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SOME STRUCTURE THEOREMS FOR COMPLETE CONSTANT MEAN CURVATURE SURFACES WITH BOUNDARY A CONVEX CURVE

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ABSTRACT. Let M be a properly embedded, connected, complete surface in $\mathbb{R}^3$ with non-zero constant mean curvature and with boundary a strictly convex plane curve C. It is shown that if M is contained in a vertical cylinder of $\mathbb{R}^3_+$, outside of some compact set of $\mathbb{R}^3$, and if M is contained in a half-space of $\mathbb{R}^3$ determined by C, then M inherits the symmetries of C. In particular, M is a Delaunay surface if C is a circle. It is also shown that if M has a finite number of vertical annular ends and the area of the flat disc D bounded by C is not "too small," then M lies in a half-space.

Let M be a properly embedded connected constant mean curvature surface (nonzero) in $\mathbf{R}^3$ with boundary a strictly convex curve C. We assume M is complete and C is contained in the horizontal plane $\mathscr{H} = \{z = 0\}$. In [1] it is shown that when M is compact and transverse to $\mathscr{H}$ along C, then M is entirely contained in a half-space of $\mathbf{R}^3$ determined by $\mathscr{H}$. Then all the symmetries of C are also symmetries of M. For example if C is a circle, then M is a spherical cap.

In this paper we continue the investigations of [1], to study complete such M that are transverse to $\mathscr{H}$ along C.

Assume M is contained in $\mathbf{R}^3_+$. First we prove that if M is contained in a vertical cylinder of $\mathbf{R}^3_+$, outside of some compact set of $\mathbf{R}^3$, then M inherits the symmetries of C (a tilted cylinder shows one needs vertical ends). In particular, M is equal to a Delaunay surface when C is a circle.

Then we study conditions that ensure M is contained in $\mathbf{R}^3_+$ when the ends of M are a finite number of vertical annular ends. We show that if the area of the flat disc D bounded by C is not "too small", then $M \subset \mathbf{R}^3_+$.

Finally we discuss some questions that arise from our work.

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Theorem 1. *Let M be a properly embedded complete constant mean curvature surface in $\mathbf{R}^3$. Suppose $\partial M = C$ is a strictly convex curve, $C \subset \mathscr{H}$, and M is transverse (in fact, $M \subset \mathbf{R}^3_+$ near C suffices) to $\mathscr{H}$ along C. If $M \subset \mathbf{R}^3_+$ and M is vertically cylindrically bounded outside of some compact set of $\mathbf{R}^3$, then M inherits the symmetries of C.*

Proof. We show M is invariant by reflection in every vertical plane V that is a plane of symmetry of C. The idea is to show that M is almost invariant by reflection in ε-tilted planes (from V), for every $\varepsilon > 0$ (this idea is used in [3]).

Let $\varepsilon > 0$ and let $P(\varepsilon)$ be a plane that makes an angle ε with the vertical plane V of symmetry of C. Since the ends of M are in some fixed vertical cylinder, one can translate $P(\varepsilon)$ to a plane P so that C and the ends of M are in distinct half-spaces of $\mathbf{R}^3$ determined by P. Let L be the half-space of $\mathbf{R}^3 - P$ that contains C, and let $D \subset \mathscr{H}$ be the disc bounded by C.

Notice that $M \cup D$ is a properly embedded submanifold of $\mathbf{R}^3$ (with a corner along C) hence separates $\mathbf{R}^3$ into two components.

Let $Q(t)$ be the foliation of L by planes parallel to P, t equal to the distance between $Q(t)$ and P. Clearly for t large, $Q(t)$ is disjoint from M.

Apply the Alexandrov reflection process to M and the planes $Q(t)$ [2]. Let $L(t)$ be the halfspace of $\mathbf{R}^3 - Q(t)$ contained in L. Let $M(t)^*$ denote the symmetry of $M \cap L(t)$ about $Q(t)$. As the planes $Q(t)$ approach P from infinity, consider the first (nontrivial) point of contact of $M(t)^*$ with M. This point cannot be the image of an interior point of M, since this would imply $M(t)^* \subset M$ by the maximum principle. Then M would be invariant by a tilted plane $Q(t)$, which is impossible (the ends of M are vertical).

