PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 113, Number 4, December 1991

ON THE SENDOV CONJECTURE FOR SIXTH DEGREE POLYNOMIALS

JOHNNY E. BROWN

(Communicated by Clifford J. Earle, Jr.)

ABSTRACT. The Sendov conjecture asserts that if $p(z) = \prod_{k=1}^{n}(z - z_k)$ is a polynomial with zeros $|z_k| \le 1$, then each disk $|z - z_k| \le 1$, $(1 \le k \le n)$ contains a zero of $p'(z)$. This conjecture has been verified in general only for polynomials of degree $n = 2, 3, 4, 5$. If $p(z)$ is an extremal polynomial for this conjecture when $n = 6$, it is known that if a zero $|z_j| \le \lambda_6 = 0.626997\ldots$ then $|z - z_j| \le 1$ contains a zero of $p'(z)$. (The conjecture for $n = 6$ would be proved if $\lambda_6 = 1$.) It is shown that λ_6 can be improved to $\lambda_6 = 63/64 = 0.984375$.

The well-known conjecture of Sendov [6, Problem 4.5] posed in 1962 states that if $p(z) = \prod_{k=1}^{n}(z - z_k)$, $|z_k| \le 1$ $(1 \le k \le n)$, then each disk $|z - z_k| \le 1$ $(1 \le k \le n)$ contains a zero of $p'(z)$. Surprisingly this conjecture has been proved only for polynomials of degree $n \le 5$ and in a few special cases [2–4, 9–15]. (See Marden [8] for an excellent expository article on this conjecture.) The case $n = 5$ was proved in 1969 [9], while the general case for $n = 6$ is still open.

The Sendov conjecture can be viewed as an extremal problem over a compact family of functions as follows. Let $\mathscr{P}_n$ denote the family of all monic polynomials of the form

$$p(z) = \prod_{k=1}^{n}(z - z_k), \qquad |z_k| \le 1 \qquad (1 \le k \le n).$$

Thus, by the classical Gauss-Lucas Theorem we have

$$p'(z) = n\prod_{j=1}^{n-1}(z - \zeta_j), \qquad |\zeta_j| \le 1 \qquad (1 \le j \le n-1).$$

Define $I(z_k) = \min_{1 \le j \le n-1}|z_k - \zeta_j|$, $I(p) = \max_{1 \le k \le n} I(z_k)$, and $I(\mathscr{P}_n) = \sup_{p \in \mathscr{P}_n} I(p)$. Phelps and Rodriguez [10] proved that there exists an extremal polynomial $p_*(z) = \prod_{k=1}^{n}(z - z_k^*) \in \mathscr{P}_n$ such that $I(p_*) = I(\mathscr{P}_n)$ (see Lemma

Received by the editors November 5, 1989; presented to the 96th Annual Meeting of the American Mathematical Society, January 1990.

1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 30C15; Secondary 30C10.

A). It thus suffices to prove the conjecture for p_* i.e., show $I(z_k^*) \le 1$ for $1 \le k \le n$. They also proved a result which implies that if $n \ge 5$ and z_j^*, for some j, satisfies $|z_j^*| \le \lambda_n$, where λ_n is the unique root of $(1+x^2)(1+x)^{n-3} - n = 0$, then $I(z_j^*) \le 1$. In the special case $n = 6$ their result gives $\lambda_6 = 0.626997\ldots$. Of course the conjecture for $n = 6$ would be proved if $\lambda_6 = 1$. The purpose of this paper is to improve the bound λ_6:

Theorem. *If $p_*(z) = \prod_{k=1}^{6}(z - z_k^*)$ is an extremal polynomial for $I(\mathscr{P}_6)$ and $|z_j^*| \le 63/64$, for some $j = 1, 2, \ldots, 6$, then the disk $|z - z_j^*| \le 1$ contains a zero of $p_*'(z)$.*

This result now reduces the conjecture for the case $n = 6$ to considering the zeros of p_* close to $|z| = 1$. In this direction, we point out that Goodman, Rahman, and Ratti [5] have proved that if z_0 is a zero of $p(z)$ and $|z_0| = 1$, then the disk $|z - z_0/2| \le \frac{1}{2}$ contains a zero of $p'(z)$. Thus for boundary zeros a result stronger than the Sendov conjecture holds.

