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"LEBESGUE MEASURE" ON $\mathbf{R}^\infty$

RICHARD BAKER

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ABSTRACT. We construct a translation invariant Borel measure λ on $\mathbf{R}^\infty = \prod_{i=1}^\infty \mathbf{R}$ such that for any infinite-dimensional rectangle $R = \prod_{i=1}^\infty (a_i, b_i)$, $-\infty < a_i \le b_i < +\infty$, if $0 \le \prod_{i=1}^\infty (b_i - a_i) < +\infty$, then $\lambda(R) = \prod_{i=1}^\infty (b_i - a_i)$. Because $\mathbf{R}^\infty$ is an infinite-dimensional locally convex topological vector space, the measure λ can not be σ-finite.

1. INTRODUCTION

If $\mathscr{X}$ is any infinite-dimensional locally convex topological vector space, then it is known that there does not exist a nontrivial translation invariant σ-finite Borel measure on $\mathscr{X}$ [2, p. 143]. Because of this impossibility theorem, it is part of the folklore of infinite-dimensional functional analysis that there is no satisfactory analogue of Lebesgue measure on the space $\mathbf{R}^\infty = \prod_{i=1}^\infty \mathbf{R}$. In this paper, without upsetting this folklore, we show that there exists a nontrivial translation invariant Borel measure λ on $\mathbf{R}^\infty$ which is analogous to Lebesgue measure in the sense that if $R = \prod_{i=1}^\infty (a_i, b_i)$ is any infinite-dimensional rectangle such that the "volume" $\prod_{i=1}^\infty (b_i - a_i)$ of R is a nonnegative real number, then

$$\lambda(R) = \prod_{i=1}^\infty (b_i - a_i).$$

Therefore, it would be appropriate to call λ *infinite-dimensional Lebesgue measure on* $\mathbf{R}^\infty$. Of course, according to the above no-go result, the measure λ can not be σ-finite.

The following theorem is the main result of this paper.

Theorem I. *Let $\mathscr{R}$ be the class of all infinite-dimensional rectangles $R \in \mathbf{R}^\infty$ of the form*

$$R = \prod_{i=1}^\infty (a_i, b_i), \qquad -\infty < a_i \le b_i < +\infty,$$

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such that $0 \leq \prod_{i=1}^{\infty}(b_i - a_i) < +\infty$. *Let* τ *be the set of function on* $\mathscr{R}$ *defined by*

$$\tau(R) = \prod_{i=1}^{\infty}(b_i - a_i), \qquad R \in \mathscr{R}.$$

Finally, let λ *be the set function defined on all subsets of* $\mathbf{R}^{\infty}$ *by*

$$\lambda(E) = \inf_{\substack{R_j \in \mathscr{R} \\ \bigcup R_j \supseteq E}} \sum_{j=1}^{\infty} \tau(R_j), \qquad E \subseteq \mathbf{R}^{\infty}.$$

We adopt the convention that any infimum taken over an empty set of real numbers has the value $+\infty$. *Then* λ *is a translation invariant Borel measure on* $\mathbf{R}^{\infty}$ *such that for all* $R = \prod_{i=1}^{\infty}(a_i, b_i) \in \mathscr{R}$, *we have*

$$\lambda(R) = \prod_{i=1}^{\infty}(b_i - a_i).$$

The plan of the paper is as follows. In §2 we present the preliminary measure-theoretic machinery that we will need for the construction of λ. In §3 we prove Theorem I.

2. Preliminaries

The purpose of this section is to collect together in one place the measure-theoretic facts that we will need for the construction of the measure λ. All of the material in this section has been transcribed from Chapter 1 of [1].

Throughout this section Ω will denote an arbitrary nonempty set. Definition 2.1 and Theorem 2.2 below give a general method for constructing outer measures on Ω.

Definition 2.1. A function τ on a class $\mathscr{C}$ of subsets of Ω will be call a *pre-measure*, if:

(a) $\varnothing \in \mathscr{C}$;
(b) $0 \leq \tau(C) \leq +\infty$ for all C in $\mathscr{C}$;
(c) $\tau(\varnothing) = 0$.