Another possibility is that M becomes orthogonal to $Q(t)$ before an interior point of contact is reached. Then the boundary maximum principle would imply that M is invariant by reflection in the titled plane $Q(t)$.

Therefore, the first point of contact must be the image of a point of C. (There must be a first point of contact of $M(t)^*$ with M before the planes $Q(t)$ have swept through C.)

Since M is transverse to $\mathscr{H}$ along C, the position of the plane $Q(t)$, at which one has a first point of contact, converges to the vertical plane V as $\varepsilon \to 0$. This means M is invariant by reflection in V. Now we prove this assertion.

Suppose, on the contrary, that M is not symmetric in V. Then there are points p, q on M such that the line segment $[p, q]$ joining p to q, is orthogonal to V, p and q are on opposite sides of V, and $d(p, V) > d(q, V)$. Thus the symmetry p^* of p through V is on the other side of q; i.e., $[p^*, p] \supset [q, p]$.

Now parallel translate V towards p, to a plane V_0, so that the symmetry p_0^* of p through V_0, still satisfies $[p_0^*, p] \supset [q, p]$. Clearly if V_0 is close enough to V, then this is always the case. We do this so that $V_0 \neq V$.

It follows that the symmetry of M through V_0 (the side of M containing p), intersects M in more than just $M \cap V_0$.

If $P(\varepsilon)$ denotes the plane V_0 tilted an angle ε, then for ε sufficiently small, the symmetry of M through $P(\varepsilon)$ still intersects M (in more than $M \cap P(\varepsilon)$). We tilt V_0 to $P(\varepsilon)$ so that $V \cap P(\varepsilon) \cap \mathbf{R}^3_+$ is empty (there are two ways to tilt V_0). This ensures that the symmetry C^* of the shorter arc of $C - P(\varepsilon)$ through $P(\varepsilon)$ is in $\mathbf{R}^3_+$.

Now M is transverse to $\mathscr{H}$ along C, so for ε sufficiently small, C^* intersects M only at the endpoints of C^*. Also, one can choose $\varepsilon > 0$ so that this last condition holds for symmetry in all planes sufficiently close to $P(\varepsilon)$ and parallel to $P(\varepsilon)$. Notice that C^* is not tangent to M at its endpoints because C is strictly convex.

Let $V(t)$ be a plane parallel to V_0 (on the same side of V_0 as p), intersecting C in two points, or one point. As in the last paragraph, there is an $\varepsilon > 0$ such that if $Q(t)$ denotes $V(t)$ tilted by ε (along the axis $V(t) \cap \mathscr{H}$), then the symmetry of the short arc of $C - Q(t)$, through $Q(t)$, intersects M only at its two endpoints. Moreover, ε can be chosen to work for all planes sufficiently close to $Q(\varepsilon)$ and parallel to $Q(\varepsilon)$.

Now by compacity of the shorter arc of $C - V_0$, there is an $\varepsilon > 0$ such that this property holds for all planes parallel to $P(\varepsilon)$ (on the same side as p). Let $Q(t)$ denote this family of planes; $Q(0) = P(\varepsilon)$, $Q(t)$ is parallel to $P(\varepsilon)$ at a distance t from $P(\varepsilon)$, and $Q(t)$ is on the other side of $P(\varepsilon)$ than V_0.

For $t = T$ sufficiently large, $Q(T)$ is disjoint from M. We let t go from T to 0 and consider the first point of contact of $M(t)^*$ (symmetry through $Q(t)$) with M. As we have shown before, this first point of contact must be the image of a point of C. However, by our choice of $P(\varepsilon)$, there is no such point of C. Therefore, M is symmetric in V and we have proved Theorem 1.

