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A NOTE ON THE HOMOLOGY OF FREE ABELIANIZED EXTENSIONS

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ABSTRACT. Let G be a group given by a free presentation $G = F/N$, and N' the commutator subgroup of N. The quotient F/N' is called a free abelianized extension of G. We study the integral homology of F/N'. In particular, if G has no elements of order p (p an odd prime), we determine the p-torsion in dimension p^2 in terms of the modulo p homology of G. This extends results of Kuz′min [5, 6] describing the p-torsion in smaller dimensions. Our approach is based on examining the homology of G with coefficients in symmetric powers of the augmentation ideal, which we relate to the integral homology of F/N'.

1. INTRODUCTION

Let G be a group and

$$1 \to N \to F \to G \to 1 \tag{1.1}$$

a free presentation of G, in which F is free and noncyclic. Let $\Phi = F/N'$, and let $M = N/N'$ be the relation module of G coming from (1.1). The group Φ is called a free abelianized extension of G, and we have the exact sequence

$$1 \to M \to \Phi \to G \to 1.$$

In his fundamental paper [5], Kuz′min has studied the integral homology groups $H_n\Phi$, and some additional information has been obtained in his recent paper [6]. Since the case $n = 1$ is trivial ($H_1\Phi = F/F'$), the case $n = 2$ is relatively easy and well understood (Hopf's formula gives $H_2\Phi \cong H_0(G, M \wedge M)$, and a description of the latter can be found in [5, 7]), we focus on $n \geq 3$. For an abelian group X, we denote by tX its torsion subgroup and, if m is a natural number, we write t_mX for the subgroup of all elements $x \in X$ with $m^rx = 0$ for some $r \geq 1$. Kuz′min has proved that the torsion subgroup $tH_n\Phi$ is of finite exponent [5, Theorem 4], and pointed out that the exponent of the odd part of $tH_n\Phi$ divides n [6, Theorem 1]. In the case when G has no p-torsion (p

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an odd prime), he established an isomorphism $t_pH_pG \cong H_{p+2}(G, \mathbb{Z}_p)$, where $\mathbb{Z}_p = \mathbb{Z}/p\mathbb{Z}$ regarded as a trivial G-module [5, Theorem 8]. In [6], Kuz′min has made the following conjectures on p-torsion $(p \geq 3)$ in $H_n\Phi$.

C1. For all $n \geq 3$, $pt_pH_n\Phi = 0$.

C2. Suppose that G has no p-torsion and let $s \geq 1$. Then

$$t_p H_{p^s}\Phi \cong \bigoplus_{i=0}^{s-1} H_{p^s+2p^i}(G, \mathbb{Z}_p). \tag{1.2}$$

The results described above confirm C1 for all n with $(n, p) = p$ or $(n, p) = 1$ (in the latter case there is no p-torsion at all), and C2 for $s = 1$. Also, Kuz′min announced without proof that

$$t_p H_{mp}\Phi \cong H_{(p+2)m}(G, \mathbb{Z}_p) \tag{1.3}$$

if $1 \leq m < p$ and G has no p-torsion [6, p. 859].

In this paper we shall prove (1.2) for the case $s = 2$, and obtain some information on the general case. Our method will also establish (1.3).

For an abelian group X, let $\bigwedge^n X$ denote the nth exterior power of X and X^n the nth symmetric power (our notation differs here from Kuz′min's). If X is a G-module, then these are G-modules via the diagonal action. Kuz′min's approach involves showing that

$$tH_n\Phi \cong t_nH_0\left(G, \bigwedge^n M\right) \tag{1.4}$$

"up to 2-torsion," meaning that the odd parts of these groups are isomorphic [5, Theorem 7]. Because of (1.4), questions about odd torsion in $H_n\Phi$ are equivalent to questions about homology of G with coefficients in exterior powers of M, which was extensively examined in [5, 6]. An alternative treatment of $H_*(G, \bigwedge^n M)$ has been given by Hannebauer and the second author [3]. In particular, they improved an "up to 2-torsion" result of [6] by showing that $H_k(G, \bigwedge^n M)$ $(k \geq 1)$ and $tH_0(G, \bigwedge^n M)$ have exponent dividing $2n(n-1)$ for all $n \geq 2$.

