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PRODUCT SHIFTS ON $B(H)$

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Abstract. A shift on $B(H)$ is a *-endomorphism $\alpha$ for which $\bigcap_r \alpha^r(B(H)) = CP$ for some projection $P$. The paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

1. Introduction

Let $H$ be a separable Hilbert space and let $B(H)$ be the algebra of all bounded linear operators on $H$. Slightly extending the definition given in [7] to include the nonunital case, a *-endomorphism $\alpha$ of $B(H)$ will be called a shift if $\bigcap_r \alpha^r(B(H)) = CP$ for some (possibly zero) projection $P$ on $B(H)$. The present paper discusses some aspects of the classification of shifts on $B(H)$ up to conjugacy by *-automorphisms, with a focus on the shifts arising from an infinite tensor product decomposition of $H$.

2. Spatial Descriptions

It was observed in [2, Proposition 2.1] that every nonzero normal *-endomorphism $\alpha$ of $B(H)$ is implemented by a sequence of isometries. An alternative proof of this result can be obtained by imitating the usual proof that *-automorphisms of $B(H)$ are inner, as described, for example, in [5, Lemma 7.5.3], and this alternative proof shows that the normality assumption can be omitted (and hence every *-endomorphism of $B(H)$ is normal). For later reference we now state this minor extension of [2, Proposition 2.1].

Proposition 2.1. Let $\alpha$ be a nonzero *-endomorphism of $B(H)$. Then there is a (finite or infinite) sequence of isometries $V_1, V_2, \ldots$ in $B(H)$ having mutually orthogonal ranges such that

$$\alpha(A) = \sum_n V_n AV_n^*$$
for each $A \in B(H)$. The linear space of operators
\[ E_\alpha = \{ T \in B(H) : \alpha(A)T = TA \text{ for all } A \in B(H) \} \]
is a Hilbert space relative to the inner product defined by
\[ T^*S = \langle S, T \rangle \]
and \{ $V_1, V_2, \ldots$ \} is an orthonormal basis for $E_\alpha$.

The following simple result relates the conjugacy of $\alpha$ and $\beta$ to the Hilbert spaces $E_\alpha$ and $E_\beta$.

**Proposition 2.2.** Let $\alpha, \beta$ be $*$-endomorphisms of $B(H)$ and let $U \in B(H)$ be unitary. Then $\alpha = (\text{Ad } U)\beta(\text{Ad } U^*)$ if and only if $E_\alpha = UE_\beta U^*$.

**Proof.** If $\alpha = (\text{Ad } U)\beta(\text{Ad } U^*)$ then an easy calculation yields $E_\alpha = UE_\beta U^*$. This same result shows that if $E_\alpha = UE_\beta U^*$ then $E_\alpha = E_\gamma$ where $\gamma = (\text{Ad } U)\beta(\text{Ad } U^*)$: hence there exist orthonormal bases \{ $V_1, V_2, \ldots$ \} and \{ $W_1, W_2, \ldots$ \} of $E_\alpha$ with $\alpha(A) = \sum V_n AV_n^*$ and $\gamma(A) = \sum W_n AW_n^*$ for each $A \in B(H)$. A simple calculation using the unitary transition matrix between the two orthonormal bases then yields the conclusion $\alpha = \gamma$. \( \square \)

A natural question arising from Propositions 2.1 and 2.2 is the extent to which the specification of the isometries $V_\iota$ up to unitary equivalence determines the corresponding $*$-endomorphism up to conjugacy. The classification of an isometry $V$ up to unitary equivalence is given by the Wold decomposition, described, for example, in [4, Problem 118], which decomposes $V$ as a direct sum of copies of the unilateral shift and a unitary mapping on a subspace; when the unitary summand is absent the isometry is said to be pure. It will now be shown that if $\alpha$ is a shift, then at most one of any corresponding family of isometries is not pure; this result extends [1, Theorem 3] which is the case of two unitarily equivalent isometries. The proof is based on the following lemma in which Cuntz’s notation $V_\mu V_\nu^*$ is used to denote $V_{\mu(1)} \cdots V_{\mu(r)} V_{\nu(s)} \cdots V_{\nu(1)}^*$, where $\mu = (\mu(1), \ldots, \mu(r))$ has $r = |\mu|$ components and $\nu = (\nu(1), \ldots, \nu(s))$ has $s = |\nu|$ components.

