SOME STRUCTURE THEOREMS
FOR COMPLETE CONSTANT MEAN CURVATURE SURFACES
WITH BOUNDARY A CONVEX CURVE

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Abstract. Let $M$ be a properly embedded, connected, complete surface in $\mathbb{R}^3$ with non-zero constant mean curvature and with boundary a strictly convex plane curve $C$. It is shown that if $M$ is contained in a vertical cylinder of $\mathbb{R}_+$, outside of some compact set of $\mathbb{R}^3$, and if $M$ is contained in a half-space of $\mathbb{R}^3$ determined by $C$, then $M$ inherits the symmetries of $C$. In particular, $M$ is a Delaunay surface if $C$ is a circle. It is also shown that if $M$ has a finite number of vertical annular ends and the area of the flat disc $D$ bounded by $C$ is not "too small," then $M$ lies in a half-space.

Let $M$ be a properly embedded connected constant mean curvature surface (nonzero) in $\mathbb{R}^3$ with boundary a strictly convex curve $C$. We assume $M$ is complete and $C$ is contained in the horizontal plane $\mathcal{H} = \{z = 0\}$. In [1] it is shown that when $M$ is compact and transverse to $\mathcal{H}$ along $C$, then $M$ is entirely contained in a half-space of $\mathbb{R}^3$ determined by $\mathcal{H}$. Then all the symmetries of $C$ are also symmetries of $M$. For example if $C$ is a circle, then $M$ is a spherical cap.

In this paper we continue the investigations of [1], to study complete such $M$ that are transverse to $\mathcal{H}$ along $C$.

Assume $M$ is contained in $\mathbb{R}_+^3$. First we prove that if $M$ is contained in a vertical cylinder of $\mathbb{R}_+^3$, outside of some compact set of $\mathbb{R}^3$, then $M$ inherits the symmetries of $C$ (a tilted cylinder shows one needs vertical ends). In particular, $M$ is equal to a Delaunay surface when $C$ is a circle.

Then we study conditions that ensure $M$ is contained in $\mathbb{R}_+^3$ when the ends of $M$ are a finite number of vertical annular ends. We show that if the area of the flat disc $D$ bounded by $C$ is not "too small", then $M \subset \mathbb{R}_+^3$.

Finally we discuss some questions that arise from our work.

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Theorem 1. Let $M$ be a properly embedded complete constant mean curvature surface in $\mathbb{R}^3$. Suppose $\partial M = C$ is a strictly convex curve, $C \subset \mathbb{R}$, and $M$ is transverse (in fact, $M \subset \mathbb{R}^3_+$ near $C$ suffices) to $\mathbb{R}$ along $C$. If $M \subset \mathbb{R}^3_+$ and $M$ is vertically cylindrically bounded outside of some compact set of $\mathbb{R}^3$, then $M$ inherits the symmetries of $C$.

Proof. We show $M$ is invariant by reflection in every vertical plane $V$ that is a plane of symmetry of $C$. The idea is to show that $M$ is almost invariant by reflection in $\epsilon$-tilted planes (from $V$), for every $\epsilon > 0$ (this idea is used in [3]).

Let $\epsilon > 0$ and let $P(\epsilon)$ be a plane that makes an angle $\epsilon$ with the vertical plane $V$ of symmetry of $C$. Since the ends of $M$ are in some fixed vertical cylinder, one can translate $P(\epsilon)$ to a plane $P$ so that $C$ and the ends of $M$ are in distinct half-spaces of $\mathbb{R}^3$ determined by $P$. Let $L$ be the half-space of $\mathbb{R}^3 - P$ that contains $C$, and let $D \subset \mathbb{R}^3$ be the disc bounded by $C$.

Notice that $M \cup D$ is a properly embedded submanifold of $\mathbb{R}^3$ (with a corner along $C$) hence separates $\mathbb{R}^3$ into two components.

Let $Q(t)$ be the foliation of $L$ by planes parallel to $P$, $t$ equal to the distance between $Q(t)$ and $P$. Clearly for $t$ large, $Q(t)$ is disjoint from $M$.