Our approach is to transform the problem to one about homology of G with coefficients in symmetric powers $(IG)^n$ of the augmentation ideal IG of $\mathbb{Z}G$, using the exact sequence

$$\begin{aligned} 0 \to \bigwedge^n M \to \bigwedge^{n-1} M \otimes P \to \cdots \to \bigwedge^{n-i} M \otimes P^i \to \cdots \\ \to M \otimes P^{n-1} \to P^n \to (IG)^n \to 0, \end{aligned} \tag{1.5}$$

which comes from the relation sequence

$$0 \to M \to P \to IG \to 0 \tag{1.6}$$

and [1, §9, ex. 2]. In the sequence (1.6), P is the free G-module $IF \otimes_G \mathbb{Z}G$. If G has no $n!$-torsion, then the modules P^i $(1 \leq i \leq n)$ are G-free [4, Lemma

4.1], and we can use dimension shifting through (1.5) to relate the homologies of G with coefficients in $\bigwedge^n M$ and $(IG)^n$. With a little more argument we get (see §2)

Proposition 1.1. *If G has no p-torsion, then*

$$t_pH_k\left(G, \bigwedge^n M\right) \cong t_pH_{k+n}(G, (IG)^n)$$

for all $k \geq 0$, $n \geq 1$.

Our main results refer to $H_*(G, (IG)^n)$ for prime powers $n = p^s$, including $p = 2$.

Theorem 1. *Let p be a prime and suppose that G has no p-torsion. Then, for all $k \geq 0$,*

$$t_pH_k(G, (IG)^{p^2}) \cong H_{k+2}(G, \mathbb{Z}_p) \oplus H_{k+2p}(G, \mathbb{Z}_p).$$

Theorem 2. *Let p be a prime and suppose that G has no p-torsion. Then, for all $k \geq 0$ and $s \geq 1$,*

$$t_pH_k(G, (IG)^{p^s}) \cong H_{k+2}(G, \mathbb{Z}_p) \oplus Y,$$

where $p^{s-1}Y = 0$.

We mention that, for any group G and any $n \geq 2$, the groups $H_k(G, (IG)^n)$ $(k \geq 1)$ and $tH_0(G, (IG)^n)$ are annihilated by $2n(n-1)$ [4, Theorem 5.1], so that $t_pH_k(G, (IG)^n) = 0$ for all $n \not\equiv 0, 1 \bmod p$. Now, if p is an odd prime, our theorems together with Proposition 1.1 and (1.4) imply immediately the following

Corollary 3. *Let p be an odd prime and suppose that G has no p-torsion. Then*

$$t_pH_{p^2}\Phi \cong H_{p^2+2}(G, \mathbb{Z}_p) \oplus H_{p^2+2p}(G, \mathbb{Z}_p),$$

and, for all $s \geq 1$,

$$t_pH_{p^s}\Phi \cong H_{p^s+2}(G, \mathbb{Z}_p) \oplus Y,$$

where $p^{s-1}Y = 0$. □

Symmetric powers of IG will be studied by embedding them in symmetric powers of $\mathbb{Z}G$ and using the G-module maps $\tau(n) : (\mathbb{Z}G)^n \to (\mathbb{Z}G)^{n-1}$ given by

$$(g_1 \circ g_2 \circ \cdots \circ g_n)\,\tau(n) = \sum_{i=1}^{n} g_1 \circ \cdots \circ \hat{g}_i \circ \cdots \circ g_n,$$

where $g_1, \ldots, g_n \in G$ and the circumflex denotes that g_i is omitted. The kernel of $\tau(n)$ is $(IG)^n$ (see §2). These maps make sense also when $\mathbb{Z}$ is replaced by any commutative ring with 1, and will not be distinguished notationally. We remark in passing that if $\bigoplus_{n=0}^{\infty}(\mathbb{Z}G)^n$, the symmetric algebra on $\mathbb{Z}G$ (with $(\mathbb{Z}G)^0 = \mathbb{Z}$), is regarded as the polynomial ring on the group elements as indeterminates, then $\bigoplus_{n=0}^{\infty}\tau(n)$ is the sum of the first partial derivatives with respect to the group elements.

