ON FIXED POINT THEOREMS OF NONEXPANSIVE MAPPINGS
IN PRODUCT SPACES

KOK-KEONG TAN AND HONG-KUN Xu

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Abstract. We prove some fixed point theorems for nonexpansive self- and non-self-mappings in product spaces; in particular, we provide a constructive proof of a result of Kirk and Martinez and a partial answer to a question of Khamsi. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

1. Introduction

Let $X$ be a Banach space and let $C$ be a nonempty subset of $X$. We recall that $C$ is said to have the fixed point property for nonexpansive mappings (FPP for short) if every nonexpansive mapping $T: C \rightarrow C$ (i.e., $\|Tx - Ty\| \leq \|x - y\|$, $x, y \in C$) has a fixed point, and the space $X$ has FPP if every weakly compact convex subset of $X$ has FPP. For the fixed point theory of nonexpansive mappings the reader is referred to Kirk [8, 9] and Goebel and Reich [4].

The purpose of this paper is to prove some fixed point theorems for nonexpansive self- and non-self-mappings in product spaces. We particularly provide a constructive proof of a result of Kirk and Martinez [12] and a partial answer to a question of Khamsi [7]. We also generalize and improve upon some results of Kirk [11] and Kirk and Sternfeld [13]. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

2. Fixed point theorems

A subset $K$ of a Banach space $X$ is said to have the Browder-Göhde (B-G) property [10] if for each nonexpansive mapping $T: K \rightarrow X$, the mapping $(I - T)$ is demiclosed on $K$, i.e., for each sequence $\{u_j\}$ in $K$, the conditions $u_j \rightarrow u$ weakly and $u_j - T(u_j) \rightarrow w$ strongly imply $u \in K$ and $u - T(u) = w$. Browder [1] proved that every bounded closed convex subset of a uniformly convex Banach space has this property.
Now suppose $E$ and $F$ are Banach spaces with $X \subset E$ and $Y \subset F$ and let $E \oplus F$ be the product space. For $1 \leq p < \infty$ and $(x, y) \in E \oplus F$, we set
\[
\|(x, y)\|_p = (\|x\|_E^p + \|y\|_F^p)^{1/p}
\]
and
\[
\|(x, y)\|_\infty = \max\{\|x\|_E, \|y\|_F\}.
\]

It was shown in Kirk and Sternfeld [13] that if $E$ is uniformly convex (or reflexive with the B-G property), if $X$ is bounded closed convex, and if $Y$ is bounded closed and separable, then the assumption that $Y$ has the fixed point property for nonexpansive mappings assures the same is true of $(X \oplus Y)_\infty$. The separability assumption on $Y$ is crucial to a diagonalization process used in the proof of [13] and it is proved in [12] by using the technique of a universal (or ultra) net that the separability can be removed if the B-G property is replaced by the stronger net B-G property (the two properties are equivalent in a strictly convex Banach space [12, Remark 2]). In this section, we first show that if $X$ is weakly compact convex with the B-G property and $Y$ has the fixed point property for nonexpansive mappings, then the same is true of $(X \oplus Y)_\infty$. Our proof of this result is constructive. We then improve a recent result of Kirk [11] for contractive mappings. We also give a partial answer to a question of Khamsi [7]. We finally prove fixed point theorems for non-self-mappings in product spaces. Our proofs are elementary in the sense that we do not use any universal (or ultra) nets.

**Theorem 2.1.** Let $E$ and $F$ be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose $X$ is weakly compact, convex, and has the B-G property. Suppose also $Y$ has the fixed point property for nonexpansive mappings. Then $(X \oplus Y)_\infty$ has the fixed point property for nonexpansive mappings.