Apply the Alexandrov reflection process to $M$ and the planes $Q(t)$ [2]. Let $L(t)$ be the halfspace of $\mathbb{R}^3 - Q(t)$ contained in $L$. Let $M(t)^*$ denote the symmetry of $M \cap L(t)$ about $Q(t)$. As the planes $Q(t)$ approach $P$ from infinity, consider the first (nontrivial) point of contact of $M(t)^*$ with $M$. This point cannot be the image of an interior point of $M$, since this would imply $M(t)^* \subset M$ by the maximum principle. Then $M$ would be invariant by a tilted plane $Q(t)$, which is impossible (the ends of $M$ are vertical).

Another possibility is that $M$ becomes orthogonal to $Q(t)$ before an interior point of contact is reached. Then the boundary maximum principle would imply that $M$ is invariant by reflection in the named plane $Q(t)$.

Therefore, the first point of contact must be the image of a point of $C$. (There must be a first point of contact of $M(t)^*$ with $M$ before the planes $Q(t)$ have swept through $C$.)

Since $M$ is transverse to $\mathbb{R}^3$ along $C$, the position of the plane $Q(t)$, at which one has a first point of contact, converges to the vertical plane $V$ as $\epsilon \to 0$. This means $M$ is invariant by reflection in $V$. Now we prove this assertion.

Suppose, on the contrary, that $M$ is not symmetric in $V$. Then there are points $p, q$ on $M$ such that the line segment $[p, q]$ joining $p$ to $q$, is orthogonal to $V$, $p$ and $q$ are on opposite sides of $V$, and $d(p, V) > d(q, V)$. Thus the symmetry $p^*$ of $p$ through $V$ is on the other side of $q$; i.e., $[p^*, p] \supset [q, p]$.

Now parallel translate $V$ towards $p$, to a plane $V_0$, so that the symmetry $p_0^*$ of $p$ through $V_0$, still satisfies $[p_0^*, p] \supset [q, p]$. Clearly if $V_0$ is close enough to $V$, then this is always the case. We do this so that $V_0 \neq V$. 

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It follows that the symmetry of \( M \) through \( V_0 \) (the side of \( M \) containing \( p \)), intersects \( M \) in more than just \( M \cap V_0 \).

If \( P(\varepsilon) \) denotes the plane \( V_0 \) tilted an angle \( \varepsilon \), then for \( \varepsilon \) sufficiently small, the symmetry of \( M \) through \( P(\varepsilon) \) still intersects \( M \) (in more than \( M \cap P(\varepsilon) \)). We tilt \( V_0 \) to \( P(\varepsilon) \) so that \( \gamma \cap P(\varepsilon) \cap R^3_+ \) is empty (there are two ways to tilt \( V_0 \)). This ensures that the symmetry \( C^* \) of the shorter arc of \( C - P(\varepsilon) \) through \( P(\varepsilon) \) is in \( R^3_+ \).

Now \( M \) is transverse to \( \mathcal{H} \) along \( C \), so for \( \varepsilon \) sufficiently small, \( C^* \) intersects \( M \) only at the endpoints of \( C^* \). Also, one can choose \( \varepsilon > 0 \) so that this last condition holds for symmetry in all planes sufficiently close to \( P(\varepsilon) \) and parallel to \( P(\varepsilon) \). Notice that \( C^* \) is not tangent to \( M \) at its endpoints because \( C \) is strictly convex.

Let \( V(t) \) be a plane parallel to \( V_0 \) (on the same side of \( V_0 \) as \( p \)), intersecting \( C \) in two points, or one point. As in the last paragraph, there is an \( \varepsilon > 0 \) such that if \( Q(t) \) denotes \( V(t) \) tilted by \( \varepsilon \) (along the axis \( V(t) \cap \mathcal{H} \)), then the symmetry of the short arc of \( C - Q(t) \), through \( Q(t) \), intersects \( M \) only at its two endpoints. Moreover, \( \varepsilon \) can be chosen to work for all planes sufficiently close to \( Q(\varepsilon) \) and parallel to \( Q(\varepsilon) \).