An interesting property of these maps that emerges during our discussion is the following.

Proposition 1.2. *Let p be a prime. Then*

$$(1.7) \quad \cdots \to (\mathbb{Z}_pG)^{rp} \to (\mathbb{Z}_pG)^{rp-1} \to (\mathbb{Z}_pG)^{(r-1)p} \to (\mathbb{Z}_pG)^{(r-1)p-1} \to \cdots$$
$$\to (\mathbb{Z}_pG)^{p} \to (\mathbb{Z}_pG)^{p-1} \to \mathbb{Z}_p \to 0,$$

where $(\mathbb{Z}_pG)^{rp} \to (\mathbb{Z}_pG)^{rp-1}$ is $\tau(rp)$ $(r = 1, 2, \ldots)$, and $(\mathbb{Z}_pG)^{rp-1} \to (\mathbb{Z}_pG)^{(r-1)p}$ is $\tau(rp-1)\tau(rp-2)\cdots\tau((r-1)p+1)$ $(r = 1, 2, \ldots)$, is a $\mathbb{Z}_pG$-resolution of $\mathbb{Z}_p$.

If G is torsion-free, the modules $(\mathbb{Z}_pG)^n$ are free $\mathbb{Z}_pG$-modules [4, Lemma 4.1] and (1.7) is a *free* $\mathbb{Z}_pG$-resolution of $\mathbb{Z}_p$. If G is merely p-torsion free, the symmetric powers of $\mathbb{Z}_pG$ need not be free, but their homology is concentrated in dimension zero (see §2).

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2. Symmetric powers of group rings and free modules

In this section, let S be a commutative ring with 1. We later use R to denote the integers localized at a fixed prime p. Let the elements of G be totally ordered in any way. Then $(SG)^n$ has an S-basis consisting of all elements

$$g_1 \circ g_2 \circ \cdots \circ g_n \qquad (g_i \in G,\ g_1 \le g_2 \le \cdots \le g_n).$$

By replacing each $g_i \neq 1$ by $1 + (g_i - 1)$ and expanding, we easily see that the elements

$$(2.1) \quad (g_1 - 1)\circ\cdots\circ(g_l - 1)\circ 1\circ\cdots\circ 1 \qquad (g_i \in G,\ g_i \neq 1,\ g_1 \le \cdots \le g_l,\ 0 \le l \le n)$$

also form an S-basis of $(SG)^n$, which we call the *augmentation basis.* In particular, the elements (2.1) with $l = n$ form an S-basis for the canonical image of $(I_SG)^n$, the nth symmetric power of the augmentation ideal I_SG of SG, in $(SG)^n$.

Consider the G-module homomorphism $\tau(n)\colon (SG)^n \to (SG)^{n-1}$ defined by

$$(\alpha_1 \circ\cdots\circ \alpha_n)\,\tau(n) = \sum_{i=1}^{n} (\alpha_i\varepsilon)\,\alpha_1 \circ\cdots\circ \hat{\alpha}_i \circ\cdots\circ \alpha_n,$$

where $\alpha_1, \ldots, \alpha_n \in SG$ and ε is the augmentation map $SG \to S$. This is clearly the $\tau(n)$ described in the introduction. On an element (2.1), $\tau(n)$ takes the very simple form

$$(2.2) \quad ((g_1-1)\circ\cdots\circ(g_l-1)\circ 1\circ\cdots\circ 1)\,\tau(n) = (n-l)\,(g_1-1)\circ\cdots\circ(g_l-1)\circ 1\circ\cdots\circ 1$$

for $l < n$, and

$$((g_1 - 1) \circ \cdots \circ (g_n - 1))\,\tau(n) = 0 \tag{2.3}$$

for $l = n$. The number of ones in (2.2) is clear from the context. Note that the right-hand side of (2.2) is a scalar multiple of an element of the augmentation basis of $(SG)^{n-1}$.

