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ON FIXED POINT THEOREMS OF NONEXPANSIVE MAPPINGS IN PRODUCT SPACES

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ABSTRACT. We prove some fixed point theorems for nonexpansive self- and non-self-mappings in product spaces; in particular, we provide a constructive proof of a result of Kirk and Martinez and a partial answer to a question of Khamsi. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

1. INTRODUCTION

Let X be a Banach space and let C be a nonempty subset of X. We recall that C is said to have the fixed point property for nonexpansive mappings (FPP for short) if every nonexpansive mapping $T: C \to C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) has a fixed point, and the space X has FPP if every weakly compact convex subset of X has FPP. For the fixed point theory of nonexpansive mappings the reader is referred to Kirk [8, 9] and Goebel and Reich [4].

The purpose of this paper is to prove some fixed point theorems for nonexpansive self- and non-self-mappings in product spaces. We particularly provide a constructive proof of a result of Kirk and Martinez [12] and a partial answer to a question of Khamsi [7]. We also generalize and improve upon some results of Kirk [11] and Kirk and Sternfeld [13]. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

2. FIXED POINT THEOREMS

A subset K of a Banach space X is said to have the Browder–Göhde (B-G) property [10] if for each nonexpansive mapping $T: K \to X$, the mapping $(I - T)$ is demiclosed on K, i.e., for each sequence $\{u_j\}$ in K, the conditions $u_j \to u$ weakly and $u_j - T(u_j) \to w$ strongly imply $u \in K$ and $u - T(u) = w$. Browder [1] proved that every bounded closed convex subset of a uniformly convex Banach space has this property.

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Now suppose E and F are Banach spaces with $X \subset E$ and $Y \subset F$ and let $E \oplus F$ be the product space. For $1 \le p < \infty$ and $(x, y) \in E \oplus F$, we set

$$\|(x, y)\|_p = (\|x\|_E^p + \|y\|_F^p)^{1/p}$$

and

$$\|(x, y)\|_\infty = \max\{\|x\|_E, \|y\|_F\}.$$

It was shown in Kirk and Sternfeld [13] that if E is uniformly convex (or reflexive with the B-G property), if X is bounded closed convex, and if Y is bounded closed and separable, then the assumption that Y has the fixed point property for nonexpansive mappings assures the same is true of $(X \oplus Y)_\infty$. The separability assumption on Y is crucial to a diagonalization process used in the proof of [13] and it is proved in [12] by using the technique of a universal (or ultra) net that the separability can be removed if the B-G property is replaced by the stronger net B-G property (the two properties are equivalent in a strictly convex Banach space [12, Remark 2]). In this section, we first show that if X is weakly compact convex with the B-G property and Y has the fixed point property for nonexpansive mappings, then the same is true of $(X \oplus Y)_\infty$. Our proof of this result is constructive. We then improve a recent result of Kirk [11] for contractive mappings. We also give a partial answer to a question of Khamsi [7]. We finally prove fixed point theorems for non-self-mappings in product spaces. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

Theorem 2.1. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose X is weakly compact, convex, and has the B-G property. Suppose also Y has the fixed point property for nonexpansive mappings. Then $(X \oplus Y)_\infty$ has the fixed point property for nonexpansive mappings.*

Proof. Let P_1 and P_2 be the natural projections of $(E \oplus F)$ onto E and F, respectively. For each fixed y in Y, we define $T_y \colon X \to X$ by

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in X.$$

Then T_y is nonexpansive. Let $S_y = (I + T_y)/2$ (I denotes the identity operator on E). Let x_0 in X be fixed. By a result of Ishikawa [6] (see also Edelstein and O'Brien [3]), we have

$$\lim_{n\to\infty} \|S_y(x_{y,n}) - x_{y,n}\|_E = 0, \tag{1}$$

where $x_{y,n} = S_y^n(x_0)$, $n = 1, 2, \ldots$. We claim that

$$\|x_{u,n} - x_{v,m}\|_E \le \|u - v\|_F, \qquad n, m = 1, 2, \ldots \tag{2}$$

for u, v in Y. Indeed, (2) is trivially valid for $n = m = 1$, since

$$\begin{aligned} \|x_{u,1} - x_{v,1}\|_E &= \tfrac{1}{2}\|T_u(x_0) - T_v(x_0)\|_E \\ &\le \tfrac{1}{2}\|T(x_0, u) - T(x_0, v)\|_\infty \\ &\le \tfrac{1}{2}\|u - v\|_F. \end{aligned}$$