Our proof of the main result involves a blend of analytic and geometric ideas. The method presented here can be used to increase the bounds for λ_n for all $n \ge 6$.

We make use of the following known results:

Lemma A (Phelps and Rodriguez [10]). *There exists an extremal polynomial $p_* \in \mathscr{P}_n$ such that $I(p_*) = I(\mathscr{P}_n)$. Moreover, $p_*(z)$ has a zero on each subarc of $|z| = 1$ of length π.*

Lemma B (Bojanov, Rahman, and Szynal [2]). *If $Q(z)$ is a monic polynomial of degree n, $Q(0) = 0$, and $Q'(z) \ne 0$ in $|z| \le R$, then*

$$|Q(z)| > R^n - (R - \lambda)^n \quad \text{for } |z| = \lambda \le R \sin \pi/n.$$

We also need the following estimates:

Lemma 1. *Let $p(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k) \in \mathscr{P}_n$, $0 < a < 1$ and $p'(z) = n\prod_{j=1}^{n-1}(z - \zeta_j)$. If $I(a) > 1$, then*

$$\left|\frac{\zeta_j - a}{a\zeta_j - 1}\right| > \frac{1}{1 + a - a^2} \qquad j = 1, 2\ldots, n-1.$$

Lemma 2. *If $p(z) = (z - a)\prod_{k=1}^{n-1}(z - z_k)$ is an extremal polynomial for $I(\mathscr{P}_n)$, $0 < a < 1$ and $p'(z) = n\prod_{j=1}^{n-1}(z - \zeta_j)$, then*

$$\prod_{j=1}^{n-1}\left|\frac{\zeta_j - a}{a\zeta_j - 1}\right| \le \left[n - (n-3)a - \frac{4a^2}{1 + a^2}\right]^{-1}.$$

Lemma 3. *Let $p(z) = (z - a)\prod_{k=1}^{5}(z - z_k)$ be an extremal polynomial for $I(\mathscr{P}_6)$.*

(i) *If $0 \le a < \frac{2}{3}$, then $I(a) \le 1$.*

(ii) *If* $\frac{2}{3} \le a \le \frac{63}{64}$ *and* $I(a) > 1$, *then there exists a zero* $\zeta_0 = a + \rho_0 e^{i\theta_0}$ *of* $p'(z)$ *such that* $\rho_0 > 1$ *and*

$$\cos\theta_0 \ge 1/25 - a.$$

Proof of Theorem. Suppose $p_*(z) = \prod_{k=1}^{6}(z - z_k^*)$ is an extremal polynomial for $I(\mathscr{P}_6)$ and suppose that $|z_j^*| \le 63/64$ for some j. By a rotation, if necessary, we may suppose that $z_j^* = a$, $0 \le a \le 63/64$. Hence

$$p_*(z) = (z-a)\prod_{k=1}^{5}(z - z_k), \qquad |z_k| \le 1 \qquad k = 1, 2, \dots, 5.$$

Assume by way of contradiction that $I(a) > 1$. By Lemma 3(i) we must then have $\frac{2}{3} \le a \le \frac{63}{64}$. Thus, making use of Lemma 3(ii) we can assert that there exists a critical point $\zeta_0 = a + \rho_0 e^{i\theta_0}$ with $\rho_0 > 1$ and $\cos\theta_0 \ge 1/25 - a$. It follows that

$$\tag{1} \begin{aligned} \left|\frac{\zeta_0 - a}{a\zeta_0 - 1}\right| &\ge \frac{1}{[(1-a^2)^2 + a^2 - 2a(1-a^2)\cos\theta_0]^{1/2}} \\ &\ge \frac{1}{[(1-a^2)^2 + a^2 - 2a(1-a^2)(1/25 - a)]^{1/2}}. \end{aligned}$$

Assuming as we may that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \le |(\zeta_j - a)/(a\zeta_j - 1)|$ for the critical points $\zeta_j \ne \zeta_0$, we apply Lemma 2 with $n = 6$ to get

$$\tag{2} \left|\frac{\zeta_1 - a}{a\zeta_1 - 1}\right|^4 \left|\frac{\zeta_0 - a}{a\zeta_0 - 1}\right| \le \prod_{j=1}^{5}\left|\frac{\zeta_j - a}{a\zeta_j - 1}\right| \le \frac{1}{6 - 3a - 4a^2/(1+a^2)}.$$