Now we study surfaces M, satisfying the hypothesis of Theorem 1, except we do not assume M is globally contained in $\mathbf{R}^3_+$. Also we assume that M has a finite number of ends, each topologically an annulus.

We parametrize the embedded Delaunay surfaces by the real parameter τ, which is the radius of a smallest parallel circle. For mean curvature H, one has $\tau \leq 1/2H$. For each annular end $\mathscr{A}$ of M there exists a Delaunay end $\mathscr{D}(\tau)$ such that $\mathscr{A}$ converges geometrically to $\mathscr{D}(\tau)$ [3]. We associate τ to $\mathscr{A}$ and write $\mathscr{A} = \mathscr{A}(\tau)$.

Theorem 2. *Let M be a constant mean curvature surface in $\mathbf{R}^3$, complete and properly embedded, with ∂M a strictly convex curve $C \subset \mathscr{H}$. Assume M is transversal to $\mathscr{H}$ along C and $M \subset \mathbf{R}^3_+$ near C.*

Also assume M is compact or M is complete with a finite number of vertical annular ends $\mathscr{A}(\tau_1), \dots, \mathscr{A}(\tau_n)$ contained in $\mathbf{R}^3_+$.

If M is compact, then $M \subset \mathbf{R}^3_+$.

If M is noncompact, then either $M \subset \mathbf{R}^3_+$ or there is a simple closed curve

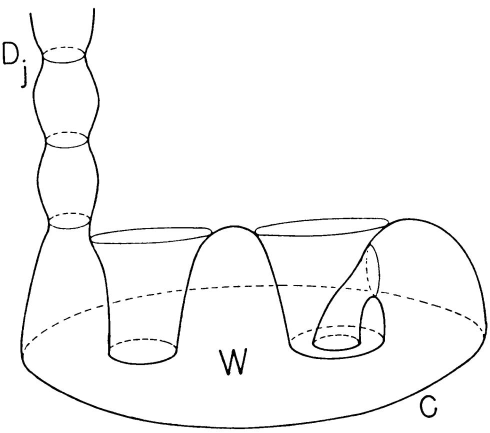

FIGURE 1

$\alpha \subset M \cap \mathscr{H} \cap \operatorname{ext}(D)$, α *generates* $\pi_1(\operatorname{ext}(D))$, *and* $\operatorname{area}(D) < \sum_{i=1}^n \pi(\frac{\tau_i}{H} - \tau_i^2)$.

Proof of Theorem 2. For each annular end $\mathscr{A} = \mathscr{A}(\tau_j)$ of M, let D_j be a horizontal disk whose boundary is a simple closed curve of $\mathscr{A}$, a generator of $\pi_1(\mathscr{A})$. Let ν_j denote the conormal to M along ∂D_j, oriented down, i.e., $\langle \nu_j, e_3 \rangle < 0$. Let ν denote the inward pointing conormal to M along $C = \partial M$.

We can assume M is transverse to $\mathscr{H}$ by displacing M slightly in the vertical direction.

First we prove that $M \cap \operatorname{ext}(D) = \varnothing$ and $M \cap \operatorname{int}(D) \neq \varnothing$ is impossible. So we assume the contrary and arrive at a contradiction.

Let M_1 be the connected component of $M \cap \mathbf{R}^3_+$ that contains C; $\partial M_1 = C \cup C_1 \cup \cdots \cup C_m$, $C_j \subset \operatorname{int} D$, $1 \leq j \leq m$. M_1 together with a proper subdomain D_0 of D, bound a 3-dimensional domain $W \subset \mathbf{R}^3_+$, not necessarily compact (cf. Figure 1).

For each annular end $\mathscr{A}(\tau_j)$ in M_1, we consider the disk D_j with ∂D_j a generator of $\pi_1(\mathscr{A}(\tau_j))$. Form a compact cycle E by removing from $M_1 \cup D_0$, the part of each $\mathscr{A}(\tau_j)$ that is above ∂D_j and attaching D_j. One could choose the D_j at different heights to ensure E is embedded, but this is not necessary.