**Lemma 2.3.** Let $V_1, V_2, \ldots$ be isometries from a Hilbert space $H$ onto a family of mutually orthogonal subspaces; $\alpha$ be the $*$-endomorphism defined by $\alpha(A) = \sum_n V_n^*AV_n$; and $h \in \bigcap_i V_i' H$.

(i) If $P_h$ is the orthogonal projection from $H$ onto the closed linear span $S$ of the vectors $V_\mu V_\nu^* h$ with $|\mu| \geq 0$ and $|\nu| \geq 0$, then $\alpha(P_h) = P_h$ and hence $P_h \in \bigcap_i \alpha'(B(H))$.

(ii) If $Q_h$ is the orthogonal projection from $H$ onto the closed linear span $T$ of the vectors $V_\mu V_\nu^* h$ with $|\mu| = |\nu| \geq 0$, then $Q_h \in \bigcap_i \alpha'(B(H))$.

**Proof.** (i) Let $|\mu| = r$, $|\nu| = s$, and $h = V_i^{s+1} k$, where $k \in H$. Then $\alpha(1) V_\mu V_\nu^* h = \sum V_n^* V_n V_\mu V_\nu^* V_i^{s+1} k = V_\mu V_\nu^* h$ (even if $r = 0$) and hence $\alpha(1) \geq P_h$. Since $S$ is invariant under both $V_n$ and $V_n^*$, it follows that $P_h V_n = V_n P_h$. 

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for each $n$ and hence $\alpha(P_h) = \sum V_n P_n V_n^* = P_h \sum V_n V_n^* = P_h \alpha(1) = P_h$.

(ii) Since $T$ is invariant under each $V_\gamma V_\delta^*$ with $|\gamma| = |\delta|$, it follows that $Q_h V_\gamma V_\delta^* = V_\gamma V_\delta^* Q_h$. Thus

$$\alpha'(V_\gamma^* Q_h V_\gamma) = \sum_{|\gamma|=r} V_\gamma V_1^* V_\gamma^* Q_h V_1^* V_\gamma^*$$

$$= \sum_{|\gamma|=r} V_\gamma V_1^* V_\gamma^* Q_h = \sum_{|\gamma|=r} V_\gamma V_\gamma^* Q_h$$

$$= \alpha'(1) Q_h.$$  

However, using part (i), $Q_h \leq P_h = \alpha'(P_h) \leq \alpha'(1)$ so $Q_h = \alpha'(V_1^* Q_h V_1^*)$ and hence $Q_h \in \bigcap_r \alpha'(B(H))$. □

**Proposition 2.4.** Let $V_1, V_2, \ldots$ be a sequence of isometries from a Hilbert space $H$ onto a family of orthogonal subspaces and let $\alpha$ be the $\ast$-endomorphism of $B(H)$ defined by $\alpha(A) = \sum_n V_n A V_n^*$. If $\alpha$ is a shift, then at most one of the isometries $V_n$ is not pure and if $V_n$ is not pure then the subspace $\bigcap_r V_n^* H$ associated with the unitary summand of $V_n$ is one-dimensional.