**Proof.** Let $P_1$ and $P_2$ be the natural projections of $(E \oplus F)$ onto $E$ and $F$, respectively. For each fixed $y$ in $Y$, we define $T_y : X \to X$ by
\[
T_y(x) = P_1 \circ T(x, y), \quad x \in X.
\]
Then $T_y$ is nonexpansive. Let $S_y = (I + T_y)/2$ ($I$ denotes the identity operator on $E$). Let $x_0$ in $X$ be fixed. By a result of Ishikawa [6] (see also Edelstein and O'Brien [3]), we have
\[
\lim_{n \to \infty} \|S_y^n(x_{y,n}) - x_{y,n}\|_E = 0,
\]
where $x_{y,n} = S_y^n(x_0), \ n = 1, 2, \ldots$. We claim that
\[
\|x_{u,m} - x_{v,m}\|_E \leq \|u - v\|_F, \quad n, m = 1, 2, \ldots
\]
for $u, v$ in $Y$. Indeed, (2) is trivially valid for $n = m = 1$, since
\[
\|x_{u,1} - x_{v,1}\|_E = \frac{1}{2}\|T_u(x_0) - T_v(x_0)\|_E \\
\leq \frac{1}{2}\|T(x_0, u) - T(x_0, v)\|_\infty \\
\leq \frac{1}{2}\|u - v\|_F.
\]
Now suppose that (2) is valid for \( n, m \leq N \). Noticing
\[
\| x_{u,n+1} - x_{v,m+1} \|_E \leq \frac{1}{2} \| x_{u,n} - x_{v,m} \|_E + \frac{1}{2} \| T_u(x_{u,n}) - T_v(x_{v,m}) \|_E \\
\leq \frac{1}{2} \| x_{u,n} - x_{v,m} \|_E + \frac{1}{2} \| T(x_{u,n}, u) - T(x_{v,m}, v) \|_\infty \\
\leq \frac{1}{2} \| x_{u,n} - x_{v,m} \|_E + \frac{1}{2} \cdot \max\{ \| x_{u,n} - x_{v,m} \|_E, \| u - v \|_F \},
\]
we find that (2) is also valid for \( n, m \leq N + 1 \). This verifies the validity of (2) for all \( n, m \geq 1 \) according to the induction principle. Now let \( y(1) \) be a weak cluster point of the sequence \( \{ x_{y,n} \} \). Then by the B-G property of \( X \) and (1), it follows that \( y(1) \) is a fixed point of \( S_y \) and hence of \( T_y \), that is, \( P_1 \circ T(y(1), y) = y(1) \). Moreover, we have from (2) that
\[
\| u(1) - v(1) \|_E \leq \limsup_{m \to \infty} \left( \limsup_{n \to \infty} \| x_{u,n} - x_{v,m} \|_E \right) \\
\leq \| u - v \|_F.
\]
Now let \( f: Y \to Y \) be defined by
\[
f(y) = P_2 \circ T(y(1), y), \quad y \in Y.
\]
Then for \( u, v \) in \( Y \), we have
\[
\| f(u) - f(v) \|_F = \| P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v) \|_F \\
\leq \| T(u(1), u) - T(v(1), v) \|_\infty \\
\leq \max\{ \| u(1) - v(1) \|_E, \| u - v \|_F \} \\
= \| u - v \|_F.
\]
Therefore \( f \) is nonexpansive on \( Y \) and thus has a fixed point \( y \in Y \). It follows that \( (y(1), y) \) is a fixed point of \( T \).

**Remark 2.1.** After completing the present paper, the authors found that Kuczumow [14] has obtained an even stronger result than Theorem 2.1. Kuczumow's proof uses Tychonoff's theorem and a technique of Bruck [2]. However, the proof given here is constructive and elementary.

For a subset \( K \) of a Banach space, a mapping \( T: K \to K \) is said to be contractive if \( \| T(x) - T(y) \| < \| x - y \|, \quad x, y \in K, \quad x \neq y \).

**Theorem 2.2.** Let \( E \) and \( F \) be Banach spaces with \( X \subset E \) and \( Y \subset F \). Suppose that both \( X \) and \( Y \) have the fixed point property for contractive mappings. Then for each \( 1 \leq p \leq \infty \), \( (X \oplus Y)_p \) has the fixed point property for contractive mappings.

**Proof.** For a fixed \( p \), \( 1 \leq p \leq \infty \), suppose \( T: (X \oplus Y)_p \to (X \oplus Y)_p \) is contractive. As before, for each \( y \) in \( Y \), we define \( T_y: X \to X \) by
\[
T_y(x) = P_1 \circ T(x, y), \quad x \in X.
\]
Then it is easily checked that in any case of \( p \), \( T_y \) is contractive and hence has a fixed point, i.e., there exists a point \( y(1) \in X \) such that \( P_1 \circ T(y(1), y) = y(1) \).

Now define \( f: Y \to Y \) by
\[
f(y) = P_2 \circ T(y(1), y), \quad y \in Y.
\]
Then for \( u, v \in Y \), \( u \neq v \), we have if \( 1 \leq p < \infty \),
\[
\| f(u) - f(v) \|_F^p = \| P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v) \|_F^p \\
= \| T(u(1), u) - T(v(1), v) \|_F^p \\
- \| P_1 \circ T(u(1), u) - P_1 \circ T(v(1), v) \|_F^p \\
< \| (u(1), u) - (v(1), v) \|_p^p - \| u(1) - v(1) \|_E^p \\
= \| u - v \|_F^p,
\]
and if \( p = \infty \),
\[
\| f(u) - f(v) \|_F = \| P_2 \circ T(u(1), u) - P_2 \circ T(v(1), v) \|_F \\
\leq \| T(u(1), u) - T(v(1), v) \|_\infty \\
< \| (u(1), u) - (v(1), v) \|_\infty \\
= \max\{\| u(1) - v(1) \|_E, \| u - v \|_F \} \\
= \| u - v \|_F,
\]
since
\[
\| u(1) - v(1) \|_E = \| P_1 \circ T(u(1), u) - P_1 \circ T(v(1), v) \|_E \\
\leq \| T(u(1), u) - T(v(1), v) \|_\infty \\
< \max\{\| u(1) - v(1) \|_E, \| u - v \|_F \} \\
= \| u - v \|_F.
\]
Therefore in any case of \( p \), \( f : Y \to Y \) is contractive and thus has a fixed point \( y \in Y \). It follows that \( (y(1), y) \) is a fixed point of \( T \). The proof is complete.