It will be convenient to introduce the following notation. Let Γ be the set of all sequences

$$\gamma = (g_1, \ldots, g_l) \tag{2.4}$$

of finite length l $(l = 0, 1, 2, \ldots)$ with $g_i \in G$, $g_i \neq 1$ and $g_1 \leq \cdots \leq g_l$, including the empty sequence γ_0 of length 0. Let $\Gamma_n \subseteq \Gamma$ denote the subset of all γ with length $\leq n$. To any $n \geq 0$ and any sequence (2.4) we assign an element $\xi(n, \gamma) \in (SG)^n$ by setting

$$\xi(n, \gamma) = (g_1 - 1) \circ \cdots \circ (g_l - 1) \circ 1 \circ \cdots \circ 1 \quad \text{for } \gamma \in \Gamma_n$$

(here we have $n - l$ ones; in particular, $\xi(n, \gamma_0) = 1 \circ 1 \circ \cdots \circ 1$), and

$$\xi(n, \gamma) = 0 \quad \text{for } \gamma \in \Gamma \backslash \Gamma_n.$$

For $n > m \geq 0$ let $\sigma(n, m)$ denote the homomorphism $(SG)^n \to (SG)^m$ defined as the composite $\sigma(n, m) = \tau(n)\tau(n-1)\cdots\tau(m+1)$. For an element (2.4) of Γ and $n > m$ we put

$$\kappa(n, m, \gamma) = (n - l)(n - l - 1)\cdots(m + 1 - l)$$

so $\kappa(n, m, \gamma) \in S$ is the product of the $n - m$ successive numbers from $m + 1 - l$ to $n - l$. Now we state some rather obvious properties of these notions.

Lemma 2.1. (i) *For any $n \geq 0$, the augmentation basis of $(SG)^n$ consists of all elements $\xi(n, \gamma)$ with $\gamma \in \Gamma_n$.*

(ii) *For $n > m \geq 0$ and $\gamma \in \Gamma$ we have*

$$\xi(n, \gamma)\sigma(n, m) = \kappa(n, m, \gamma)\xi(m, \gamma). \tag{2.5}$$

(iii) *If Ω is a finite subset of Γ_n, then a linear combination*

$$\sum_{\gamma \in \Omega} \alpha(\gamma)\xi(n, \gamma) \in (SG)^n$$

$(\alpha(\gamma) \in S,\ \alpha(\gamma) \neq 0)$ is in the kernel of $\sigma(n, m)$ if and only if for all $\gamma \in \Omega$ either $\gamma \in \Gamma_n \backslash \Gamma_m$ or $\alpha(\gamma)\kappa(n, m, \gamma) = 0$ in S.

(iv) *If S is a domain of characteristic zero, then the elements $\xi(n, \gamma)$ with $\gamma \in \Gamma_n \backslash \Gamma_m$ form an S-basis for the kernel of $\sigma(n, m)$. In particular, the kernel of $\tau(n) = \sigma(n, n-1)$ coincides with the canonical image of $(I_S G)^n$ in $(SG)^n$.*

Proof. (i) is clear from the definition. For $m = n - 1$ $(\sigma(n, n-1) = \tau(n))$, (2.5) is essentially the same as (2.2), (2.3). The case $m < n - 1$ follows by trivial induction on $n-m$. Finally, (iii) and (iv) are obvious consequences of (i) and (ii). $\square$

Proposition 1.2 is now easy to establish.