Now suppose that (2) is valid for $n, m \le N$. Noticing

$$\begin{aligned}\|x_{u,n+1} - x_{v,m+1}\|_E &\le \tfrac{1}{2}\|x_{u,n} - x_{v,m}\|_E + \tfrac{1}{2}\|T_u(x_{u,n}) - T_v(x_{v,m})\|_E \\ &\le \tfrac{1}{2}\|x_{u,n} - x_{v,m}\|_E + \tfrac{1}{2}\|T(x_{u,n}, u) - T(x_{v,m}, v)\|_\infty \\ &\le \tfrac{1}{2}\|x_{u,n} - x_{v,m}\|_E + \tfrac{1}{2}\cdot\max\{\|x_{u,n} - x_{v,m}\|_E, \|u-v\|_F\},\end{aligned}$$

we find that (2) is also valid for $n, m \le N+1$. This verifies the validity of (2) for all $n, m \ge 1$ according to the induction principle. Now let $y(1)$ be a weak cluster point of the sequence $\{x_{y,n}\}$. Then by the B-G property of X and (1), it follows that $y(1)$ is a fixed point of S_y and hence of T_y, that is, $P_1 \circ T(y(1), y) = y(1)$. Moreover, we have from (2) that

$$\begin{aligned}\|u(1) - v(1)\|_E &\le \limsup_{n\to\infty}\left(\limsup_{m\to\infty}\|x_{u,n} - x_{v,m}\|_E\right) \\ &\le \|u - v\|_F.\end{aligned}$$

Now let $f\colon Y \to Y$ be defined by

$$f(y) = P_2 \circ T(y(1), y), \qquad y \in Y.$$

Then for u, v in Y, we have

$$\begin{aligned}\|f(u) - f(v)\|_F &= \|P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v)\|_F \\ &\le \|T(u(1), u) - T(v(1), v)\|_\infty \\ &\le \max\{\|u(1) - v(1)\|_E, \|u - v\|_F\} \\ &= \|u - v\|_F.\end{aligned}$$

Therefore f is nonexpansive on Y and thus has a fixed point $y \in Y$. It follows that $(y(1), y)$ is a fixed point of T.

Remark 2.1. After completing the present paper, the authors found that Kuczumow [14] has obtained an even stronger result than Theorem 2.1. Kuczumow's proof uses Tychonoff's theorem and a technique of Bruck [2]. However, the proof given here is constructive and elementary.

For a subset K of a Banach space, a mapping $T\colon K \to K$ is said to be contractive if $\|T(x) - T(y)\| < \|x - y\|$, $x, y \in K$, $x \ne y$.

Theorem 2.2. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose that both X and Y have the fixed point property for contractive mappings. Then for each $1 \le p \le \infty$, $(X \oplus Y)_p$ has the fixed point property for contractive mappings.*

Proof. For a fixed p, $1 \le p \le \infty$, suppose $T\colon (X \oplus Y)_p \to (X \oplus Y)_p$ is contractive. As before, for each y in Y, we define $T_y\colon X \to X$ by