Theorem 2.2. *If* τ *is a pre-measure defined on a class* $\mathscr{C}$ *of subsets of* Ω, *the set function*

$$\mu(E) = \inf_{\substack{C_j \in \mathscr{C} \\ \bigcup C_j \supseteq E}} \sum_{j=1}^{\infty} \tau(C_j)$$

is an outer measure on Ω.

Theorem 2.3. *Let* μ *be an outer measure on* Ω. *Then* μ *is the outer measure, constructed via Theorem* 2.2, *from the pre-measure* τ, *defined on the class of all subsets of* Ω *to coincide with* μ.

Now assume that Ω is a metric space with metric ρ. The next theorem gives another general method of constructing an outer measure on Ω.

Theorem 2.4. *If τ is a pre-measure defined in a class $\mathscr{C}$ of sets, in a metric space Ω with metric ρ, the set function*

$$\mu(E) = \sup_{\delta>0} \mu_\delta(E),$$

where

$$\mu_\delta(E) = \inf_{\substack{C_j \in \mathscr{C},\ d(C_j) \le \delta \\ \bigcup C_j \supseteq E}} \sum_{j=1}^{\infty} \tau(C_j)$$

is an outer measure on Ω. Here $d(C)$ is the diameter of C with respect to ρ, $C \subseteq \mathbf{R}^\infty$.

The last two theorems of this section, Theorems 2.7 and 2.8, state that the outer measure μ constructed in Theorem 2.4 has the desirable property that every Borel subset of Ω is μ-measurable.

Definition 2.5. If A and B are disjoint nonempty sets in a metric space Ω with metric ρ, A and B are said to be *positively separated* if the distance

$$\inf_{a \in A,\ b \in B} \rho(a, b)$$

separating A and B is positive.

Definition 2.6. An outer measure μ defined on a metric space Ω is called a *metric outer measure*, if

$$\mu(A \cup B) = \mu(A) + \mu(B),$$

for every pair of disjoint nonempty sets A, B that are positively separated.

Theorem 2.7. *if μ is a metric outer measure on a metric space Ω, then all Borel sets in Ω are μ-measurable. Recall that a subset $E \subseteq \Omega$ is μ-measurable if for all sets $A \subseteq E$, $B \subseteq \Omega \backslash E$, we have $\mu(A \cup B) = \mu(A) + \mu(B)$; moreover, the system $\mathscr{M}$ of μ-measurable sets is a σ-field containing the null sets, and the restriction of μ to $\mathscr{M}$ is countably additive on $\mathscr{M}$.*

Theorem 2.8. *If τ is pre-measure defined on a class $\mathscr{C}$ of subsets of a metric space Ω, and if μ is the outer measure on Ω constructed from τ via Theorem 2.4, then μ is a metric outer measure on Ω.*

3. The measure λ on $\mathbf{R}^\infty$

In this section we construct an outer measure λ on $\mathbf{R}^\infty$ such that λ has all the properties stated in Theorem I.

Let $\mathscr{R}$, τ, and λ be defined as in Theorem I. Then by Theorem 2.2, λ is an outer measure on $\mathbf{R}^\infty$. The first two theorems of this section establish the key fact that for all $R \in \mathscr{R}$, $\tau(R) = \lambda(R)$.

Theorem 3.1. *Let* $I = \prod_{i=1}^{\infty}[a_i, b_i]$, $-\infty < a_i \le b_i < +\infty$, *be an infinite-dimensional compact rectangle in* $\mathbf{R}^{\infty}$ *such that* $0 \le \prod_{i=1}^{\infty}(b_i - a_i) < +\infty$, *then*

$$\prod_{i=1}^{\infty}(b_i - a_i) \le \lambda(I).$$

Proof. It will suffice to show that if $I \subseteq \bigcup_{j=1}^{\infty} R_j$, where $R_j \in \mathscr{R}$, then

$$(*) \qquad \tau(I) \le \sum_{j=1}^{\infty} \tau(R_j),$$

where $\tau(I) = \prod_{i=1}^{\infty}(b_i - a_i)$. We may assume that $0 < \tau(I)$ and $\sum_{j=1}^{\infty} \tau(R_j) < \infty$. For all $j, n \ge 1$, let

$$R_j = \prod_{i=1}^{\infty}(a_{ij}, b_{ij}), \qquad R_{nj} = \prod_{i=1}^{n}(a_{ij}, b_{ij}) \times \prod_{i=n+1}^{\infty} \mathbf{R}.$$