Using (1) and (2) we obtain

$$\tag{3} \left|\frac{\zeta_1 - a}{a\zeta_1 - 1}\right|^4 \le \frac{[(1-a^2)^2 + a^2 - 2a(1-a^2)(1/25 - a)]^{1/2}}{6 - 3a - 4a^2/(1+a^2)} \equiv \phi(a).$$

Simple numerical calculations show that

$$\phi(a) \le \frac{1}{(1 + a - a^2)^4} \quad \text{for } \frac{2}{3} \le a \le \frac{63}{64}.$$

From (3) we conclude that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \le 1/(1 + a - a^2)$, contradicting Lemma 1. □

We now turn to the proofs of the lemmas.

Proof of Lemma 1. Assume by way of contradiction that $|(\zeta_1 - a)/(a\zeta_1 - 1)| \le 1/(1 + a - a^2)$. Hence we get $(\zeta_1 - a)/(a\zeta_1 - 1) = re^{i\theta}$, where $0 \le r \le 1/(1 + a - a^2)$. This gives $\zeta_1 = (a - re^{i\theta})/(1 - are^{i\theta})$ and so

$$|\zeta_1 - a| = \frac{r(1-a^2)}{|1 - are^{i\theta}|} \le \frac{r(1-a^2)}{1 - ar} \le 1,$$

which contradicts $I(a) > 1$. □

Proof of Lemma 2. Suppose that $p(z)$ is extremal and

$$p(z) = (z-a)\prod_{k=1}^{n-1}(z-z_k), \qquad |z_k| \le 1 \qquad (1 \le k \le n-1), \tag{4}$$

$$p'(z) = n\prod_{j=1}^{n-1}(z-\zeta_j), \qquad |\zeta_j| \le 1 \qquad (1 \le j \le n-1), \tag{5}$$

and $0 < a < 1$. Let $z = T(w) = (w-a)/(aw-1)$ and note that

$$p(T(w)) = p_0(w)(aw-1)^{-n},$$

where

$$p_0(w) = Aw(w^{n-1} + b_{n-1}w^{n-2} + \cdots + b_1).$$

The zeros of $p_0(w)$ are $0, w_1, w_2, \ldots, w_{n-1}$, where $w_k = T(z_k)$, $1 \le k \le n-1$. Hence we see that

$$b_1 = (-1)^{n-1}\prod_{k=1}^{n-1} w_k \tag{6}$$

and

$$b_{n-1} = -\sum_{k=1}^{n-1} w_k. \tag{7}$$

Differentiating $p(T(w))$ gives

$$\frac{dp(T(w))}{dw} = \frac{dp(T(w))}{dz}\cdot\frac{dz}{dw} = A_{\frac{1}{a}}p_0(w)[-a(aw-1)^{-n-1}],$$

where

$$A_{1/a}p_0(w) = np_0(w) + \left(\frac{1}{a} - w\right)p_0'(w) = B\prod_{j=1}^{n-1}(w-\gamma_j) \tag{8}$$

is the polar derivative of $p_0(w)$ with respect to $1/a$ (see Marden [7, p. 44]). Thus we get

$$p'(T(w)) = A_{1/a}p_0(w)[-a(aw-1)^{-n-1}dw/dz] \tag{9}$$

(where $'$ denotes differentiation with respect to z). It follows from (5), (8), and (9) that the zeros of $p'(z)$ and $A_{1/a}p_0(w)$ are related by

$$\gamma_j = \frac{\zeta_j - a}{a\zeta_j - 1}, \qquad 1 \le j \le n-1. \tag{10}$$

Next, we see after a simple check that

$$A_{1/a}p_0(w) = B\left[w^{n-1} + \cdots + \left(\frac{b_1}{n + ab_{n-1}}\right)\right] = B\prod_{j=1}^{n-1}(w-\gamma_j)$$

and so

$$\prod_{j=1}^{n-1} |\gamma_j| = \left| \frac{b_1}{n + ab_{n-1}} \right|. \tag{11}$$

By Lemma A there is a zero on each subarc of $|z| = 1$ of length π. Hence without loss of generality

$$\begin{cases} z_{n-1} = e^{i\theta_0}, & 0 \le \theta_0 \le \pi/2 \\ z_{n-2} = e^{i\theta_1}, & \theta_0 + \pi \le \theta_1 \le 2\pi. \end{cases}$$