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in X.$$

Then it is easily checked that in any case of p, T_y is contractive and hence has a fixed point, i.e., there exists a point $y(1) \in X$ such that $P_1 \circ T(y(1), y) = y(1)$. Now define $f\colon Y \to Y$ by

$$f(y) = P_2 \circ T(y(1), y), \qquad y \in Y.$$

Then for $u, v \in Y$, $u \neq v$, we have if $1 \leq p < \infty$,

$$\begin{aligned}\|f(u) - f(v)\|_F^p &= \|P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v)\|_F^p \\ &= \|T(u(1), u) - T(v(1), v)\|_p^p \\ &\quad - \|P_1 \circ T(u(1), u) - P_1 \circ T(v(1), v)\|_E^p \\ &< \|(u(1), u) - (v(1), v)\|_p^p - \|u(1) - v(1)\|_E^p \\ &= \|u - v\|_F^p,\end{aligned}$$

and if $p = \infty$,

$$\begin{aligned}\|f(u) - f(v)\|_F &= \|P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v)\|_F \\ &\leq \|T(u(1), u) - T(v(1), v)\|_\infty \\ &< \|(u(1), u) - (v(1), v)\|_\infty \\ &= \max\{\|u(1) - v(1)\|_E, \|u - v\|_F\} \\ &= \|u - v\|_F,\end{aligned}$$

since

$$\begin{aligned}\|u(1) - v(1)\|_E &= \|P_1 \circ T(u(1), u) - P_1 \circ T(v(1), v)\|_E \\ &\leq \|T(u(1), u) - T(v(1), v)\|_\infty \\ &< \max\{\|u(1) - v(1)\|_E, \|u - v\|_F\} \\ &= \|u - v\|_F.\end{aligned}$$

Therefore in any case of p, $f: Y \to Y$ is contractive and thus has a fixed point $y \in Y$. It follows that $(y(1), y)$ is a fixed point of T. The proof is complete.

Remark 2.2. Theorem 2.2 says that Theorem 2.3 of Kirk and Martinez [12] holds true for contractive mappings. It also says that the weak compactness on X of Theorem 2.4 of Kirk [11] can be removed. Moreover, the technique of universal nets for contractive mappings is not necessary.

Next we consider the fixed point property for nonexpansive mappings in product spaces. Let E be a Banach space. We shall say that E has the property (P) if for any nonconstant sequence $\{x_n\}$ in E converging weakly to x, we have $\liminf_{n\to\infty} \|x_n - x\| < \operatorname{diam}(x_n)$.

Remark 2.3. It is easy to see that E satisfies the property (P) above if it satisfies one of the following properties:

(i) E has uniformly normal structure (cf. [15]), e.g., E is uniformly convex, or uniformly smooth, or k-uniformly rotund [17] for an integer $k > 1$.

(ii) E satisfies the Opial's property, that is, for any sequence $\{x_n\}$ of E, the condition $x_n \to x$ weakly implies $\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$ for y in E, $y \neq x$.

(iii) The Maluta's constant $D(E)$ of E (see [15]) is less than one, e.g., E is nearly uniformly convex [5].

The following provides a partial answer to a question of Khamsi [7, p. 999].

Theorem 2.3. *Let E be a Banach space with the property (P) above, let F be a finite-dimensional space, and let $E \oplus F$ be the product space endowed with a norm satisfying the following properties:*

(a) *The restrictions of the norm on $E \oplus F$ to E and F are the initial norms of E and F.*

(b) *The natural projections P_1 and P_2 associated to $E \oplus F$ have norm* 1.

Then $E \oplus F$ has the fixed point property for nonexpansive mappings.

Proof. Let C be a weakly compact convex subset of $E \oplus F$ and let $T: C \to C$ be a nonexpansive mapping. Choose a subset K of C which is minimal with respect to being nonempty, closed, convex, and T-invariant. Let $\{z_n\}$ be a sequence in K such that $\lim_{n\to\infty} \|T(z_n) - z_n\| = 0$. Then by a result of Karlovitz (cf. [4]), we have

$$\lim_{n\to\infty} \|z_n - z\| = \operatorname{diam}(K) \quad \text{for all } z \text{ in } K.$$