Proof of Proposition 1.2. We work here with $S = \mathbb{Z}_p$. Let $n \geq 1$. In view of Lemma 2.1(iii), the kernel of $\tau(n)\colon (\mathbb{Z}_pG)^n \to (\mathbb{Z}_pG)^{n-1}$ has a $\mathbb{Z}_p$-basis consisting of all elements $\xi(n, \gamma)$ such that $\gamma \in \Gamma_n$ and $\kappa(n, n-1, \gamma) \equiv 0 \bmod p$ (on noting that $\gamma \in \Gamma_n \setminus \Gamma_{n-1}$ implies $\kappa(n, n-1, \gamma) = 0$). Now

$$\xi(n+p-1, \gamma)\,\sigma(n+p-1, n) = \kappa(n+p-1, n, \gamma)\,\xi(n, \gamma). \tag{2.6}$$

If γ has length l, one has $\kappa(n, n-1, \gamma) = n-l$ and

$$\kappa(n+p-1, n, \gamma) = (n+p-1-l)(n+p-2-l)\cdots(n+1-l).$$

Consequently, $\kappa(n, n-1, \gamma) \equiv 0 \bmod p$ implies that $\kappa(n+p-1, n, \gamma)$ is a unit in $\mathbb{Z}_p$, and (2.6) gives $\operatorname{im}\sigma(n+p-1, n) = \ker\tau(n)$ for all $n \geq 1$. Similarly, we get $\operatorname{im}\tau(n+1) = \ker\sigma(n, n-p+1)$ for all $n \geq p-1$. In particular, (1.7) is exact. $\square$

The symmetric powers $(\mathbb{Z}_pG)^n$ $(n \geq 1)$ are $\mathbb{Z}_pG$-free if G is torsion-free [4, Lemma 4.1]. So we can state the following

Corollary 2.2. *If G is torsion-free, then* (1.7) *is a free $\mathbb{Z}_pG$-resolution of $\mathbb{Z}_p$.* $\square$

In the sequel we will work with groups that are p-torsion free, that is, they have no elements of order p. For such groups we have

Lemma 2.3. *Let p be a prime, B a free SG-module. Suppose that G is p-torsion free and every prime $q \neq p$ is invertible in S. Then $H_k(G, D \otimes_S B^n) = 0$ for all SG-modules D, $k \geq 1$ and $n \geq 1$.*

Proof. The symmetric power B^n is a permutation module and the stabilizers of all elements of its canonical permutation basis are finite subgroups of G [4, §4]. Consequently, B^n decomposes into the direct sum of induced modules

$$\operatorname{Ind}_H^G S = S \otimes_H \qquad H \leq G,\ |H| < \infty$$

(see for example [2, III.5]). Therefore, to prove our lemma it suffices to show that $H_k(G, D \otimes_S \operatorname{Ind}_H^G S) = 0$ for all $k \geq 1$ and finite $H \leq G$. Now

$$D \otimes_S \operatorname{Ind}_H^G S \cong \operatorname{Ind}_H^G \left(\operatorname{Res}_H^G D \otimes_S S\right).$$

By Shapiro's Lemma,

$$H_k\left(G, D \otimes_S \operatorname{Ind}_H^G S\right) \cong H_k\left(H, \operatorname{Res}_H^G D\right).$$

The right-hand side is annihilated by $|H|$ if $k \geq 1$. The hypotheses imply that $|H|$ is invertible in S, whence the assertion. $\square$

Recall that R denotes the integers localized at p, and that tensoring with R, that is, localizing at p, is an exact functor on the category of abelian groups.

Thus $H_k(G, X \otimes R) \cong H_k(G, X) \otimes R$ for any G-module X. We deduce

Lemma 2.4. $t_p H_k(G, X) \cong tH_k(G, X \otimes R)$ *for any G-module X.* □

For this reason, we usually work over R instead of $\mathbb{Z}$ from now on.