Let Y be a constant vector field of $\mathbf{R}^3$. The flux of Y across E is zero and this yields the "balancing formula $(Y = e_3)$," [1, 3]:

$$\frac{1}{2H} \int_{\partial D_0} e_3 \cdot \nu = \operatorname{area}(D_0) + \sum_j \pi \left(\frac{\tau_j}{H} - \tau_j^2 \right). \tag{2.1}$$

The sum is taken over all ends of M_1. Notice that this sum is nonnegative since each $\tau_j \leq 1/2H$. We indicate how (2.1) is derived: since the flux of Y is zero across E,

$$\int_{D_0} Y \cdot n_{D_0} + \int_{M_2} Y \cdot n_{M_2} + \sum_j \int_{D_j} Y \cdot n_{D_j} = 0.$$

Here $M_2 = E - (D_0 \cup \bigcup_j D_j)$, $n_{D_0} = e_3$, $n_{D_j} = -e_3$, $j \geq 1$, and $n_{M_2} = \vec{H}/H$. The reason $n_{D_0} = e_3$ is that $\vec{H}$ points towards D_0 along ∂M; there are no exterior intersection curves of M with $\mathscr{H}$ by assumption.

One has $\int_{M_2} Y \cdot n_{M_2} = -(1/2H) \int_{\partial M_2} Y \cdot \nu$, ν the inward pointing conormal to M_2 (this formula follows from $\Delta X = -2Hn_{M_2}$, X the position vector field).

Each annular end $\mathscr{A}(\tau_j)$ converges geometrically to a Delaunay end $\mathscr{D}(\tau_j)$. Thus $\int_{D_j} e_3 \cdot n_{D_j} - (1/2H) \int_{\partial D_j} e_3 \cdot \nu_j = \pi\tau_j/H - \pi\tau_j^2$. To prove this last formula, one constructs a compact cycle and applies the divergence theorem. Choose a horizontal disk $\widetilde{D}_j$, whose boundary approximates a "shortest" parallel circle of $\mathscr{D}(\tau_j)$ and do this so that $\partial\widetilde{D}_j \cup \partial D_j$ bound an embedded annulus on $A(\tau_j)$. Then

$$\int_{D_j} e_3 \cdot n_{D_j} - \frac{1}{2H} \int_{\partial D_j} e_3 \cdot \nu_j = \int_{\widetilde{D}_j} e_3 \cdot n_{\widetilde{D}_j} - \frac{1}{2H} \int_{\partial \widetilde{D}_j} e_3 \cdot \tilde{\nu}_j .$$

As the $\widetilde{D}_j$ get higher, $\partial\widetilde{D}_j$ converges geometrically to the shortest parallel circle of $\mathscr{D}(\tau_j)$. Therefore the right side of the above equation is constant and equals $\pi(\tau_j/H - \tau_j^2)$. Let $Y = e_3$ and put these equations together; one obtains (2.1).

Now $\nu \cdot e_3 > 0$ along ∂D_0, so (2.1) implies:

$$\frac{1}{2H} \int_C e_3 \cdot \nu < \operatorname{area}(D_0) + \sum_j \pi \left(\frac{\tau_j}{H} - \tau_j^2 \right). \tag{2.2}$$

Next consider the cycle $M \cap D$. The flux of e_3 across this cycle is also zero; more precisely, one obtains a compact cycle from $M \cup D$ as before, attaching D_i to each annular end $\mathscr{A}(\tau_i)$ of M and removing the part of $\mathscr{A}(\tau_i)$ that is above ∂D_i. The same type of calculation as above yields:

$$\frac{1}{2H} \int_C e_3 \cdot \nu = \operatorname{area}(D) + \sum_{i=1}^n \pi \left(\frac{\tau_i}{H} - \tau_i^2 \right). \tag{2.3}$$