**Proof.** Suppose, to obtain a contradiction, that there exist nonzero vectors $h \in \bigcap_r V_n^* H$ and $k \in \bigcap_r V_j^* H$ with $\langle h, k \rangle = 0$. Then, by part (ii) of the lemma, there exist associated nonzero projections $Q_h, Q_k$ in $\bigcap_r \alpha'(B(H))$. Let $|\mu| = |\nu|$ and $|\delta| = |\gamma|$. The proof will be completed if it is shown that $\langle V_\mu V_\nu^* h, V_\gamma V_\delta^* k \rangle = 0$ for then $Q_h Q_k = 0$ and, therefore, $\alpha$ is not a shift. To this end let $V_\delta V_\gamma^* V_\nu V_\mu^* = V_\rho V_\sigma^*$; $|\rho| = |\sigma| = |\nu| = r = r'; h = V_{r'+1} h'$; and $k = V_{r'+1} k'$. Then $\langle V_\mu V_\nu^* h, V_\delta V_\gamma V_\delta^* k \rangle = \langle V_\rho V_\sigma^* h, k \rangle = \langle V_\sigma^* h, V_\rho^* k \rangle = \delta_{\sigma(1, \ldots, i), \delta(\rho(j, \ldots, \mu))} \langle V_i^* h', V_j^* k' \rangle$. But, when $i \neq j$, $\langle V_i^* h', V_j^* k' \rangle = \langle h', V_i^* V_j k' \rangle = 0$ and, when $i = j$, $\langle V_i^* h', V_j^* k' \rangle = \langle V_i^* V_i^* h, V_i^* k' \rangle = \langle h, V_i^{r'+1} k' \rangle = \langle h, k \rangle = 0$, as required. □

Using Proposition 2.4, it is possible to give a complete classification up to conjugacy of the simplest shifts $\alpha$, for which $E_\alpha$ is one-dimensional.

**Proposition 2.5.** (i) Let $V$ be an isometry on a separable infinite-dimensional Hilbert space $H$ and let $\alpha$ be the $\ast$-endomorphism of $B(H)$ defined by $\alpha(A) = V A V^*$. Then $\alpha$ is a shift if and only if $V$ is either a pure isometry or $\bigcap_r V^* H$ is one-dimensional.

(ii) Let $\alpha, \beta$ be shifts on $B(H)$ defined by $\alpha(A) = V A V^*$ and $\beta(A) = W A W^*$, where $V$ and $W$ are isometries on $H$. Then $\alpha$ is conjugate to $\beta$ if and only if $V$ is unitarily equivalent to $\lambda W$ for some $\lambda \in S^1$ i.e. if and only if $V$ and $W$ have the same Wold decomposition.

**Proof.** (i) By Proposition 2.4, the conditions on $V$ are necessary for $\alpha$ to be a shift. Conversely, if $V$ satisfies the given conditions, then $V$ can be assumed to be a direct sum of copies of the unilateral shift and (possibly) a unitary map on a
one-dimensional subspace. Let \( \{e_i : i \in \mathbb{N}\} \) be the orthonormal basis associated with one of the shift summands. If \( T \in \alpha(B(H)) \) then \( T = V'SV*' \) for some \( S \in B(H) \) and hence \( Te_i = 0 \) for \( 1 \leq i \leq r \). It follows that, if \( T \in \bigcap_r \alpha(B(H)) \) then the restriction of \( T \) to \( \bigcap_r V'rH \perp \) is 0. The same argument applies to \( T* \) and hence \( T \) leaves \( \bigcap_r V'rH \) globally invariant. Therefore \( \bigcap_r \alpha(B(H)) = CP \) where \( P \) is the projection from \( H \) onto the zero or one-dimensional space \( \bigcap_r V'rH \).

(ii) This follows from Proposition 2.2. \( \square \)

The case in which \( E_\alpha \) has dimension one is very special. For example, in contrast to the result of Proposition 2.5(i), if \( E_\alpha \) has dimension greater than one, then each isometry \( V \) in \( E_\alpha \) is onto a space of infinite codimension and hence must be equivalent to an infinite direct sum of copies of the unilateral shift, possibly with an additional one-dimensional unitary summand.

Later in the paper an example will be produced to show that it may occur that \( \alpha(A) = \sum V_nAV_n* = \sum W_nAW_n* \) where all the \( V_n \) are pure and one of the \( W_n \) is not. It will also be shown that the condition that all the \( V_n \) are pure is not sufficient to ensure that \( \alpha \) is a shift. Both these examples rely on the notion of a product shift, that will now be introduced.