Proof of Proposition 1.1. By Lemma 2.4 it suffices to show that

$$tH_k\left(G, \bigwedge^n(M \otimes R)\right) \cong tH_{k+n}(G, (I_R G)^n).$$

Consider the exact sequence (1.5) localized at p. Then Lemma 2.3 tells us that all its terms except $\bigwedge^n(M \otimes R)$ and $(I_R G)^n$ have zero homology in positive dimensions. So for $k \geq 1$, Proposition 1.1 follows by dimension shifting through (1.5) localized at p. For $k = 0$, note that $\bigwedge^{n-1} M \otimes P$ is actually free. So, by dimension shifting we get

$$\ker\left(H_0\left(G, \bigwedge^n(M \otimes R)\right) \to H_0\left(G, \bigwedge^{n-1} M \otimes P \otimes\right)\right) \cong H_n(G, (I_R G)^n).$$

The proposition follows on noting that $tH_0(G, \bigwedge^n(M \otimes R))$ is contained in the kernel on the left-hand side (indeed, the corresponding image lies in the free R-module $H_0(G, \bigwedge^{n-1} M \otimes P \otimes R)$). □

3. Homology of symmetric powers of IG

In this section we are concerned with discussing p-torsion in $H_*(G, (IG)^n)$, where p is an arbitrary prime. We shall prove our two theorems and establish (1.3). By Lemma 2.4, there is an isomorphism

$$t_p H_k(G, (IG)^n) \cong tH_k(G, (IG)^n \otimes R), \tag{3.1}$$

so we can work with symmetric powers of $I_R G$, the augmentation ideal of the localized group ring RG. Throughout this section we will assume that G is p-torsion free.

By Lemma 2.1(iv) we have a 4-term exact sequence

$$0 \to (I_R G)^n \to (RG)^n \xrightarrow{\tau(n)} (RG)^{n-1} \to \operatorname{coker} \tau(n) \to 0.$$

By Lemma 2.3, the homology of the middle terms is zero in positive dimensions, and we get by dimension shifting

$$H_k(G, (I_R G)^n) \cong H_{k+2}(G, \operatorname{coker} \tau(n)) \qquad (k \geq 1,\ n \geq 2). \tag{3.2}$$

If $n = p^s$ $(s \geq 1)$, we also have

$$tH_0(G, (I_R G)^n) \cong tH_2(G, \operatorname{coker} \tau(n)). \tag{3.3}$$

This can be proved along the same lines as the case $k = 0$ in the proof of Proposition 1.1 on noting that $(RG)^{p^s}$ is free [4, Lemma 4.1]. Thus, we transfer our attention to $\operatorname{coker} \tau(n)$, the R-structure of which is quite transparent. Recall that $(RG)^{n-1}$ has an R-basis consisting of all elements $\xi(n-1, \gamma)$ with $\gamma \in \Gamma_{n-1}$ (see §2). Let $\pi(m)$ denote the largest power of p dividing the natural number m. The following is clear from Lemma 2.1.

Lemma 3.1. *The cokernel of $\tau(n)$ is a direct sum of finite cyclic p-groups generated by the elements $\xi(n-1, \gamma) + \operatorname{im}\tau(n)$, $\gamma \in \Gamma_{n-1}$, such that p divides $\kappa(n, n-1, \gamma)$. The order of $\xi(n-1, \gamma) + \operatorname{im}\tau(n)$ is $\pi(\kappa(n, n-1, \gamma))$.* □

Now consider the commutative diagram

$$\begin{array}{ccccc} (\mathbb{Z}_p G)^n & \xrightarrow{\overline{\tau}(n)} & (\mathbb{Z}_p G)^{n-1} & \longrightarrow & \operatorname{coker}\overline{\tau}(n) \\ \uparrow & & \uparrow & & \uparrow \\ (RG)^n & \xrightarrow{\tau(n)} & (RG)^{n-1} & \longrightarrow & \operatorname{coker}\tau(n) \end{array}$$