For future reference we remark that if the mean curvature vector $\vec{H}$ points towards $\operatorname{ext}(D)$, along C, then (2.3) becomes:

$$\frac{1}{2H} \int_C e_3 \cdot \nu = -\operatorname{area}(D) + \sum_{i=1}^n \pi \left(\frac{\tau_i}{H} - \tau_i^2 \right). \tag{2.4}$$

When M is compact, the term $\sum_{i=1}^n \pi(\frac{\tau_i}{H} - \tau_i^2)$ does not appear in (2.3) or (2.4). Clearly (2.2) and (2.3) are impossible since area $(D_0) < \operatorname{area}(D)$ and

$$\sum_j \pi \left(\frac{\tau_j}{H} - \tau_j^2 \right) \leq \sum_{i=1}^n \pi \left(\frac{\tau_i}{H} - \tau_i^2 \right).$$

This proves that $M \cap \operatorname{ext}(D) = \varnothing$ and $M \cap \operatorname{int}(D) \neq \varnothing$ is impossible.

So we may assume $M \cap \operatorname{ext}(D) \neq \varnothing$. We will show that $M_1 \cap \operatorname{ext}(D)$ is exactly one Jordan curve that generates $\pi_1(\operatorname{ext}(D))$, and $\operatorname{area}(D) < \sum_{i=1}^n \pi(\frac{\tau_i}{H} - \tau_i^2)$.

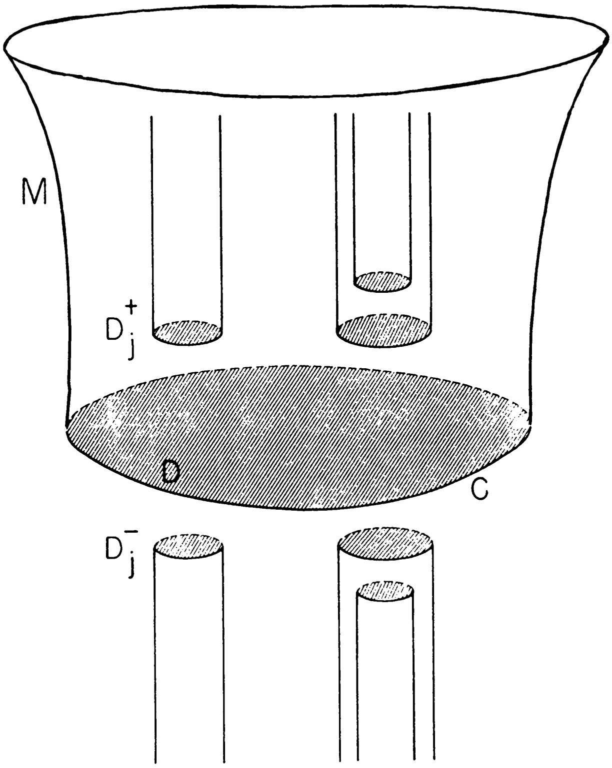

FIGURE 2

The idea is to apply the Alexandrov reflection principle using ε-tilted planes. First we must do some cutting and pasting along the cycles of $M \cap \text{int}(D)$, to obtain a manifold that separates $\mathbf{R}^3$; this enables us to be sure the mean curvature vectors are pointing in the same direction when we do Alexandrov reflection.

Let $C_1, \ldots, C_m$ be the Jordan curves of $M \cap \text{int}(D)$. For each C_j, let $C_j^+(\varepsilon)$ be the planar curve on M, near C_j, obtained by intersecting M with the plane $\{z = \varepsilon\}$. Similarly, let $C_j^-(\varepsilon)$ be the curve $M \cap \{z = -\varepsilon\}$ that is near C_j.

We form an embedded surface N by removing from M the annuli bounded by $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ and attaching the horizontal planar domains $D_j^+ \cup D_j^-$ bounded by $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ (cf. [1]). Also we attach D to M along C.