Now by compacity of the shorter arc of \( C - V_0 \), there is an \( \varepsilon > 0 \) such that this property holds for all planes parallel to \( P(\varepsilon) \) (on the same side as \( p \)). Let \( Q(t) \) denote this family of planes; \( Q(0) = P(\varepsilon) \), \( Q(t) \) is parallel to \( P(\varepsilon) \) at a distance \( t \) from \( P(\varepsilon) \), and \( Q(t) \) is on the other side of \( P(\varepsilon) \) than \( V_0 \).

For \( t = T \) sufficiently large, \( Q(T) \) is disjoint from \( M \). We let \( t \) go from \( T \) to 0 and consider the first point of contact of \( M(t)^\ast \) (symmetry through \( Q(t) \)) with \( M \). As we have shown before, this first point of contact must be the image of a point of \( C \). However, by our choice of \( P(\varepsilon) \), there is no such point of \( C \). Therefore, \( M \) is symmetric in \( V \) and we have proved Theorem 1.

Now we study surfaces \( M \), satisfying the hypothesis of Theorem 1, except we do not assume \( M \) is globally contained in \( R^3_+ \). Also we assume that \( M \) has a finite number of ends, each topologically an annulus.

We parametrize the embedded Delaunay surfaces by the real parameter \( \tau \), which is the radius of a smallest parallel circle. For mean curvature \( H \), one has \( \tau \leq 1/2H \). For each annular end \( \mathscr{A} \) of \( M \) there exists a Delaunay end \( \mathscr{D}(\tau) \) such that \( \mathscr{A} \) converges geometrically to \( \mathscr{D}(\tau) \) [3]. We associate \( \tau \) to \( \mathscr{A} \) and write \( \mathscr{A} = \mathscr{A}(\tau) \).

**Theorem 2.** Let \( M \) be a constant mean curvature surface in \( R^3_+ \), complete and properly embedded, with \( \partial M \) a strictly convex curve \( C \subset \mathcal{H} \). Assume \( M \) is transversal to \( \mathcal{H} \) along \( C \) and \( M \subset R^3_+ \) near \( C \).

Also assume \( M \) is compact or \( M \) is complete with a finite number of vertical annular ends \( \mathscr{A}(\tau_1), \ldots, \mathscr{A}(\tau_n) \) contained in \( R^3_+ \).

If \( M \) is compact, then \( M \subset R^3_+ \).

If \( M \) is noncompact, then either \( M \subset R^3_+ \) or there is a simple closed curve
\( \alpha \subset M \cap \mathcal{H} \cap \text{ext}(D), \) \( \alpha \) generates \( \pi_1(\text{ext}(D)) \), and \( \text{area}(D) < \sum_{i=1}^{n} \pi \left( \frac{\tau_i}{H} - \tau_i^2 \right) \).

**Proof of Theorem 2.** For each annular end \( \mathcal{A} = \mathcal{A}(\tau_j) \) of \( M \), let \( D_j \) be a horizontal disk whose boundary is a simple closed curve of \( \mathcal{A} \), a generator of \( \pi_1(\mathcal{A}) \). Let \( \nu_j \) denote the conormal to \( M \) along \( \partial D_j \), oriented down, i.e., \( \langle \nu_j, e_3 \rangle < 0 \). Let \( \nu \) denote the inward pointing conormal to \( M \) along \( C = \partial M \).

We can assume \( M \) is transverse to \( \mathcal{H} \) by displacing \( M \) slightly in the vertical direction.

First we prove that \( M \cap \text{ext}(D) = \emptyset \) and \( M \cap \text{int}(D) \neq \emptyset \) is impossible. So we assume the contrary and arrive at a contradiction.

Let \( M_1 \) be the connected component of \( M \cap \mathbb{R}^3 \) that contains \( C \); \( \partial M_1 = C \cup C_1 \cup \cdots \cup C_m \), \( C_j \subset \text{int} D \), \( 1 \leq j \leq m \). \( M_1 \) together with a proper subdomain \( D_0 \) of \( D \), bound a 3-dimensional domain \( W \subset \mathbb{R}^3 \), not necessarily compact (cf. Figure 1).