where the vertical maps are reduction $\bmod p$ and the notation $\overline{\tau}(n)$ is introduced to distinguish the appropriate versions of $\tau(n)$. If $n = rp$ $(r \geq 1)$, the homology of $\operatorname{coker}\overline{\tau}(n)$ can be computed by dimension shifting through (1.7) (using Lemma 2.3), and in view of $\operatorname{coker}\tau(n) \otimes \mathbb{Z}_p \cong \operatorname{coker}\overline{\tau}(n)$ we get

$$H_k(G, \operatorname{coker}\tau(rp) \otimes \mathbb{Z}_p) \cong H_{k+2(r-1)}(G, \mathbb{Z}_p) \qquad (k \geq 1,\ r \geq 1). \tag{3.4}$$

If $r < p$, Lemma 3.1 tells us that $\operatorname{coker}\tau(rp)$ is of exponent p. Hence

$$\operatorname{coker}\tau(rp) \cong \operatorname{coker}\tau(rp) \otimes \mathbb{Z}_p,$$

and from (3.1), (3.2) and (3.4) we deduce

Proposition 3.2. *Let G be p-torsion free and $1 \leq r < p$. Then*

$$t_p H_k(G, (IG)^{rp}) \cong H_{k+2r}(G, \mathbb{Z}_p) \qquad (k \geq 1). \ \square$$

In view of (1.4) and Proposition 1.1 this gives

Corollary 3.3 (Kuz′min [6]). *Suppose that p is an odd prime, G is p-torsion free and $1 \leq r < p$. Then $t_p H_{rp}\Phi \cong H_{(p+2)r}(G, \mathbb{Z}_p)$.* □

We conclude with the proofs of our two theorems. Since the case $n = p$ is established in Proposition 3.2 (for $r = 1$ the assertion extends to $k \geq 0$; and previously in [3, 5]), we can assume that $n = p^s$ with $s \geq 2$.

Proof of Theorem 2. Consider the commutative diagram

$$\begin{array}{ccccc} (RG)^{p^s} & \xrightarrow{\sigma(p^s,\, p^s - p^{s-1})} & (RG)^{p^s - p^{s-1}} & \longrightarrow & \operatorname{coker}\sigma(p^s,\, p^s - p^{s-1}) \\ \| & & \uparrow{\scriptstyle \sigma(p^s,\, p^s - p^{s-1})} & & \uparrow{\scriptstyle \varphi} \\ (RG)^{p^s} & \xrightarrow{\tau(p^s)} & (RG)^{p^s - 1} & \longrightarrow & \operatorname{coker}\tau(p^s) \end{array} \tag{3.5}$$

where φ is the induced homomorphism. By Lemma 2.1 we have for all $\gamma \in \Gamma_{p^s}$,

$$\xi(p^s, \gamma)\sigma(p^s, p^s - p^{s-1}) = \kappa(p^s, p^s - p^{s-1}, \gamma)\xi(p^s - p^{s-1}, \gamma),$$

where $\kappa(p^s, p^s - p^{s-1}, \gamma)$ is a product of p^{s-1} successive numbers $\leq p^s$ (and $\geq -p^{s-1}$). Let $t = \pi(p^{s-1}!)$. Then $\pi(\kappa(p^s, p^s - p^{s-1}, \gamma_0)) = p^{t+1}$ and

$\pi(\kappa(p^s, p^s - p^{s-1}, \gamma)) = p^t$ for all $\gamma \in \Gamma_{p^s - p^{s-1}} \backslash \gamma_0$. Hence, for the cokernel of $\sigma(p^s, p^s - p^{s-1})$ there is an exact sequence

$$0 \to \mathbb{Z}_p \to \operatorname{coker} \sigma(p^s, p^s - p^{s-1}) \to (RG)^{p^s - p^{s-1}} \otimes \mathbb{Z}_{p^t} \to 0,$$

where the embedding of $\mathbb{Z}_p$ into the cokernel is given by

$$1 \mapsto p^t \xi(p^s - p^{s-1}, \gamma_0) + \operatorname{im} \sigma(p^s, p^s - p^{s-1}).$$