To ensure that N is embedded, one uses different values of ε, when several C_j are concentric (cf. Figure 2). N is a properly embedded submanifold (with corners) of $\mathbf{R}^3$, $\partial N = \varnothing$, hence each connected component of N separates $\mathbf{R}^3$ into two connected components.

Let $\widetilde{M}$ be the component of N that contains C. We orient $\widetilde{M}$ be the mean curvature vector $\vec{H}$ of M. Notice that this makes sense since, abstractly, $\widetilde{M}$ is a submanifold L of M (hence $\vec{H}$ is defined over L) to which one has attached D and the discs $D_j^{\pm}(\varepsilon)$. Clearly $\vec{H}$ extends across these disks to define a normal field to $\widetilde{M}$. The corners of $\widetilde{M}$ along the boundaries of the disks, do not affect this.

Now when $\widetilde{M}$ is compact, one can use Alexandrov reflection in vertical planes to prove Theorem 2, i.e., $\widetilde{M} \subset \mathbf{R}^3_+$. This case was treated in [1]. For the readers benefit, we give the argument here as well.

Consider a family of vertical planes $Q(t)$ coming from infinity and apply the Alexandrov technique to $\widetilde{M}$. Observe that $\widetilde{M} \cap \operatorname{ext} D$ can have no cycle β that is null homotopic in $\operatorname{ext} D$, since one could come from infinity in a direction such that for some position of the vertical plane $Q(t)$ (before reaching C), a symmetry of β by $Q(t)$ would touch β at an interior point. Thus $\widetilde{M}$ would have a plane of symmetry before reaching C, a contradiction; $\widetilde{M}$ is a graph on one side of this plane and contains C on the other side. We used C convex in this argument. Now if $\widetilde{M} \cap \operatorname{ext}(D)$ contained two cycles α_1, α_2, both generators of $\pi_1(\operatorname{ext} D)$, then again one uses vertical planes $Q(t)$ coming from infinity. Clearly there is some position of $Q(t)$, before reaching C, where a symmetry of α_1 by $Q(t)$ meets a point of α_2 (assuming α_1 is the exterior cycle). Thus $\widetilde{M}$ would have a vertical plane of symmetry before reaching C.

This proves that $M_1 \cap \operatorname{ext}(D)$ is at most one cycle α and α generates $\pi_1(\operatorname{ext} D)$. Since the mean curvature vector $\vec{H}$ points into the bounded component of $\mathbf{R}^3 - \widetilde{M}$, $\vec{H}$ points towards C along α and hence, $\vec{H}$ points towards $\operatorname{ext}(D)$ along C.

When $\widetilde{M}$ is compact this implies $n_D = -e_3$, which contradicts the balancing formula: $(1/2H)\int_C e_3 \cdot \nu = -\operatorname{area}(D)$.

Now suppose $\widetilde{M}$ is not compact. Let $Q(t)$ be a family of parallel planes, tilted an angle $\varepsilon > 0$ from the vertical. The ends of $\widetilde{M}$ are contained in some fixed vertical cylinder.

As in the case $\widetilde{M}$ compact, there are no cycles in $\operatorname{ext}(D) \cap \widetilde{M}$ that are null homotopic in $\operatorname{ext}(D)$. For if β were such a cycle, we could approach $\widetilde{M}$ by $Q(t)$ from infinity, in a direction such that a position of $Q(t)$ would be attained, before reaching C, where a symmetry of β would touch M at an interior point of M (near β). Thus there would be a plane of symmetry of $\widetilde{M}$, among the $Q(t)$, before reaching C. This is impossible.

Also, $\widetilde{M} \cap \operatorname{ext}(D)$ can contain at most one cycle that generates $\pi_1(\operatorname{ext}(D))$. For if α_1, α_2 were two such cycles, then coming from infinity with the tilted planes $Q(t)$, there would be a $Q(t)$ where a symmetry of α_1 would touch $\widetilde{M}$ at a point near α_2. This occurs before reaching C so one would have a plane of symmetry before reaching C; a contradiction.