Now let $0 < \varepsilon < 1$ be arbitrary, and let $\mathscr{F}$ be the family of all those R_{nj} such that $\prod_{i=n+1}^{\infty}(b_i - a_i) < 1 + \varepsilon$ and either $\prod_{i=n+1}^{\infty}(b_{ij} - a_{ij}) > 1 - \varepsilon$ or $\prod_{i=1}^{n}(b_{ij} - a_{ij}) < \varepsilon/2^j$. Because $0 < \tau(I) < +\infty$ and $0 \le \tau(R_j) < +\infty$, it is clear that $\mathscr{F}$ covers I. The members of $\mathscr{F}$ are open and I is compact, hence there exists a finite subfamily $\{R_{n_p j_p} | 1 \le p \le k\}$ of $\mathscr{F}$ that covers I. For $n > \max\{n_1, \dots, n_k\}$ and $1 \le p \le k$, let

$$I_n = \prod_{i=1}^{n}[a_i, b_i], \qquad S_{np} = \prod_{i=1}^{n_p}(a_{ij_p}, b_{ij_p}) \times \prod_{i=n_p+1}^{n}[a_i, b_i].$$

Because $I \subseteq \bigcup_{p=1}^{k} R_{n_p j_p}$, we have $I_n \subseteq \bigcup_{p=1}^{k} S_{np}$. Let λ_n be Lebesgue measure on $\mathbf{R}^n$, then we have

$$\begin{aligned}\prod_{i=1}^{n}(b_i - a_i) = \lambda_n(I_n) &\le \sum_{p=1}^{k} \lambda_n(S_{np}) \\ &= \sum_{p=1}^{k} \left\{ \prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}) \cdot \prod_{i=n_p+1}^{n}(b_i - a_i) \right\}.\end{aligned}$$

Taking the limit $n \to \infty$, we get

$$\begin{aligned}\tau(I) &\le \sum_{p}^{k} \left\{ \prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}) \cdot \prod_{i=n_p+1}^{\infty}(b_i - a_i) \right\} \\ &\le (1+\varepsilon) \sum_{p=1}^{k} \prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}) \\ &\le (1+\varepsilon) \sum_{1 \le p \le k}{}' \prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}) + (1+\varepsilon) \sum_{1 \le p \le k}{}'' \prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}),\end{aligned}$$

where $\sum'$ is the sum over those p for which $\prod_{i=n_p+1}^{\infty}(b_{ij_p} - a_{ij_p}) > 1 - \varepsilon$ and $\sum''$ is the sum over those p for which $\prod_{i=1}^{n_p}(b_{ij_p} - a_{ij_p}) < \varepsilon/2^{j_p}$. It follows that

$$\begin{aligned}\tau(I) &\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sum_{1\le p\le k}{}' \tau(R_{j_p}) + (1+\varepsilon)\varepsilon\sum_{1\le p\le k}{}''(1/2^{j_p})\\ &\le \left(\frac{1+\varepsilon}{1-\varepsilon}\right)\sum_{j=1}^{\infty}\tau(R_j) + (1+\varepsilon)\varepsilon.\end{aligned}$$

Because ε is arbitrary, we see that $(*)$ holds. □

Theorem 3.2. *For every $R \in \mathscr{R}$, we have $\tau(R) = \lambda(R)$.*

Proof. Let $R \in \mathscr{R}$. It is clear that $\lambda(R) \le \tau(R)$, and hence we may assume that $\tau(R) > 0$. Let $0 < \varepsilon < 1$ be arbitrary. Then it is not hard to see that there exists a compact rectangle $I = \prod_{i=1}^{\infty}[a_i, b_i] \subseteq R$ such that $\prod_{i=1}^{\infty}(b_i - a_i) = (1-\varepsilon)\tau(R)$. By Theorem 3.1, $\prod_{i=1}^{\infty}(b_i - a_i) \le \lambda(I)$, therefore we have $(1-\varepsilon)\tau(R) \le \lambda(R)$. Because ε is arbitrary, this proves the theorem. □