(If $\operatorname{Im} z_{n-1} < 0$, simply consider $\overline{p(\overline{z})}$.) It is easy to check that

$$\begin{aligned} \operatorname{Re}\{T(z_{n-1}) + T(z_{n-2})\} &= \operatorname{Re}\left\{ \left(\frac{e^{i\theta_0} - a}{ae^{i\theta_0} - 1} \right) + \left(\frac{e^{i\theta_1} - a}{ae^{i\theta_1} - 1} \right) \right\} \\ &\le \frac{2a - (1 - a^2)\cos\theta_0}{(1 + a^2) - 2a\cos\theta_0} + \frac{2a + (1 + a^2)\cos\theta_0}{(1 + a^2) + 2a\cos\theta_0} \\ &= \frac{4a(1 + a^2)(1 - \cos^2\theta_0)}{(1 + a^2)^2 - 4a^2\cos^2\theta_0} \le \frac{4a}{1 + a^2}. \end{aligned}$$

Using (6), (7), (11), and the above inequality, we obtain

$$\begin{aligned} \prod_{j=1}^{n-1} \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| &\le \frac{1}{|n - a\sum_{k=1}^{n-1} T(z_k)|} \\ &\le \frac{1}{n - a\sum_{k=1}^{n-1} \operatorname{Re} T(z_k)} \\ &\le \frac{1}{n - 4a^2/(1 + a^2) - a(n - 3)}. \end{aligned}$$

The proof of the lemma is complete. □

Note that (11) immediately gives $\prod_{j=1}^{n-1} |\gamma_j| \le 1/(n - (n-1)a)$. However this estimate is not good enough for our purposes.

Proof of Lemma 3. Let $p(z) = (z - a)\prod_{k=1}^{5}(z - z_k)$ be extremal.

(i) $0 \le a < \frac{2}{3}$. Clearly $I(0) \le 1$ by the classical Gauss-Lucas Theorem. Next, from Lemma 2 with $n = 6$ we see that

$$\prod_{j=1}^{5} \left| \frac{\zeta_j - a}{a\zeta_j - 1} \right| \le \frac{1}{6 - 3a - 4a^2/(1 + a^2)}. \tag{12}$$

Assuming $|(\zeta_1 - a)/(a\zeta_1 - 1)| \le |(\zeta_j - a)/(a\zeta_j - 1)|$ for $j = 1, 2, \ldots, 5$ we have from (12) and an easy calculation that

$$\left| \frac{\zeta_1 - a}{a\zeta_1 - 1} \right| \le \frac{1}{(6 - 3a - 4a^2/(1 + a^2))^{1/5}} \le \frac{1}{1 + a - a^2}.$$

Thus, by Lemma 1, we get $I(a) \le 1$.

(ii) $\frac{2}{3} \le a \le \frac{63}{64}$ and $I(a) > 1$. For fixed a we apply Lemma B with $Q(z) = p(z+a)$, $n = 6$, $R = 1$, and $\lambda = 1 - (1-a)^{1/6}$ to conclude that

$$|p(z)| > 1 - (1-\lambda)^6 = a \ge |p(0)| \tag{13}$$

for $|z - a| = \lambda = 1 - (1-a)^{1/6}$. Note that $\lambda \le \frac{1}{2}$. Since $p'(z) \ne 0$ in $|z-a| \le 1$ and the degree of $p(z)$ is six, we see that by Alexander's Theorem (See Marden [7, p. 110, exercise 2]) that $p(z)$ is univalent in $|z-a| \le \frac{1}{2}$. Thus from (13) we see that there exists a *unique* z_0 such that $|z_0 - a| \le \lambda$ with $p(z_0) = p(0)$. We may assume that $\operatorname{Im} z_0 \ge 0$ (if not, consider $\overline{p(\overline{z})}$). Let Γ_0 be the perpendicular bisector of the segment from 0 to z_0, and let H_0^+ and H_0^- be the closed halfplanes bounded by Γ_0. By a variant of the Grace-Heawood Theorem (see [1, Lemma 1] for example) we see that $p'(z)$ has a zero in both H_0^+ and H_0^-. Let $\omega_0 = \mu_0(x_0, y_0) + i\nu_0(x_0, y_0)$ be the intersection of Γ_0 with the circle $|z-a| = 1$ with $\nu_0 > 0$. It follows that $p'(z)$ has a zero $\zeta_0 = a + \rho_0 e^{i\theta_0}$ with $\rho_0 > 1$ and $\cos\theta_0 \ge \mu_0(x_0, y_0) - a$. (See Figure 1.) It suffices to prove that $\mu_0(x_0, y_0) \ge 1/25$.