Suppose $\operatorname{diam}(K) > 0$. We may assume without loss of any generality that $\operatorname{diam}(K) = 1$. Let $z_n = x_n + y_n$ with $x_n \in E$ and $y_n \in F$. We may also assume that $x_n \to x_0$ weakly and $y_n \to y_0$ strongly. Then $z_n \to z_0$ weakly, where $z_0 = x_0 + y_0 \in K$. Noticing $\operatorname{diam}(x_n) \le \operatorname{diam}(z_n)$ by property (b), we derive from the property (P) of E that

$$\begin{aligned} 1 &= \lim_{n\to\infty} \|z_n - z_0\| \le \liminf_{n\to\infty} \|x_n - x_0\|_E + \lim_{n\to\infty} \|y_n - y_0\|_F \\ &= \liminf_{n\to\infty} \|x_n - x_0\|_E < \operatorname{diam}(x_n) \le \operatorname{diam}(z_n) = 1. \end{aligned}$$

This is a contradiction. Therefore, K must be a singleton and the proof is complete.

Since for each p, $1 \le p \le \infty$, the norm $\|\cdot\|_p$ satisfies the properties (a) and (b) above, we have the following:

Corollary 2.1. *Let E and F be as in Theorem* 2.3. *Then the product space $E \oplus F$ with any L^p norm for $1 \le p \le \infty$ has the fixed point property for nonexpansive mappings.*

Finally, we prove fixed point theorems for non-self-mappings in product spaces. Recall for a closed subset C of a Banach space E, the inward set of C at an x in C, $I_C(x)$, is defined by

$$I_C(x) = \{z \in E \colon z = x + a(y - x) \text{ for some } y \text{ in } C \text{ and } a \ge 0\}.$$

A mapping $T: C \to E$ is said to be weakly inward if for each x in C, $T(x)$ belongs to $\operatorname{cl} I_C(x)$, the closure of $I_C(x)$. We will say that C has the fixed point property for nonexpansive weakly inward mappings if every nonexpansive weakly inward mapping $T: C \to E$ has a fixed point. If C is also convex, then it is known (cf. [4]) that this property is equivalent to the fixed point property for nonexpansive mappings of C.

Theorem 2.4. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Let $E \oplus F$ be the product space with an L^p norm, $1 \le p < \infty$. Suppose both X and Y have the fixed point property for nonexpansive weakly inward mappings. Then $(X \oplus Y)_p$ also has the fixed point property for nonexpansive weakly inward mappings.*

Proof. Let $T\colon (X \oplus Y)_p \to (E \oplus F)_p$ be a nonexpansive and weakly inward mapping. For a fixed y in Y, we define, as before, the mapping $T_y\colon X \to E$ by

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in X.$$

Then T_y is nonexpansive. It is also weakly inward. Indeed, for each $x \in X$, since $T(x, y) \in \mathrm{cl}\, I_{X\oplus Y}(x, y)$, we have

$$T_y(x) \in P_1(\mathrm{cl}\, I_{X\oplus Y}(x, y)) \subseteq \mathrm{cl}\, I_X(x)$$

as required. Hence T_y has a fixed point $y(1) \in X$. Now define $f\colon Y \to F$ by

$$f(y) = P_2 \circ T(y(1), y), \qquad y \in Y.$$

Then it is easy to see that f is nonexpansive. We now check it is also weakly inward. In fact, $f(y) \in P_2(\mathrm{cl}\, I_{X\oplus Y}(y(1), y)) \subseteq \mathrm{cl}\, I_Y(y)$. Therefore, f has a fixed point $y \in Y$ and $(y(1), y)$ is a fixed point of T. The proof is complete.

In our final theorem, we assume X has the net B-G property [12], i.e., if $T\colon X \to E$ is nonexpansive and if $\{x_\alpha\}$ is a net in X for which $x_\alpha \to x$ weakly and $x_\alpha - T(x_\alpha) \to w$ strongly, then $x \in X$ and $x - T(x) = w$. A uniformly convex Banach space has this property.