For each annular end \( \mathcal{A}(\tau_j) \) in \( M_1 \), we consider the disk \( D_j \) with \( \partial D_j \) a generator of \( \pi_1(\mathcal{A}(\tau_j)) \). Form a compact cycle \( E \) by removing from \( M_1 \cup D_0 \), the part of each \( \mathcal{A}(\tau_j) \) that is above \( \partial D_j \) and attaching \( D_j \). One could choose the \( D_j \) at different heights to ensure \( E \) is embedded, but this is not necessary.

Let \( Y \) be a constant vector field of \( \mathbb{R}^3 \). The flux of \( Y \) across \( E \) is zero and this yields the “balancing formula \( \langle Y = e_3 \rangle \),” [1, 3]:

\[
(2.1) \quad \frac{1}{2H} \int_{\partial D_0} e_3 \cdot \nu = \text{area}(D_0) + \sum_j \pi \left( \frac{\tau_j}{H} - \tau_j^2 \right).
\]

The sum is taken over all ends of \( M_1 \). Notice that this sum is nonnegative since each \( \tau_j \leq 1/2H \). We indicate how (2.1) is derived: since the flux of \( Y \) is zero across \( E \),

\[
\int_{D_0} Y \cdot n_{D_0} + \int_{D_j} Y \cdot n_{D_j} + \sum_j \int_{D_j} Y \cdot n_{D_j} = 0.
\]
Here $M_2 = E - (D_0 \cup \bigcup D_j)$, $n_{D_0} = e_3$, $n_{D_j} = -e_3$, $j \geq 1$, and $n_{M_2} = \frac{\vec{H}}{H}$.

The reason $n_{D_0} = e_3$ is that $\vec{H}$ points towards $D_0$ along $\partial M$; there are no exterior intersection curves of $M$ with $\mathcal{H}$ by assumption.

One has $\int_{M_2} e_3 \cdot n_{M_2} = -\frac{1}{2H} \int_{\partial M} e_3 \cdot \nu$, $\nu$ the inward pointing conormal to $M_2$ (this formula follows from $\Delta X = -2Hn_{M_2}$, $X$ the position vector field).

Each annular end $\mathcal{A}(\tau_j)$ converges geometrically to a Delaunay end $\mathcal{D}(\tau_j)$. Thus $\int_{D_j} e_3 \cdot n_{D_j} - \frac{1}{2H} \int_{\partial D_j} e_3 \cdot \nu = \pi \tau_j / H - \pi \tau_j^2$. To prove this last formula, one constructs a compact cycle and applies the divergence theorem. Choose a horizontal disk $\tilde{D}_j$, whose boundary approximates a “shortest” parallel circle of $\mathcal{D}(\tau_j)$ and do this so that $\partial \tilde{D}_j \cup \partial D_j$ bound an embedded annulus on $A(\tau_j)$. Then

$$\int_{D_j} e_3 \cdot n_{D_j} - \frac{1}{2H} \int_{\partial D_j} e_3 \cdot \nu = \int_{\tilde{D}_j} e_3 \cdot n_{\tilde{D}_j} - \frac{1}{2H} \int_{\partial \tilde{D}_j} e_3 \cdot \tilde{\nu}_j.$$  

As the $\tilde{D}_j$ get higher, $\partial \tilde{D}_j$ converges geometrically to the shortest parallel circle of $\mathcal{D}(\tau_j)$. Therefore the right side of the above equation is constant and equals $\pi (\tau_j / H - \tau_j^2)$. Let $Y = e_3$ and put these equations together; one obtains (2.1).

Now $\nu \cdot e_3 > 0$ along $\partial D_0$, so (2.1) implies:

$$\frac{1}{2H} \int_C e_3 \cdot \nu < \text{area}(D_0) + \sum_{j} \pi \left( \frac{\tau_j}{H} - \tau_j^2 \right).$$

Next consider the cycle $M \cap D$. The flux of $e_3$ across this cycle is also zero; more precisely, one obtains a compact cycle from $M \cup D$ as before, attaching $D_i$ to each annular end $\mathcal{A}(\tau_i)$ of $M$ and removing the part of $\mathcal{A}(\tau_i)$ that is above $\partial D_i$. The same type of calculation as above yields:

$$\frac{1}{2H} \int_C e_3 \cdot \nu = \text{area}(D) + \sum_{i=1}^{n} \pi \left( \frac{\tau_i}{H} - \tau_i^2 \right).$$

