-2}u) \geqq 0$ and

$$|\operatorname{Im}(A_p u, |u|^{p-2}u)| \leqq \frac{M|p-2|}{2C_0\sqrt{p-1}} \operatorname{Re}(A_p u, |u|^{p-2}u), \tag{4}$$

where $M = \max\{|a_{jk}(x)|;\, x \in \overline{\Omega},\ 1 \leqq j, k \leqq m\}$ (see [7, Proof of Theorem 7.3.6]).

The purpose of this note is to show that (4) holds independently of the constants C_0 and M:

$$|\operatorname{Im}(A_p u, |u|^{p-2}u)| \leqq \frac{|p-2|}{2\sqrt{p-1}} \operatorname{Re}(A_p u, |u|^{p-2}u) \tag{5}$$

for $u \in D(A_p)$. So we see that (5) holds even in the degenerate elliptic case. Then it follows that the "generalized" numerical range of A_p is contained in the sector

$$|\arg \zeta| \leqq \omega_p := \tan^{-1}\left(|p-2|/2\sqrt{p-1}\right),$$

i.e., A_p is *sectorial* of type $S(\tan \omega_p)$ in the sense of Goldstein [2, Definition 1.5.8] (see also Kato [5, V–§3.10]).

Let $C_0 > 0$. Then we show further that when $p \geqq 2$,

$$\begin{aligned} \operatorname{Re}(A_p u, |u|^{p-2}u) &\geqq C_0 \operatorname{Re}(-\Delta u, |u|^{p-2}u) \\ &= C_0 \int_\Omega |u(x)|^{p-2} |\operatorname{grad} u(x)|^2 dx \\ &\quad + (p-2)C_0 \int_\Omega |u(x)|^{p-2} |\operatorname{grad} |u(x)||^2 dx, \end{aligned} \tag{6}$$

and when $1 < p < 2$,

$$\begin{aligned} &\operatorname{Re}(A_p u, |u|^{p-2}u) \\ &\quad \geqq (p-1)C_0 \lim_{\delta \downarrow 0} \int_\Omega \left(|u(x)|^2 + \delta\right)^{(p-2)/2} |\operatorname{grad} u(x)|^2\, dx. \end{aligned} \tag{7}$$

Here Δ is the Laplacian; note that $A(x, D) = -\Delta$ if $a_{jk}(x) = \delta_{jk}$ (the Kronecker delta). Since A_p is m-accretive (see Pazy [7, Theorem 7.3.6]), A_p is m-sectorial in the sense of [2, 5]. Therefore, $-A_p$ generates an analytic (or a holomorphic) contraction semigroup $\{T_p(t)\}$ on $L^p(\Omega)$, i.e., $T_p(t)$ is analytic in the sector $|\arg t| < (\pi/2) - \omega_p$ and

$$\|T_p(t)\|_p \leqq 1 \quad \text{for } |\arg t| \leqq (\pi/2) - \omega_p \tag{8}$$

(see [2, Theorem 1.5.9; 5, Theorem IX–1.24]). Twenty years ago Stein noticed in [8, III–§2, Theorem 1, and the proof on p. 71] that

$$\|T_p(t)\|_p \leqq 1 \quad \text{for } |\arg t| \leqq \frac{\pi}{2}\left(1 - \left|\frac{2}{p} - 1\right|\right) \tag{9}$$

as a consequence of his interpolation theorem. After a simple computation we see that

$$\tan^{-1}\left(\frac{|p-2|}{2\sqrt{p-1}}\right) \leqq \frac{\pi}{2}\left|\frac{2}{p}-1\right|.$$

Namely, (8) is an improvement of (9).

2. Sectorial estimates

Let A_p be defined by (2) and (3). Then for $u \in D(A_p)$,

$$\begin{aligned}(A_p u, |u|^{p-2}u) &= -\int_\Omega |u(x)|^{p-2}\overline{u(x)} \sum_{j,k=1}^m \frac{\partial}{\partial x_k}\left[a_{jk}(x)\frac{\partial u}{\partial x_j}\right] dx \\ &= \lim_{\delta \downarrow 0} I_p(u, \delta),\end{aligned}$$

where

$$I_p(u, \delta) = -\int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-2}{2}} \overline{u(x)} \sum_{j,k=1}^m \frac{\partial}{\partial x_k}\left[a_{jk}(x)\frac{\partial u}{\partial x_j}\right] dx.$$

Here we have to take $\delta > 0$ when $1 < p < 2$, and $\delta = 0$ when $p \geqq 2$. Since $u \in W_0^{1,p}(\Omega)$, we have