### 3. Product shifts

Let \( K \) be a Hilbert space and, for each \( i \in \mathbb{N} \), let \( k_i \in K \) be a unit vector such that \( \sum_i |1 - \langle k_i, k_{i+1}\rangle| < \infty \). Recall that, as described in [3] or [6], the incomplete infinite tensor product \( \bigotimes^{k_i} K \) is the Hilbert space inductive limit of the sequence \( K \to K \otimes K \to K \otimes K \otimes K \to \cdots \), where the \( i \)-th map takes \( h \) to \( h \otimes k_{i+1} \). Let \( \phi_i \) be the natural map from the \( i \)-fold tensor product \( K \otimes K \otimes \cdots \otimes K \) into \( \bigotimes^{k_i} K \) and let \( \psi_i \) be the corresponding map from \( K \otimes K \otimes \cdots \otimes K \) into \( \bigotimes^{k_i+1} K \). Then, by [3, Proposition 1.1], for each \( h \) in the \( r \)-fold tensor product \( K \otimes K \otimes \cdots \otimes K \), the sequence \( \phi_i(h), \phi_{r+1}(h \otimes k_{r+2}), \phi_{r+2}(h \otimes k_{r+2} \otimes k_{r+3}), \cdots \) has a limit in \( \bigotimes^{k_i} K \) and, by [3, Proposition 1.3], the map taking \( \psi_i(h) \) to this limit extends to an isomorphism \( \theta \) from \( \bigotimes^{k_i+1} K \) onto \( \bigotimes^{k_i} K \). Informally, we can describe \( \theta \) as taking \( \bigotimes x_i \) (where \( x_i = k_{i+1} \) for almost all \( i \)) to \( \bigotimes x_i \) (interpreting the last expression as the limit of \( \phi_1(x_1), \phi_2(x_1 \otimes x_2), \cdots \)).

**Proposition 3.1.** Let \( P \) be a projection on \( K \); \( k_i \) be a sequence of unit vectors in \( K \) with \( \sum_i |1 - \langle k_i, k_{i+1}\rangle| < \infty \); and \( \theta \) be the isomorphism from \( \bigotimes^{k_i+1} K \) onto \( \bigotimes^{k_i} K \) taking \( \bigotimes x_i \) (where \( x_i = k_{i+1} \) for almost all \( i \)) to \( \bigotimes x_i \). Then the map \( \alpha_{k_i, P} \) defined on \( B(H) = B(\bigotimes^{k_i} K) \) by \( \alpha_{k_i, P}(T) = \theta(P \otimes T)\theta^{-1} \) is a shift.

**Proof.** The map \( \alpha_{k_i, P} \) is clearly a \(*\)-endomorphism. In the unital case, \( \alpha_{k_i, P}'(B(H))' = B(K \otimes \cdots \otimes K) \otimes \mathbb{C}1 \) (where the tensor product has \( r \) factors) and hence \( \bigcup_i \alpha_{k_i, P}'(B(H))' \) is weakly dense in \( B(H) \), from which it follows that \( \alpha_{k_i, P} \) is a shift. In the nonunital case let \( q \) be the greatest lower bound
of the projections $\alpha_{k,p}(1)$. If $q = 0$ then, since $\alpha_{k,p}(1) = T$ for each $T \in \bigcap \alpha_{k,p}(B(H))$, $T = qT = 0$ for all such $T$ and hence $\alpha_{k,p}$ is a shift. If $q \neq 0$ then $q\alpha_{k,1}(T) = q\alpha_{k,p}(T)q$ for all $T$ because, as can be checked by induction, $\alpha_{k,1}(T)\alpha_{k,p}(T) = \alpha_{k,p}(T)$. Thus, since $\alpha_{k,p}(q) = q$, $\alpha_{k,p}(qB(H)q) = q\alpha_{k,p}(B(H))q = q\alpha_{k,1}(B(H))q$ and so, when both algebras are restricted to $qH$, $[\alpha_{k,p}(qB(H))q]' = q\alpha_{k,1}(B(H))'q$. Hence, as in the unital case, $\bigcap r\alpha_{k,p}(qB(H)) = Cq$; however, $\bigcap r\alpha_{k,p}(B(H)) \subseteq qB(H)q$ and, therefore, $\bigcap r\alpha_{k,p}(B(H)) = Cq$, as required. □