For future reference we remark that if the mean curvature vector $\vec{H}$ points towards $\text{ext}(D)$, along $C$, then (2.3) becomes:

$$\frac{1}{2H} \int_C e_3 \cdot \nu = -\text{area}(D) + \sum_{i=1}^{n} \pi \left( \frac{\tau_i}{H} - \tau_i^2 \right).$$

When $M$ is compact, the term $\sum_{i=1}^{n} \pi (\frac{\tau_i}{H} - \tau_i^2)$ does not appear in (2.3) or (2.4). Clearly (2.2) and (2.3) are impossible since $\text{area}(D_0) < \text{area}(D)$ and

$$\sum_{j} \pi \left( \frac{\tau_j}{H} - \tau_j^2 \right) \leq \sum_{i=1}^{n} \pi \left( \frac{\tau_i}{H} - \tau_i^2 \right).$$

This proves that $M \cap \text{ext}(D) = \emptyset$ and $M \cap \text{int}(D) \neq \emptyset$ is impossible.

So we may assume $M \cap \text{ext}(D) \neq \emptyset$. We will show that $M \cap \text{ext}(D)$ is exactly one Jordan curve that generates $\pi_1(\text{ext}(D))$, and $\text{area}(D) < \sum_{i=1}^{n} \pi (\frac{\tau_i}{H} - \tau_i^2)$.  

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The idea is to apply the Alexandrov reflection principle using $\varepsilon$-tilted planes. First we must do some cutting and pasting along the cycles of $M \cap \text{int}(D)$, to obtain a manifold that separates $\mathbb{R}^3$; this enables us to be sure the mean curvature vectors are pointing in the same direction when we do Alexandrov reflection.

Let $C_1, \ldots, C_m$ be the Jordan curves of $M \cap \text{int}(D)$. For each $C_j$, let $C_j^+(\varepsilon)$ be the planar curve on $M$, near $C_j$, obtained by intersecting $M$ with the plane $\{z = \varepsilon\}$. Similarly, let $C_j^-(\varepsilon)$ be the curve $M \cap \{z = -\varepsilon\}$ that is near $C_j$.

We form an embedded surface $N$ by removing from $M$ the annuli bounded by $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ and attaching the horizontal planar domains $D_j^+ \cup D_j^-$ bounded by $C_j^+(\varepsilon) \cup C_j^-(\varepsilon)$ (cf. [1]). Also we attach $D$ to $M$ along $C$.

To ensure that $N$ is embedded, one uses different values of $\varepsilon$, when several $C_j$ are concentric (cf. Figure 2). $N$ is a properly embedded submanifold (with corners) of $\mathbb{R}^3$, $\partial N = \emptyset$, hence each connected component of $N$ separates $\mathbb{R}^3$ into two connected components.

Let $\tilde{M}$ be the component of $N$ that contains $C$. We orient $\tilde{M}$ be the mean
curvature vector $\vec{H}$ of $M$. Notice that this makes sense since, abstractly, $\tilde{M}$ is a submanifold $L$ of $M$ (hence $\vec{H}$ is defined over $L$) to which one has attached $D$ and the discs $D_j^\pm(\varepsilon)$. Clearly $\vec{H}$ extends across these disks to define a normal field to $\tilde{M}$. The corners of $\tilde{M}$ along the boundaries of the disks, do not affect this.

Now when $\tilde{M}$ is compact, one can use Alexandrov reflection in vertical planes to prove Theorem 2, i.e., $\tilde{M} \subset \mathbb{R}^3_+$. This case was treated in [1]. For the readers benefit, we give the argument here as well.