(10)

$$\begin{aligned}I_p(u, \delta) = \int_\Omega &\left(|u(x)|^2 + \delta\right)^{\frac{p-2}{2}} \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_k}\, dx \\ &+ (p-2)\int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|\overline{u(x)} \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial |u|}{\partial x_k}\, dx;\end{aligned}$$

note that $|u| \in W_0^{1,p}(\Omega)$ (Stampacchia's Lemma; see Gilbarg-Trudinger [3, Lemma 7.6]). The first term on the right-hand side of (10) is real. In fact, it is equal to

$$\begin{aligned}\int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-2}{2}} \sum_{j,k=1}^m a_{jk}(x) &\left[\left(\operatorname{Re}\frac{\partial u}{\partial x_j}\right)\left(\operatorname{Re}\frac{\partial u}{\partial x_k}\right)\right. \\ &\left. + \left(\operatorname{Im}\frac{\partial u}{\partial x_j}\right)\left(\operatorname{Im}\frac{\partial u}{\partial x_k}\right)\right] dx,\end{aligned}$$

and by the ellipticity (1), it is larger than

$$C_0 \int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-2}{2}} |\operatorname{grad} u(x)|^2\, dx.$$

Setting

$$H_p(u, \delta) = \int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|\overline{u(x)} \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial |u|}{\partial x_k}\, dx,$$

and noting that $2\,\mathrm{Re}\,\overline{u(x)}\frac{\partial u}{\partial x_j} = \frac{\partial}{\partial x_j}|u(x)|^2$, we have

$$\mathrm{Re}\,H_p(u\,,\,\delta) = \int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|^2 \sum_{j,k=1}^m a_{jk}(x)\frac{\partial |u|}{\partial x_j}\frac{\partial |u|}{\partial x_k}\,dx. \tag{11}$$

It follows from (10) that

$$\begin{aligned}&\mathrm{Re}\,I_p(u\,,\,\delta) - (p-2)\,\mathrm{Re}\,H_p(u\,,\,\delta)\\ &\quad= \int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-2}{2}} \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_k}\,dx,\end{aligned} \tag{12}$$

$$\mathrm{Im}\,I_p(u\,,\,\delta) = (p-2)\,\mathrm{Im}\,H_p(u\,,\,\delta). \tag{13}$$

Here we can prove that

$$\mathrm{Re}\,H_p(u\,,\,\delta) \leqq \int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|^2 \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_k}\,dx. \tag{14}$$

In fact, we have

$$\begin{aligned}|\,\mathrm{Re}\,H_p(u\,,\,\delta)|^2 &\leqq |H_p(u\,,\,\delta)|^2\\ &\leqq \left[\int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|^2 \times \left|\sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial |u|}{\partial x_k}\right| dx\right]^2.\end{aligned}$$

Since $(a_{jk}(x))$ is a nonnegative definite symmetric matrix, we can apply the Schwarz inequality:

$$\begin{aligned}&\left|\sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\partial |u|}{\partial x_k}\right|^2\\ &\quad\leqq \left(\sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_k}\right)\left(\sum_{j,k=1}^m a_{jk}(x)\frac{\partial |u|}{\partial x_j}\frac{\partial |u|}{\partial x_k}\right).\end{aligned}$$

Therefore it follows from (11) that

$$\begin{aligned}&|\,\mathrm{Re}\,H_p(u\,,\,\delta)|^2 \leqq |H_p(u\,,\,\delta)|^2\\ &\quad\leqq \left[\int_\Omega \left(|u(x)|^2 + \delta\right)^{\frac{p-4}{2}} |u(x)|^2 \sum_{j,k=1}^m a_{jk}(x)\left|\frac{\partial u}{\partial x_j}\frac{\overline{\partial u}}{\partial x_k}\right| dx\right]\mathrm{Re}\,H_p(u\,,\,\delta).\end{aligned} \tag{15}$$

Hence we obtain (14). Now let $p \geqq 2$. Then we see from (11), (12), and (1) that

$$\begin{aligned}\mathrm{Re}\,I_p(u\,,\,\delta) &\geqq C_0\int_\Omega |u(x)|^{p-2}|\,\mathrm{grad}\,u(x)|^2 dx\\ &\qquad + (p-2)C_0\int_\Omega |u(x)|^{p-2}|\,\mathrm{grad}\,|u(x)||^2 dx,\\ &= C_0\,\mathrm{Re}(-\Delta u\,,\,|u|^{p-2}u).\end{aligned}$$

Next let $1 < p < 2$. Then, since $p-2<0$, it follows from (12), (14), and (1) that

$$\begin{aligned}\operatorname{Re} I_p(u,\delta) &\geqq (p-1)\int_\Omega \left(|u(x)|^2+\delta\right)^{\frac{p-2}{2}} \sum_{j,k=1}^m a_{jk}(x)\frac{\partial u}{\partial x_j}\overline{\frac{\partial u}{\partial x_k}}\,dx \\ &\geqq (p-1)C_0\int_\Omega \left(|u(x)|^2+\delta\right)^{\frac{p-2}{2}} |\operatorname{grad} u(x)|^2\,dx.\end{aligned}$$

Thus we can obtain (6) and (7).