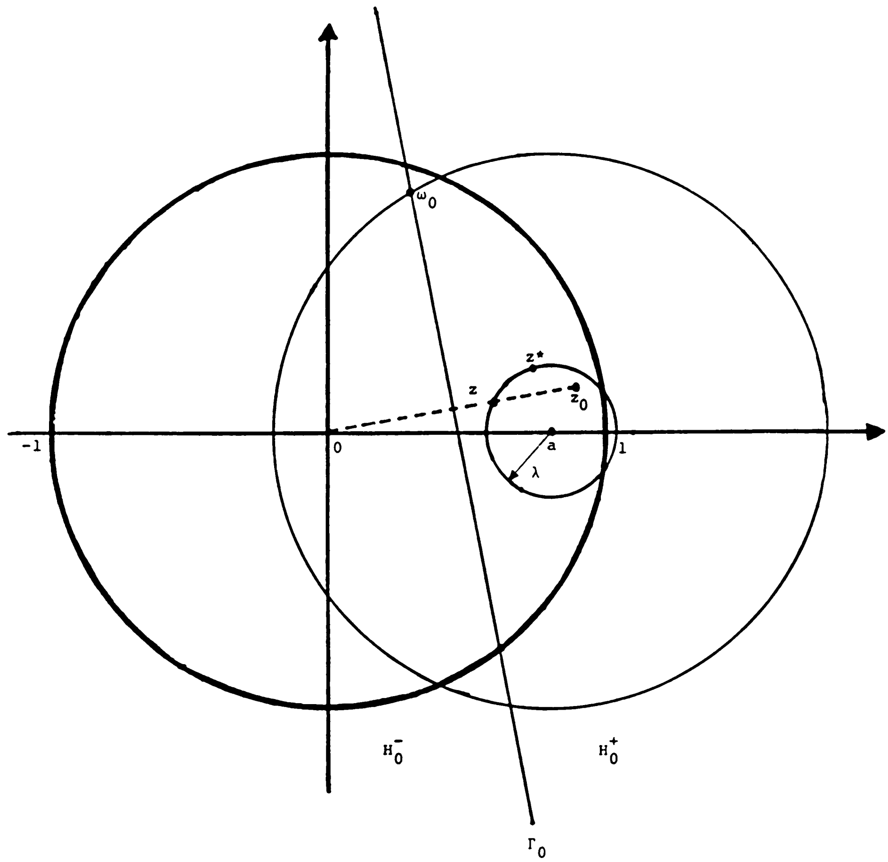

FIGURE 1

Let $z^* = x^* + iy^*$ be the point on $|z-a| = \lambda$ such that the line through 0 and z^* is tangent to $|z-a| = \lambda$ and $y^* > 0$. Hence $x^* = (a^2 - \lambda^2)/a$. If $z_0 = r_0 e^{it_0}$ then by letting $z = re^{it_0} = x + iy$ $(r \le r_0)$ be the point on $|z-a| = \lambda$ with $y > 0$ and $x \le x^*$, we conclude that $\mu_0(x_0, y_0) \ge \mu_0(x, y)$. Thus, it suffices to prove that $\mu_0(x, y) \ge 1/25$ where $(x-a)^2 + y^2 = \lambda^2$, $a - \lambda \le x \le x^*$, and $y \ge 0$.

A calculation shows that

$$\mu_0(x, y) = \frac{x(\lambda^2 + 3a^2) + 2a\beta - \sqrt{y^2(8ax + 4\beta - \beta^2)}}{2(2ax + \beta)}, \tag{14}$$

where $\beta = \lambda^2 - a^2$. Observe that since $a - \lambda \le x \le x^*$ we get

$$y^2 = \lambda^2 - (x-a)^2 \le 2\lambda(x - a + \lambda) \quad \text{and} \quad 2ax + \beta \le -\beta.$$

From (14) we conclude that

$$\mu_0(x, y) \ge \frac{x(\lambda^2 + 3a^2) + 2a\beta - \sqrt{2\lambda(x - a + \lambda)(8ax + 4\beta - \beta^2)}}{-2\beta} \equiv \mu_1.$$

It is enough to show that $\mu_1 \ge 1/25$.