If there exists a unitary $U$ from a Hilbert space $H$ onto an infinite tensor product $\bigotimes K$ with the properties above, then any *-endomorphism $\alpha$ of $B(H)$ of the form $(Ad U^*)\alpha_{k,p}(Ad U)$ will be called a product shift on $B(H)$.

Not every shift on $B(H)$ is a product as can be seen from Proposition 2.4. Any product shift $\alpha_{k,p}$ has $\alpha_{k,p}(1)$ of infinite codimension whereas if $\alpha$ is the shift corresponding to a single unilateral shift $V$ then $\alpha(1)$ has codimension 1. Nevertheless the author is not aware of examples of nonproduct unital shifts on $B(H)$: the following proposition shows that any sequence of isometries defining such a shift must be all pure.

Proposition 3.2. Let $\alpha$ be a unital shift on $B(H)$ defined by $\alpha(A) = \sum \lambda_n A V_n^*$ where $V = V_1$ is not pure. Then $\alpha$ is a product shift.

Proof. Let $h$ be a unit vector in $\bigcap V'H$ and note that, by Proposition 2.4, $\bigcap V'H$ is one-dimensional. It follows that $Vh = \lambda h$ for some $\lambda \in S^1$ and, replacing $V$ by $\lambda V$, we can assume that $Vh = h$.

Let $c_{00}$ denote the space of functions $f: \mathbb{N} \to \mathbb{N}$ such that $f(r) = 1$ for all sufficiently large $r$ and, for each $f \in c_{00}$, let $b_f = \phi_s(V_f(1) \otimes \cdots \otimes V_f(s)) \in \bigotimes V E$, where $\phi_s$ is the natural map from the $s$-fold tensor product $E \otimes \cdots \otimes E$ into $\bigotimes V E$ and where $s$ is chosen sufficiently large so that $f(r) = 1$ for all $r \geq s$. Then, by [6, Lemma 4.1.4], $\{b_f: f \in c_{00}\}$ is an orthonormal basis for $\bigotimes V E$. Define $\psi: \bigotimes V E \to H$ extending $\psi(b_f) = V_f(1) \cdots V_f(s)h$ (which is well defined since $Vh = h$). Then, for $t \geq s$, $\langle \psi(b_r), \psi(b_s) \rangle = \langle V_f(1) \cdots V_f(s)h, V_g(1) \cdots V_g(t)h \rangle = \delta_{f,g}(1) \cdots \delta_{f,g}(s) \delta_{g,(s+1)}(V_f^* \cdots V_g^* h, h) = \delta_{f,g}(h, h) = \delta_{f,g}$ so that $\psi$ defines a unitary mapping from $\bigotimes V E$ onto a subspace $K$ of $H$.

For each $\mu, \nu$ with $|\mu| \geq 0$ and $|\nu| = r \geq 0$, $V_\mu V_\nu^* h = V_\mu V_\nu^* V_\nu^* h = \delta_{(1,\ldots,1,1)} V_\mu^* h$, so that $K$ is equal to the closed linear span of the vectors $V_\mu V_\nu^* h$ with $|\mu| \geq 0$ and $|\nu| \geq 0$. By Lemma 2.3 (i) it follows that the projection onto $K$ belongs to $\bigcap r\alpha(B(H))$ and hence, since $\alpha$ is a unital shift, $K = H$.