**Remark 2.2.** Theorem 2.2 says that Theorem 2.3 of Kirk and Martinez [12] holds true for contractive mappings. It also says that the weak compactness on \( X \) of Theorem 2.4 of Kirk [11] can be removed. Moreover, the technique of universal nets for contractive mappings is not necessary.

Next we consider the fixed point property for nonexpansive mappings in product spaces. Let \( E \) be a Banach space. We shall say that \( E \) has the property \( (P) \) if for any nonconstant sequence \( \{x_n\} \) in \( E \) converging weakly to \( x \), we have
\[
\liminf_{n \to \infty} \| x_n - x \| < \text{diam}(x_n).
\]

**Remark 2.3.** It is easy to see that \( E \) satisfies the property \( (P) \) above if it satisfies one of the following properties:

(i) \( E \) has uniformly normal structure (cf. [15]), e.g., \( E \) is uniformly convex, or uniformly smooth, or \( k \)-uniformly rotund [17] for an integer \( k > 1 \).

(ii) \( E \) satisfies the Opial's property, that is, for any sequence \( \{x_n\} \) of \( E \), the condition \( x_n \to x \) weakly implies \( \liminf_{n \to \infty} \| x_n - x \| < \liminf_{n \to \infty} \| x_n - y \| \) for \( y \) in \( E \), \( y \neq x \).

(iii) The Maluta's constant \( D(E) \) of \( E \) (see [15]) is less than one, e.g., \( E \) is nearly uniformly convex [5].

The following provides a partial answer to a question of Khamsi [7, p. 999].
Theorem 2.3. Let $E$ be a Banach space with the property (P) above, let $F$ be a finite-dimensional space, and let $E \oplus F$ be the product space endowed with a norm satisfying the following properties:

(a) The restrictions of the norm on $E \oplus F$ to $E$ and $F$ are the initial norms of $E$ and $F$.

(b) The natural projections $P_1$ and $P_2$ associated to $E \oplus F$ have norm 1. Then $E \oplus F$ has the fixed point property for nonexpansive mappings.

Proof. Let $C$ be a weakly compact convex subset of $E \oplus F$ and let $T : C \to C$ be a nonexpansive mapping. Choose a subset $K$ of $C$ which is minimal with respect to being nonempty, closed, convex, and $T$-invariant. Let $\{z_n\}$ be a sequence in $K$ such that $\lim_{n \to \infty} \|T(z_n) - z_n\| = 0$. Then by a result of Karlovitz (cf. [4]), we have

$$\lim_{n \to \infty} \|z_n - z\| = \text{diam}(K) \quad \text{for all } z \in K.$$ 

Suppose $\text{diam}(K) > 0$. We may assume without loss of any generality that $\text{diam}(K) = 1$. Let $z_n = x_n + y_n$ with $x_n \in E$ and $y_n \in F$. We may also assume that $x_n \to x_0$ weakly and $y_n \to y_0$ strongly. Then $z_n \to z_0$ weakly, where $z_0 = x_0 + y_0 \in K$. Noticing $\text{diam}(x_n) \leq \text{diam}(z_n)$ by property (b), we derive from the property (P) of $E$ that

$$1 = \lim_{n \to \infty} \|z_n - z_0\| \leq \liminf_{n \to \infty} \|x_n - x_0\|_E + \lim_{n \to \infty} \|y_n - y_0\|_F$$

$$= \liminf_{n \to \infty} \|x_n - x_0\|_E < \text{diam}(x_n) \leq \text{diam}(z_n) = 1.$$ 

This is a contradiction. Therefore, $K$ must be a singleton and the proof is complete.

Since for each $p$, $1 \leq p \leq \infty$, the norm $\| \cdot \|_p$ satisfies the properties (a) and (b) above, we have the following:

Corollary 2.1. Let $E$ and $F$ be as in Theorem 2.3. Then the product space $E \oplus F$ with any $L^p$ norm for $1 \leq p \leq \infty$ has the fixed point property for nonexpansive mappings.