Now $\mu_1 \ge 1/25$ if and only if

$$F(x) \equiv c_1 x^2 + c_2 x + c_3 \ge 0, \qquad a - \lambda \le x \le x^*, \tag{15}$$

where

$$\begin{aligned} c_1 &= (\lambda^2 + 3a^2)^2 - 16a\lambda, \\ c_2 &= 2\beta(2a + 0.08)(\lambda^2 + 3a^2) - 2\lambda\beta(4 - \beta) - 16a\lambda(\lambda - a), \\ c_3 &= \beta^2(2a + 0.08)^2 - 2\lambda\beta(4 - \beta)(\lambda - a), \end{aligned}$$

$(\beta = \lambda^2 - a^2)$. An easy check shows that $c_1 > 0$ and so (15) follows if $\Delta \equiv c_2^2 - 4c_1c_3 < 0$. A brief calculation shows that $\Delta = 4\lambda(\lambda - a)^2\Delta_0$, where

$$\begin{aligned} \Delta_0 = {} & [(\lambda + a)(4 - \beta) + 8a][8a\lambda + (\lambda + a)\{\lambda(4 - \beta) - (4a + 0.16)(\lambda^2 + 3a^2)\}] \\ & + (\lambda + a)[16a(\lambda + a)(2a + 0.08)^2 + 2(4 - \beta)(\lambda^2 + 3a^2)^2 - 32a\lambda(4 - \beta)]. \end{aligned}$$

Finally, a computation shows that $\Delta_0 < 0$ for $\frac{2}{3} \le a \le \frac{63}{64}$. (It can easily be checked numerically that $\Delta_0 < -0.009$.) Thus (15) holds and hence $\mu_0 \ge 1/25$. □

The proof of the Sendov conjecture has been elusive for more than twenty-five years and only verified in a few special cases. The method of proof in the case $n = 5$ does not seem to be useful for $n \ge 6$. It is not surprising to see the different ideas used to prove our results.

REFERENCES

1. A. Aziz, *On the zeros of a polynomial and its derivative*, Bull. Austral. Math. Soc. **31** (1985), 245–255.
2. B. Bojanov, Q. I. Rahman, and J. Szynal, *On a conjecture of Sendov about the critical points of a polynomial*, Math. Z. **190** (1985), 281–285.
3. D. A. Brannan, *On a conjecture of Ilieff*, Proc. Cambridge Philos. Soc. **64** (1968), 83–85.
4. J. E. Brown, *On the Ilieff-Sendov conjecture*, Pacific J. Math. **135** (1988), 223–232.
5. A. W. Goodman, Q. I. Rahman, and J. Ratti, *On the zeros of a polynomial and its derivative*, Proc. Amer. Math. Soc. **21** (1969), 273–274.
6. W. K. Hayman, *Research problems in function theory*, Athlone Press, London, 1967, 56 pp.
7. M. Marden, *Geometry of polynomials*, Amer. Math. Soc. Surveys, no. 3, 1966.
8. ——, *Conjectures on the critical points of a polynomial*, Amer. Math. Monthly **90** (1980), 267–276.
9. A. Meir and A. Sharma, *On Ilyeff's conjecture*, Pacific J. Math. **31** (1969), 459–467.
10. D. Phelps and R. Rodriguez, *Some properties of extremal polynomials for the Ilieff Conjecture*, Kodai Math. Sem. Report **24** (1972), 172–175.
11. Z. Rubinstein, *On a problem of Ilyeff*, Pacific J. Math. **26** (1968), 159–161.
12. E. B. Saff and J. Twomey, *A note on the location of critical points of polynomials*, Proc. Amer. Math. Soc. **27** (1971), 303–308.
13. G. Schmeisser, *Bermerkungen zu einer Vermutung von Ilieff*, Math. Z. **111** (1969), 121–125.
14. ——, *Zur Lage der knitischen Punkte eines polynomes*, Rend. Sem. Mat. Univ. Padova **46** (1971), 405–415.
15. ——, *On Ilieff's conjecture*, Math. Z. **156** (1977), 165–173.

DEPARTMENT OF MATHEMATICS, PURDUE UNIVERSITY, WEST LAFAYETTE, INDIANA 47907