Now we estimate the $\operatorname{Im} I_p(u,\delta)$. If follows from (12), (13), and (15) that

$$\begin{aligned}(p-2)^{-2}|\operatorname{Im} I_p(u,\delta)|^2 &= |H_p(u,\delta)|^2 - |\operatorname{Re} H_p(u,\delta)|^2 \\ &\leqq \left[\operatorname{Re} I_p(u,\delta)-(p-2)\operatorname{Re} H_p(u,\delta)\right]\operatorname{Re} H_p(u,\delta) - |\operatorname{Re} H_p(u,\delta)|^2 \\ &\leqq |\operatorname{Re} H_p(u,\delta)|\cdot|\operatorname{Re} I_p(u,\delta)| - (p-1)|\operatorname{Re} H_p(u,\delta)|^2 \\ &\leqq \frac{1}{4(p-1)}|\operatorname{Re} I_p(u,\delta)|^2\end{aligned}$$

and hence

$$|\operatorname{Im} I_p(u,\delta)| \leqq \left(|p-2|/2\sqrt{p-1}\right)\operatorname{Re} I_p(u,\delta).$$

Going to the limit $\delta \downarrow 0$, we obtain (5).

3. Remarks

1. First we note that $\tan^{-1} x + \tan^{-1}(1/x) = \pi/2$ $(x>0)$. Consequently, we have

$$\omega_p = \tan^{-1}\left(\frac{|p-2|}{2\sqrt{p-1}}\right) = \frac{\pi}{2} - \tan^{-1}\left(\frac{2\sqrt{p-1}}{|p-2|}\right).$$

Set $\varphi_p := \frac{\pi}{2} - \omega_p$ $(1<p<\infty)$. Then φ_p is found in Fattorini [1, Theorem 4.3.1, p. 188]; note that

$$\frac{2\sqrt{p-1}}{|p-2|} = \left[\left(\frac{p}{p-2}\right)^2 - 1\right]^{1/2} \qquad (p\neq 2).$$

Fattorini considered a class of second order ordinary differential operators in $L^p(0,l)$, under various boundary conditions, as the infinitesimal generators of (C_0)-semigroups $\{S_p(\zeta)\}$ such that $S_p(\zeta)$ is analytic in the sector $|\arg\zeta| < \varphi_p$, and for $\omega>0$, $e^{-\omega\zeta}S_p(\zeta)$ is a contraction in a "smaller" sector. This is caused by the generality of the operators (see also Kato [5, Example V–3.34]).

2. Set $p' := p(p-1)^{-1}$. Then it follows that

$$\frac{|p'-2|}{2\sqrt{p'-1}} = \frac{|p-2|}{2\sqrt{p-1}} \qquad (1<p<\infty);$$

namely, the constant is the same in the adjoint space $L^{p'}(\Omega)$.

3. We have started with the operator A_p defined by (2) and (3). But the boundary condition was used just to get the equality (10). So we see that

the sectorial estimate (5) is true for those operators satisfying the boundary conditions which lead to (10).

4. The estimates (5) and (8) hold even if $\Omega = R^m$ and , for example, $A_p = -\Delta$ with $D(A_p) = W^{2,p}(R^m)$ and $\{T_p(t)\}$ is the semigroup generated by $-A_p = \Delta$; note that $C_0^\infty(R^m)$ is a core for A_p $(1 < p < \infty)$, and the (normalized) duality map is continuous (see Pascali-Sburlan [6, III–§2]). (8) may be used to improve an estimate in Hempel-Voigt [4, Theorem 2.2(c)]. In fact, they used Stein's estimate (9) in the proof.

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Added in proof. For the Laplacian, the constant $\tan\omega_p = (1/2)|p-2|/\sqrt{p-1}$ has already appeared in [10, p. 32] (see also [9] which was a preprint when the book [1] was being written). In this connection we note that the constant is best possible in the following sense: Suppose that there is a constant $C > 0$ such that

$$|\operatorname{Im}(-\Delta u, |u|^{p-2}u)| \leqq C \operatorname{Re}(-\Delta u, |u|^{p-2}u), \qquad u \in W^{2,p}(\Omega) \cap W_0^{1,p}(\Omega).$$

Substituting a suitable function in this inequality, we can obtain $C \geqq \tan\omega_p$. This fact was first pointed out by Prof. S. Miyajima (Science Univ. of Tokyo) in the case of $\Omega = R^2$. The author expresses his hearty thanks to Prof. Miyajima.

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