Note that
\[
(\psi^* V_n \psi)(\phi_r(V_f(1) \otimes \cdots \otimes V_f(r))) = \psi^* V_n V_f(1) \cdots V_f(r)h = \phi_{r+1}(V_n \otimes V_f(1) \otimes \cdots \otimes V_f(r)),
\]
from which it follows that
\[(\psi^* V_n^* \psi)(\phi_r(V_{f(1)} \otimes \cdots \otimes V_{f(s)})) = (V_{f(1)}, V_n)\phi_{r-1}(V_{f(2)} \otimes \cdots \otimes V_{f(r)});\]
hence
\[(\text{Ad } \psi^*)\alpha(\text{Ad } \psi)(T_1 \otimes T_2 \otimes \cdots \otimes T_s \otimes 1) = \sum_n (\psi^* V_n \psi)(T_1 \otimes \cdots \otimes T_s \otimes 1)(\psi^* V_n \psi)^* = 1 \otimes T_1 \otimes \cdots \otimes T_s \otimes 1\]
and thus that \(\alpha\) is a product shift. \(\square\)

It will now be shown how product shifts can be defined to illustrate the comments made after Proposition 2.5.

Example 3.3. Let \(K = \mathbb{C}^2\); \(k_i = (1, 0)\) for each \(i\); and the isometries \(V_1, V_2, W_1, W_2\) be defined on \(H = \bigotimes^h_k K\) by
\[
V_1(h) = (1/\sqrt{2}, 1/\sqrt{2}) \otimes h, \quad V_2(h) = (1/\sqrt{2}, -1/\sqrt{2}) \otimes h,
\]
\[
W_1(h) = (1, 0) \otimes h, \quad W_2(h) = (0, 1) \otimes h.
\]
Then \(\alpha_{k,1}(A) = V_1 A V_1^* + V_2 A V_2^* = W_1 A W_1^* + W_2 A W_2^*\) for each \(A \in B(H)\). However, both \(V_1\) and \(V_2\) are pure isometries (as is \(W_2\)) but \(\bigcap_n W_1^n H = \mathbb{C} k\), where \(k = \bigotimes(1, 0)\). \(\square\)

Example 3.4. Let \(H, V_1, V_2\), and \(\alpha_{k,1}\) be as defined in Example 3.3 and let \(\mathcal{H}\) be any Hilbert space. Then \(\alpha = \alpha_{k,1} \otimes \text{id}\) on \(B(H) \otimes B(\mathcal{H})\) is not a shift; however,
\[
\alpha(A) = (V_1 \otimes 1)A(V_1^* \otimes 1) + (V_2 \otimes 1)A(V_2^* \otimes 1)
\]
for each \(A\), and both \(V_1 \otimes 1\) and \(V_2 \otimes 1\) are pure isometries. Hence the converse of Proposition 2.4 is false. \(\square\)

The next main result gives a criterion for two unital product shifts to be conjugate. The proof is based on the following lemma:

Lemma 3.5. Let \(W : \bigotimes^k K \to \bigotimes^h K\) be a unitary for which \(\text{Ad } W)\alpha_{k,1}(\text{Ad } W^*) = \alpha_{h,1}\). Then there exists a unitary \(V\) on \(K\) such that, for each \(r \geq 1\) and each \(T_1, \ldots, T_r \in B(K)\), \(\text{Ad } W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1 V^*) \otimes (VT_2 V^*) \otimes \cdots \otimes (VT_r V^*) \otimes 1\).

Proof. For notational convenience, let \(H_h = \bigotimes^h K\) and \(H_k = \bigotimes^k K\). Then \(\alpha_{h,1}(B(H_h))^t = \text{Ad } W)([\alpha_{k,1}(B(H_k))^t]\) and hence, for each \(T \in B(K)\),
\[
(\text{Ad } W)(T \otimes 1 \otimes 1 \otimes \cdots) = VTV^* \otimes 1 \otimes 1 \otimes \cdots
\]
for some unitary \(V\) on \(K\). Then, for each \(r \geq 0\),
\[
(\text{Ad } W)\alpha_{k,1}^t(T \otimes 1 \otimes 1 \otimes \cdots) = \alpha_{h,1}(\text{Ad } W)(T \otimes 1 \otimes 1 \otimes \cdots)
\]
\[
= \alpha_{h,1}^t(VTV^* \otimes 1 \otimes 1 \otimes \cdots).
\]
Hence, for each $r \geq 0$, \((\text{Ad} \, W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^*) \otimes (VT_2V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1\). \hspace{1cm} \Box

**Theorem 3.6.** The product shifts $\alpha_{k,1}$ and $\alpha_{h,1}$ are conjugate if and only if there exists a unitary map $V$ on $K$ with $\sum |1 - |\langle V k_i, h_j \rangle|| < \infty$.