By Lemma 3.1, $\operatorname{coker} \tau(p^s)$ is a direct sum of cyclic p-groups. Note that the order of $\xi(p^s - 1, \gamma_0) + \operatorname{im} \tau(p^s)$ is p^s, whereas the orders of the remaining generators of $\operatorname{coker} \tau(p^s)$ are $< p^s$. Hence, we have an exact sequence

$$0 \to \mathbb{Z}_p \to \operatorname{coker} \tau(p^s) \to \operatorname{coker} \tau(p^s) \otimes \mathbb{Z}_{p^{s-1}} \to 0,$$

where the embedding $\mathbb{Z}_p \to \operatorname{coker} \tau(p^s)$ is given by

$$1 \mapsto p^{s-1} \xi(p^{s-1} - 1, \gamma_0) + \operatorname{im} \tau(p^s).$$

Now consider (3.5). We have

$$\begin{aligned} p^{s-1} &\xi(p^s - 1, \gamma_0) \sigma(p^s - 1, p^s - p^{s-1}) \\ &= p^{s-1} \kappa(p^s - 1, p^s - p^{s-1}) \xi(p^s - p^{s-1}, \gamma_0) \\ &= p^t u \xi(p^s - p^{s-1}, \gamma_0), \end{aligned}$$

where u is a unit in R. Hence the induced homomorphism φ maps $p^{s-1} \times \operatorname{coker} \tau(p^s) (\cong \mathbb{Z}_p)$ isomorphically onto $p^t \operatorname{coker} \sigma(p^s, p^s - p^{s-1}) (\cong \mathbb{Z}_p)$, so φ induces a homomorphism of exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \mathbb{Z}_p \arrow[r] & \operatorname{coker} \sigma(p^s, p^s - p^{s-1}) \arrow[r] & (RG)^{p^s - p^{s-1}} \otimes \mathbb{Z}_{p^t} \arrow[r] & 0 \\ 0 \arrow[r] & \mathbb{Z}_p \arrow[r] \arrow[u] & \operatorname{coker} \tau(p^s) \arrow[r] \arrow[u, "\varphi"] & \operatorname{coker} \tau(p^s) \otimes \mathbb{Z}_{p^{s-1}} \arrow[r] \arrow[u] & 0 \end{tikzcd}$$

where the first vertical arrow is an isomorphism. By applying the homology functor we get the commutative diagram

$$\begin{tikzcd} & 0 \arrow[r] & H_k(G, \mathbb{Z}_p) \arrow[r] & H_k(G, \operatorname{coker} \sigma(p^s, p^s - p^{s-1})) \arrow[r] & 0 & \\ \cdots \arrow[r] & & H_k(G, \mathbb{Z}_p) \arrow[r] \arrow[u] & H_k(G, \operatorname{coker} \tau(p^s)) \arrow[r] \arrow[u] & H_k(G, \operatorname{coker} \tau(p^s) \otimes \mathbb{Z}_{p^{s-1}}) \arrow[r] & H_{k-1}(G, \mathbb{Z}_p) \arrow[r] & \cdots \end{tikzcd}$$

in which $H_k(G, \mathbb{Z}_p) \to H_k(G, \mathbb{Z}_p)$ is an isomorphism. This yields

$$H_k(G, \operatorname{coker} \tau(p^s)) \cong H_k(G, \mathbb{Z}_p) \oplus H_k(G, \operatorname{coker} \tau(p^s) \otimes \mathbb{Z}_{p^{s-1}}) \tag{3.6}$$

for all $k \geq 0$. Since $H_k(G, \operatorname{coker} \tau(p^s) \otimes \mathbb{Z}_{p^{s-1}})$ is annihilated by p^{s-1}, Theorem 2 follows in view of (3.1), (3.2) and (3.3). □

Proof of Theorem 1. Here we have $s = 2$ and the theorem follows by (3.6) and (3.4) with $r = p$. □

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