Thus if α is a cycle in $\widetilde{M} \cap \operatorname{ext}(D)$ that generates $\pi_1(\operatorname{ext} D)$, we conclude,

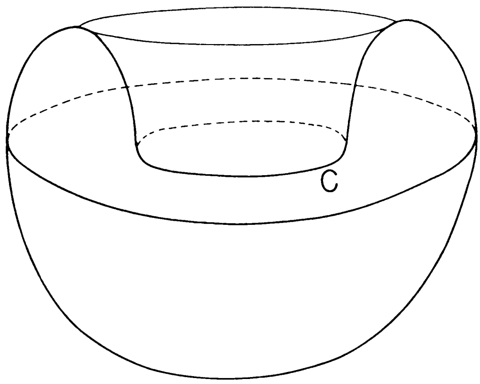

FIGURE 3

as in the case $\widetilde{M}$ compact, that the mean curvature vector $\vec{H}$ of M points towards $\text{ext}(D)$ along C.

By the balancing formula (2.4):

$$\frac{1}{2H}\int_C Y\cdot\nu = \sum_{i=1}^{n}\pi\left(\frac{\tau_i}{H}-\tau_i^2\right) - \text{area}(D).$$

Since $Y = e_3$, $e_3\cdot\nu > 0$ along C. Hence $\text{area}(D) < \sum_{i=1}^{n}\pi(\tau_i/H - \tau_i^2)$.

This completes the proof of Theorem 2.

Corollary. *Assume M satisfies the hypothesis of Theorem 2 and ∂M is a circle of radius r. Then if $r \geq \sqrt{n}/2H$, M equals a Delaunay surface.*

The problem posed by Theorem 2 is whether examples exist that satisfy the hypothesis of Theorem 2, and that are not contained in $\mathbf{R}^3_+$.

Here are some examples that we cannot exclude by our techniques. First consider the compact surface M_0 of Figure 3.

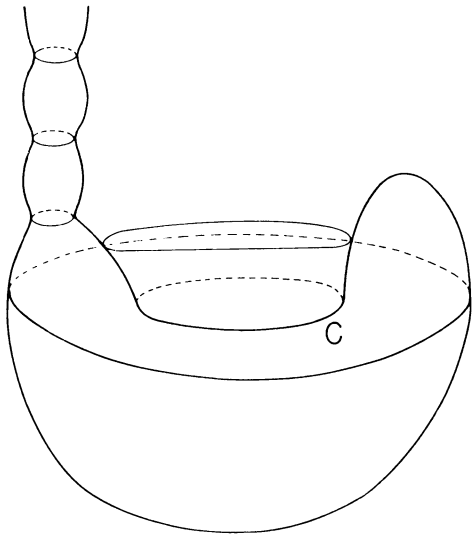

FIGURE 4

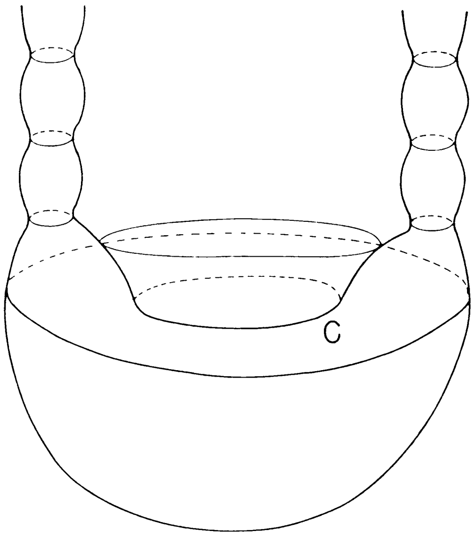

FIGURE 5

This surface cannot have constant mean curvature (Theorem 2, or more simply the balancing formula (2.4)). However perhaps one can puncture M_0 at one (or more) points, at the top, and attach vertical Delaunay type ends at the punctures. In Figures 4, 5 we indicate some possibilities.

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