**Proof.** If $\sum |1 - |\langle V k_i, h_j \rangle|| < \infty$ then, by [3, Proposition 1.3], the map taking $\otimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\otimes x_i$ extends to an isomorphism from $\otimes^k K$ onto $\otimes^h K$, where $\alpha_j$ is the complex number of unit modulus such that $\langle k_i, V^*h_j \rangle = \langle k_i, \alpha_j h_j \rangle$. However there is an isomorphism from $\otimes^a V^* h_j K$ onto $\otimes^h K$ taking $\otimes x_i$ to $\otimes \alpha_i V x_i$ and hence there is an isomorphism $W$ from $\otimes^k K$ onto $\otimes^h K$ taking $\otimes x_i$ (where $x_i = k_i$ for almost all $i$) to $\otimes \alpha_i V x_i$. Then, for $T_1, \ldots, T_r \in B(K)$,

$$((\text{Ad} \, W)\alpha_{k,1}((\text{Ad} \, W^*) (T_1 \otimes \cdots \otimes T_r \otimes 1) = \alpha_{h,1} (T_1 \otimes \cdots \otimes T_r \otimes 1),$$

from which it follows, by linearity and continuity that $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$.

Conversely, if $\alpha_{k,1}$ is conjugate to $\alpha_{h,1}$ then, by Lemma 3.5, there exists an isomorphism $W$ from $\otimes^k K$ onto $\otimes^h K$ and a unitary $V$ on $K$ such that

$$(\text{Ad} \, W)(T_1 \otimes \cdots \otimes T_r \otimes 1) = (VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1 \text{ for all } T_1, \ldots, T_r \in B(K).$$

In particular, this equation holds when $T_i$ is the one-dimensional projection onto the subspace spanned by $k_i$. In this case, the decreasing sequence of projections $T_1 \otimes \cdots \otimes T_r \otimes 1$ is weakly convergent as $r \to \infty$; by considering the effect of $T_1 \otimes \cdots \otimes T_r \otimes 1$ on an orthonormal basis $\{e_{i_1} \otimes e_{i_2} \otimes \cdots \otimes i_1, i_2, \ldots \in \mathbb{N}\}$ (where $e_{r_1} = k_r$ for each $r$ and $i_k = 1$ for all sufficiently large $k$) it can be seen that the weak limit is the one-dimensional projection onto the subspace spanned by $k = \otimes k_i$. Hence the projections $(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1$ are weakly convergent to a one-dimensional projection in $\otimes^h K$; let $x$ be a unit vector in the corresponding one-dimensional subspace. If $\phi_r$ denotes the natural map from the $r$-fold tensor product $K \otimes \cdots \otimes K$ into $\otimes^h K$ then there exists $s \in \mathbb{N}$ and a unit vector $x_s$ in $\phi_s(K \otimes \cdots \otimes K)$ such that $\|x - x_s\| < \frac{1}{2}$. Then, for each $r$,

$$\|(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1\| x_s \| \geq \|x\| - \|x - [(VT_1V^*) \otimes \cdots \otimes (VT_rV^*) \otimes 1] x_s \| > \frac{1}{2}.$$

It follows that $\prod_{i=m+1}^{r} |\langle k_i, V^* h_j \rangle| \to 0$ as $r \to \infty$ and hence, by Lemma 2.4.1 (II) of [6], $\sum |1 - |\langle k_i, V^* h_j \rangle|| < \infty$, as required. \hspace{1cm} \Box