Definition 3.3. Let ρ be the metric on $\mathbf{R}^\infty$ defined as

$$\rho(x, y) = \sum_{i=1}^{\infty}\frac{(1/2^i)|x_i - y_i|}{\{1 + |x_i - y_i|\}}, \qquad x = (x_i), y = (y_i) \in \mathbf{R}^\infty.$$

Observe that ρ induces the product topology on $\mathbf{R}^\infty$.

Theorems 3.4 and 3.5 below, show that λ has the important property of being a metric outer measure on $\mathbf{R}^\infty$.

Theorem 3.4. *Let ν be the outer measure on $\mathbf{R}^\infty$ constructed from the pair τ, ρ via Theorem 2.4. Then for all $R \in \mathscr{R}$, we have $\nu(R) = \tau(R)$.*

Proof. Let $R \in \mathscr{R}$. By Theorem 3.2, $\lambda(R) = \tau(R)$. It is clear that $\lambda(R) \le \nu(R)$, hence it will suffice to show that

$$(**) \qquad \nu(R) \le \tau(R).$$

Write

$$R = \prod_{i=1}^{\infty}(a_i, b_i), \qquad -\infty < a_i \le b_i < +\infty.$$

If $R = \varnothing$, then $\nu(R) = \tau(R) = 0$, hence we may assume that $R \ne \varnothing$. Then for all i, $a_i < b_i$. Therefore, if $\tau(R) = 0$, then for all n, we have $\prod_{i=n+1}^{\infty}(b_i - a_i) = 0$. On the other hand, if $\tau(R) > 0$, then $\lim_{n\to\infty}\prod_{i=n+1}^{\infty}(b_i - a_i) = 1$. Let $\delta, \varepsilon > 0$ be arbitrary, then there exists an $n \ge$ such that $\prod_{i=n+1}^{\infty}(b_i - a_i) < 1 + \varepsilon$ and $\sum_{i=1+n}^{\infty} 2^{-i} < \delta/2$. Define $R_n = \prod_{i=1}^{n}(a_i, b_i)$. For $x = (x_i)_{i=1}^n$, $y = (y_i)_{i=1}^n \in \mathbf{R}^n$, let $\rho_n(x, y)$ be defined as

$$\rho_n(x, y) = \sum_{i=1}^{n}\left(\frac{1}{2^i}\right)\frac{|x_i - y_i|}{\{1 + |x_i - y_i|\}}.$$

Cover the rectangle R_n by rectangles $R_{n1}, \dots, R_{nm}$ in $\mathbf{R}^n$ such that

(a) For $1 \le j \le m$, $R_{nj} = \prod_{i=1}^n (a_{ij}, b_{ij})$, $-\infty < a_{ij} \le b_{ij} < +\infty$.

(b) For $1 \le j \le m$, $\sup_{x, y \in R_{nj}} \|x - y\| < \delta/2n$, where $\|\cdot\|$ is the Euclidean norm on $\mathbf{R}^n$.

(c) $\sum_{j=1}^m \lambda_n(R_{nj}) < \prod_{i=1}^n (b_i - a_i) + \varepsilon$.

For $1 \le j \le m$, define $R_j = R_{nj} \times \prod_{i=n+1}^\infty (a_i, b_i)$. Let $x = (x_i)$, $y = (y_i) \in R_j$, and define $x^{(n)} = (x_i)_{i=1}^n$, $y^{(n)} = (y_i)_{i=1}^n$, then, by (b), we have

$$\begin{aligned}
\rho_n(x, y) &= \rho_n(x^{(n)}, y^{(n)}) + \sum_{i=n+1}^\infty \left(\frac{1}{2^i}\right) \frac{|x_i - y_i|}{\{1 + |x_i - y_i|\}} \\
&< \rho_n(x^{(n)}, y^{(n)}) + \delta/2 \\
&\le n\|x^{(n)} - y^{(n)}\| + \delta/2 \\
&< \delta.
\end{aligned}$$