Consider a family of vertical planes $Q(t)$ coming from infinity and apply the Alexandrov technique to $\tilde{M}$. Observe that $\tilde{M} \cap \text{ext} D$ can have no cycle $\beta$ that is null homotopic in $\text{ext} D$, since one could come from infinity in a direction such that for some position of the vertical plane $Q(t)$ (before reaching $C$), a symmetry of $\beta$ by $Q(t)$ would touch $\beta$ at an interior point. Thus $\tilde{M}$ would have a plane of symmetry before reaching $C$, a contradiction; $\tilde{M}$ is a graph on one side of this plane and contains $C$ on the other side. We used $C$ convex in this argument. Now if $\tilde{M} \cap \text{ext}(D)$ contained two cycles $\alpha_1, \alpha_2$, both generators of $\pi_1(\text{ext} D)$, then again one uses vertical planes $Q(t)$ coming from infinity. Clearly there is some position of $Q(t)$, before reaching $C$, where a symmetry of $\alpha_1$ by $Q(t)$ meets a point of $\alpha_2$ (assuming $\alpha_1$ is the exterior cycle). Thus $\tilde{M}$ would have a vertical plane of symmetry before reaching $C$.

This proves that $\tilde{M} \cap \text{ext}(D)$ is at most one cycle $\alpha$ and $\alpha$ generates $\pi_1(\text{ext} D)$. Since the mean curvature vector $\vec{H}$ points into the bounded component of $\mathbb{R}^3 - \tilde{M}$, $\vec{H}$ points towards $C$ along $\alpha$ and hence, $\vec{H}$ points towards $\text{ext}(D)$ along $C$.

When $\tilde{M}$ is compact this implies $n_D = -e_3$, which contradicts the balancing formula: $(1/2H) \int_{\zeta} e_3 \cdot \nu = -\text{area}(D)$.

Now suppose $\tilde{M}$ is not compact. Let $Q(t)$ be a family of parallel planes, tilted an angle $\varepsilon > 0$ from the vertical. The ends of $\tilde{M}$ are contained in some fixed vertical cylinder.

As in the case $\tilde{M}$ compact, there are no cycles in $\text{ext}(D) \cap \tilde{M}$ that are null homotopic in $\text{ext}(D)$. For if $\beta$ were such a cycle, we could approach $\tilde{M}$ by $Q(t)$ from infinity, in a direction such that a position of $Q(t)$ would be attained, before reaching $C$, where a symmetry of $\beta$ would touch $M$ at an interior point of $M$ (near $\beta$). Thus there would be a plane of symmetry of $\tilde{M}$, among the $Q(t)$, before reaching $C$. This is impossible.

Also, $\tilde{M} \cap \text{ext}(D)$ can contain at most one cycle that generates $\pi_1(\text{ext}(D))$. For if $\alpha_1, \alpha_2$ were two such cycles, then coming from infinity with the tilted planes $Q(t)$, there would be a $Q(t)$ where a symmetry of $\alpha_1$ would touch $\tilde{M}$ at a point near $\alpha_2$. This occurs before reaching $C$ so one would have a plane of symmetry before reaching $C$; a contradiction.

Thus if $\alpha$ is a cycle in $\tilde{M} \cap \text{ext}(D)$ that generates $\pi_1(\text{ext} D)$, we conclude,
as in the case $\tilde{M}$ compact, that the mean curvature vector $\overline{H}$ of $M$ points towards $\text{ext}(D)$ along $C$.

By the balancing formula (2.4):

$$\frac{1}{2H} \int_C Y \cdot \nu = \sum_{i=1}^{n} \pi \left( \frac{\tau_i}{H} - \tau_i^2 \right) - \text{area}(D).$$

Since $Y = e_3$, $e_3 \cdot \nu > 0$ along $C$. Hence $\text{area}(D) < \sum_{i=1}^{n} \pi(\tau_i/H - \tau_i^2)$.

This completes the proof of Theorem 2.

**Corollary.** Assume $M$ satisfies the hypothesis of Theorem 2 and $\partial M$ is a circle of radius $r$. Then if $r \geq \sqrt{n}/2H$, $M$ equals a Delaunay surface.

The problem posed by Theorem 2 is whether examples exist that satisfy the hypothesis of Theorem 2, and that are not contained in $\mathbb{R}^3$.

Here are some examples that we cannot exclude by our techniques. First consider the compact surface $M_0$ of Figure 3.
This surface cannot have constant mean curvature (Theorem 2, or more simply the balancing formula (2.4)). However perhaps one can puncture $M_0$ at one (or more) points, at the top, and attach vertical Delaunay type ends at the punctures. In Figures 4, 5 we indicate some possibilities.

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