Theorem 3.6 can be used to produce examples of nonconjugate but outer-conjugate unital shifts. Let $\log^k$ denote the $k$-fold composite $\log \circ \cdots \circ \log$ and, for each $r \in \mathbb{N}$, define a sequence $f_r$ by

$$f_r(n) = \begin{cases} 1 & \text{if } \log^r(n) \leq 0, \\ 1/(n \log(n) \log^2(n) \cdots \log^r(n)) & \text{if } \log^r(n) > 0. \end{cases}$$
Let $K$ be a separable Hilbert space (of dimension at least 2) with an orthonormal basis $\{e_i\}$ and, for each $n \in \mathbb{N}$, define $k_{r,n} \in K$ by $k_{r,n} = (1 - f_r(n))e_1 + [f_r(n)(2 - f_r(n))]^{1/2}e_2$.

**Lemma 3.7.**

(i) $\|k_{r,n}\| = 1$ for each $r$ and $n$.

(ii) $\sum_n |1 - \langle k_{r,n}, k_{r,n+1} \rangle| < \infty$ for each $r$.

**Proof.** Claim (i) is easily verified. To check claim (ii), note that

$$o < |1 - \langle k_{r,n}, k_{r,n+1} \rangle| < \|k_{r,n}\|\|k_{r,n+1}\| = 1$$

and hence that

$$0 \leq 1 - \langle k_{r,n}, k_{r,n+1} \rangle = f_r(n) + f_r(n+1) - f_r(n)f_r(n+1) - [f_r(n)f_r(n+1)(2 - f_r(n))(2 - f_r(n+1))]^{1/2}$$

$$\leq 2f_r(n) - f_r(n)f_r(n+1) - 2f_r(n+1) = 2(f_r(n) - f_r(n+1)).$$

However $\sum_n 2(f_r(n) - f_r(n+1))$ is convergent, from which the result follows. \(\Box\)

**Proposition 3.8.** For each $r \in \mathbb{N}$, let $\alpha_r$ be the product shift $\alpha_{k_{r,1}}$ on $\prod_{n=1}^{\infty} k_{r,n} K$, where $k_{r,n}$ is defined above. Then, if $r \neq s$, $\alpha_r$ is outer conjugate but not conjugate to $\alpha_s$.

**Proof.** For definiteness suppose that $r > s$. By construction $\alpha_r$ and $\alpha_s$ have the same multiplicity $\dim(K)$ and hence are outer conjugate by [7, Theorem 2.4]. If they are conjugate then, by Theorem 3.6, there exists a unitary $V$ on $K$ such that $\sum_n |1 - \langle V k_{r,n}, k_{s,n} \rangle| < \infty$. It follows that $\langle V k_{r,n}, k_{s,n} \rangle \to 1$ as $n \to \infty$ and hence that $\langle V e_1, e_1 \rangle = 1$, so that $Ve_1 = \alpha e_1$ for some $\alpha \in S^1$.

Hence,

$$\langle V k_{r,n}, k_{s,n} \rangle = |(1 - f_r(n))(1 - f_s(n))\langle \alpha e_1, e_1 \rangle + [f_r(n)f_s(n)(2 - f_r(n))(2 - f_s(n))]^{1/2}\langle Ve_2, e_2 \rangle|$$

$$\leq (1 - f_r(n))(1 - f_s(n)) + 2f_s(n)^{1/2}f_r(n)^{1/2}$$

$$= 1 - f_s(n)[1 - f_r(n) + f_r(n)f_s(n)-1 - 2f_r(n)^{1/2}f_s(n)^{-1/2}].$$

However, for all sufficiently large $n$, $f_r(n)f_s(n)^{-1} = 1/(\log^{s+1}(n) \cdots \log^r(n)) \to 0$ as $n \to \infty$ and therefore, for all sufficiently large $n$, $|\langle V k_{r,n}, k_{s,n} \rangle| \leq 1 - f_s(n)/2$ and therefore, $\sum |1 - |\langle V k_{r,n}, k_{s,n} \rangle||$ is not convergent, giving a contradiction. \(\Box\)

**References**


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