Theorem 2.5. *Let E and F be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose X is weakly compact, convex, and has the net B-G property. Suppose also Y has the fixed point property for nonexpansive weakly inward mappings. Then $(X \oplus Y)_\infty$ has the fixed point property for nonexpansive weakly inward mappings.*

Proof. Let $T\colon (X \oplus Y)_\infty \to (E \oplus F)_\infty$ be a nonexpansive and weakly inward mapping. Then for each $y \in Y$, the mapping $T_y\colon X \to E$ defined by

$$T_y(x) = P_1 \circ T(x, y), \qquad x \in X$$

is nonexpansive and weakly inward. Hence, the contraction $(1-t)z + tT_y$ is weakly inward, where z is a fixed element of X and $0 < t < 1$. Let $y(t)$ be the unique fixed point of this contraction, i.e.,

$$y(t) = (1-t)z + tP_1 \circ T(y(t), y).$$

Now by the same way as Kirk and Martinez [12, Theorem 2.3], there exists a fixed point $y(1)$ of T_y such that

$$\|u(1) - v(1)\|_E \le \|u - v\|_F$$

for all u, v in Y. Now it is easily checked that the mapping $f\colon Y \to E$ defined by

$$f(y) = P_2 \circ T(y(1), y), \qquad y \in Y$$

is nonexpansive and weakly inward. Thus, f has a fixed point $y \in Y$. It follows that $(y(1), y)$ is a fixed point of T. This completes the proof.

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REFERENCES

1. F. E. Browder, *Semicontractive and semiaccretive nonlinear mappings in Banach spaces*, Bull. Amer. Math. Soc. **74** (1968), 660–665.
2. R. E. Bruck, *Properties of fixed point sets of nonexpansive mappings in Banach spaces*, Trans. Amer. Math. Soc. **179** (1973), 251–262.
3. M. Edelstein and R. C. O'Brien, *Nonexpansive mappings, asymptotic regularity and successive approximations*, J. London Math. Soc. **17** (1978), 547–554.
4. K. Goebel and S. Reich, *Uniform convexity, hyperbolic geometry, and nonexpansive mappings*, Marcel Dekker, New York, Basel, 1984.
5. R. Huff, *Banach spaces which are nearly uniformly convex*, Rocky Mountain J. Math. **10** (1980), 743–749.
6. S. Ishikawa, *Fixed points and iteration of a nonexpansive mapping in a Banach space*, Proc. Amer. Math. Soc. **59** (1976), 65–71.
7. M. A. Khamsi, *On normal structure, fixed point property and contractions of type* (γ), Proc. Amer. Math. Soc. **106** (1989), 995–1001.
8. W. A. Kirk, *Fixed point theory for nonexpansive mappings*, Lecture Notes in Math., vol. 886, Springer-Verlag, 1981, pp. 484–505.
9. ——, *Fixed point theory for nonexpansive mappings* II, Contemp. Math. **18** (1983), 121–140.
10. ——, *Nonexpansive mappings in product spaces, set-valued mappings and* k*-uniform rotundity*, Nonlinear Functional Analysis (F. E. Browder, ed.), Proc. Sympos. Pure Math., vol. 45, Part 2, Amer. Math. Soc., 1986, pp. 51–64.
11. ——, *An iteration process for nonexpansive mappings with applications to fixed point theory in product spaces*, Proc. Amer. Math. Soc. **107** (1989), 411–415.
12. W. A. Kirk and C. Martinez Yanez, *Nonexpansive and locally nonexpansive mappings in product spaces*, Nonlinear Anal. **12** (1988), 719–725.
13. W. A. Kirk and Y. Sternfeld, *The fixed point property for nonexpansive mappings in certain product spaces*, Houston J. Math. **10** (1984), 207–214.
14. T. Kuczumow, *Fixed point theorems in product spaces*, Proc. Amer. Math. Soc. **108** (1990), 727–729.
15. E. Maluta, *Uniformly normal structure and related coefficients*, Pacific J. Math. **111** (1984), 357–369.
16. Z. Opial, *Weak convergence of the sequence of successive approximations for nonexpansive mappings*, Bull. Amer. Math. Soc. **73** (1967), 595–597.
17. F. Sullivan, *A generalization of uniformly rotund Banach spaces*, Canad. J. Math. **31** (1979), 628–636.

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