Finally, we prove fixed point theorems for non-self-mappings in product spaces. Recall for a closed subset $C$ of a Banach space $E$, the inward set of $C$ at an $x$ in $C$, $I_C(x)$, is defined by

$$I_C(x) = \{z \in E: z = x + a(y - x) \text{ for some } y \in C \text{ and } a \geq 0\}.$$ 

A mapping $T : C \to E$ is said to be weakly inward if for each $x$ in $C$, $T(x)$ belongs to $\overline{I_C(x)}$, the closure of $I_C(x)$. We will say that $C$ has the fixed point property for nonexpansive weakly inward mappings if every nonexpansive weakly inward mapping $T : C \to E$ has a fixed point. If $C$ is also convex, then it is known (cf. [4]) that this property is equivalent to the fixed point property for nonexpansive mappings of $C$. 

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Theorem 2.4. Let $E$ and $F$ be Banach spaces with $X \subset E$ and $Y \subset F$. Let $E \oplus F$ be the product space with an $L^p$ norm, $1 \leq p < \infty$. Suppose both $X$ and $Y$ have the fixed point property for nonexpansive weakly inward mappings. Then $(X \oplus Y)_p$ also has the fixed point property for nonexpansive weakly inward mappings.

Proof. Let $T : (X \oplus Y)_p \to (E \oplus F)_p$ be a nonexpansive and weakly inward mapping. For a fixed $y$ in $Y$, we define, as before, the mapping $T_y : X \to E$ by

$$T_y(x) = P_1 \circ T(x, y), \quad x \in X.$$ 

Then $T_y$ is nonexpansive. It is also weakly inward. Indeed, for each $x \in X$, since $T(x, y) \in \overline{I_{X \oplus Y}(x, y)}$, we have

$$T_y(x) \in P_1(\overline{I_{X \oplus Y}(x, y)}) \subseteq \overline{I_X(x)}$$

as required. Hence $T_y$ has a fixed point $y(1) \in X$. Now define $f : Y \to F$ by

$$f(y) = P_2 \circ T(y(1), y), \quad y \in Y.$$ 

Then it is easy to see that $f$ is nonexpansive. We now check it is also weakly inward. In fact, $f(y) \in P_2(\overline{I_{Y}(y(1), y)}) \subseteq \overline{I_Y(y)}$. Therefore, $f$ has a fixed point $y \in Y$ and $(y(1), y)$ is a fixed point of $T$. The proof is complete.

In our final theorem, we assume $X$ has the net B-G property [12], i.e., if $T : X \to E$ is nonexpansive and if $\{x_\alpha\}$ is a net in $X$ for which $x_\alpha \to x$ weakly and $x_\alpha - T(x_\alpha) \to w$ strongly, then $x \in X$ and $x - T(x) = w$. A uniformly convex Banach space has this property.

Theorem 2.5. Let $E$ and $F$ be Banach spaces with $X \subset E$ and $Y \subset F$. Suppose $X$ is weakly compact, convex, and has the net B-G property. Suppose also $Y$ has the fixed point property for nonexpansive weakly inward mappings. Then $(X \oplus Y)_\infty$ has the fixed point property for nonexpansive weakly inward mappings.

Proof. Let $T : (X \oplus Y)_\infty \to (E \oplus F)_\infty$ be a nonexpansive and weakly inward mapping. Then for each $y \in Y$, the mapping $T_y : X \to E$ defined by

$$T_y(x) = P_1 \circ T(x, y), \quad x \in X$$

is nonexpansive and weakly inward. Hence, the contraction $(1 - t)z + tT_y$ is weakly inward, where $z$ is a fixed element of $X$ and $0 < t < 1$. Let $y(t)$ be the unique fixed point of this contraction, i.e.,

$$y(t) = (1 - t)z + tP_1 \circ T(y(t), y).$$

Now by the same way as Kirk and Martinez [12, Theorem 2.3], there exists a fixed point $y(1)$ of $T_y$ such that

$$\|u(1) - v(1)\|_E \leq \|u - v\|_F$$

for all $u, v$ in $Y$. Now it is easily checked that the mapping $f : Y \to E$ defined by

$$f(y) = P_2 \circ T(y(1), y), \quad y \in Y$$

is nonexpansive and weakly inward. Indeed, for each $x \in X$,

$$f(y) \in P_2(\overline{I_Y(y(1), y)}) \subseteq \overline{I_Y(y)}.$$ 

The proof is complete.
is nonexpansive and weakly inward. Thus, $f$ has a fixed point $y \in Y$. It follows that $(y(1), y)$ is a fixed point of $T$. This completes the proof.

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**DEPARTMENT OF MATHEMATICS, STATISTICS AND COMPUTING SCIENCE, DALHOUSIE UNIVERSITY, HALIFAX, NOVA SCOTIA, CANADA B3H 3J5**

**DEPARTMENT OF MATHEMATICS, EAST CHINA UNIVERSITY OF CHEMICAL TECHNOLOGY, SHANGHAI 200237, CHINA**

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