Hence, for all $1 \le j \le m$, we have $d(R_j) < \delta$. It is clear that $R \subseteq \bigcup_{j=1}^m R_j$, hence by definition, $\nu_\delta(R) \le \sum_{j=1}^m \tau(R_j)$. By (c),

$$\begin{aligned}
\sum_{j=1}^m \tau(R_j) &= \sum_{j=1}^m \left\{ \lambda_n(R_{nj}) \cdot \prod_{i=n+1}^\infty (b_i - a_i) \right\} \\
&\le \left\{ \prod_{i=1}^n (b_i - a_i) + \varepsilon \right\} \cdot \prod_{i=n+1}^\infty (b_i - a_i) \\
&= \tau(R) + \varepsilon \prod_{i=n+1}^\infty (b_i - a_i) \\
&< \tau(R) + \varepsilon(1 + \varepsilon).
\end{aligned}$$

Because ε is arbitrary, we get $\nu_\delta(R) \le \tau(R)$. But δ is also arbitrary, hence $\nu(R) \le \tau(R)$. Therefore $(**)$ holds. □

Theorem 3.5. *We have $\lambda(E) = \nu(E)$, for all $E \subseteq \mathbf{R}^\infty$. Hence, by Theorem* 2.8, *λ is a metric outer measure on $\mathbf{R}^\infty$.*

Proof. For $E \subseteq \mathbf{R}^\infty$, we have $\lambda(E) \le \nu(E)$, hence it suffices to prove that

$$(\dagger) \qquad \nu(E) \le \lambda(E), \qquad E \subseteq \mathbf{R}^\infty.$$

Fix $E \subseteq \mathbf{R}^\infty$. By Theorem 2.3, we have

$$\nu(E) = \inf_{\substack{C_j \subseteq \mathbf{R}^\infty \\ \bigcup C_j \supseteq E}} \sum_{j=1}^\infty \nu(C_j).$$

Hence we see that

$$\nu(E) \le \inf_{\substack{R_j \in \mathscr{R} \\ \bigcup R_j \supseteq E}} \sum_{j=1}^\infty \nu(R_j).$$

For all $R \in \mathscr{R}$, Theorem 3.5 implies that $\nu(R) = \tau(R)$, therefore we have $\nu(E) \leq \lambda(E)$. This proves (†). □

Theorem 3.6. *The outer measure λ is translation invariant on $\mathbf{R}^\infty$.*

Proof. This follows from the fact that if $R = \prod_{i=1}^\infty (a_i, b_i) \in \mathscr{R}$ and $x \in \mathbf{R}^\infty$, then $R + x \in \mathscr{R}$ and $\tau(R + x) = \tau(R)$. □

We conclude with Theorem I:

Theorem 3.7. *The outer measure τ has all the properties stated in Theorem* I.

Proof. This is the content of Theorems 2.7, 2.8, 3.3, 3.5, and 3.6. □

Remarks. We conclude the paper with the following observations. These observations are presented without proof. (a) The measure λ is *Borel regular*, i.e., if $E \subseteq \mathbf{R}^\infty$, then there is a Borel subset $R \subseteq \mathbf{R}^\infty$ such that

$$E \subseteq R \quad \text{and} \quad \lambda(E) = \lambda(R).$$

For a proof of this, see [1, p. 34].

(b) The following change of variable formula holds: Let $T^n\colon \mathbf{R}^n \to \mathbf{R}^n$, $n \geq 1$, be a linear transformation with Jacobian $\Delta \neq 0$, and let $T^\infty\colon \mathbf{R}^\infty \to \mathbf{R}^\infty$ be the map defined by

$$T^\infty(x) = (T^n(x_1, \dots, x_n), x_{n+1}, x_{n+2}, \dots), \qquad x = (x_i) \in \mathbf{R}^\infty.$$

Then for each $E \subseteq \mathbf{R}^\infty$, $\lambda(T^\infty(E)) = |\Delta|\lambda(E)$. The proof of this theorem is based on the same type of arguments used in the proof of Theorem 3.4.

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Department of Mathematics, University of California, Berkeley, California 94720