A GLOBAL PINCHING THEOREM
FOR COMPACT MINIMAL SURFACES IN $S^3$

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Abstract. Let $M$ be a compact minimally immersed surface in the unit sphere $S^3$, and let $S$ denote the square of the length of the second fundamental form of $M$. We prove that if $\|S\|_2 < 2\sqrt{2\pi}$, then $M$ is either the equatorial sphere or the Clifford torus.

Let $M$ be a compact minimally immersed hypersurface in the unit sphere $S^{n+1}$. Denote by $S$ the square of the length of the second fundamental form of $M$. It is well known that if $0 \leq S \leq n$, then $M$ is either the equatorial sphere or a Clifford torus [1]. Recently, C. L. Shen [4, Theorem 2] proved that if $M$ is a compact embedded minimal surface of nonnegative Gauss curvature in the unit sphere $S^3$ with $\|S\|_2 < 1/(6912\sqrt{2\pi}(g + 1))$, then $M$ is the equatorial sphere, where $g$ denotes the genus of $M$. The purpose of this note is to improve this theorem and obtain the best constant. The following is our main result:

Theorem. Let $M$ be a compact minimally immersed surface in the unit sphere $S^3$. Then $\|S\|_2 \geq 2\sqrt{2\pi n}$. The equality sign holds if and only if $M$ is either the equatorial sphere or the Clifford torus. In particular, if $\|S\|_2 \leq 2\sqrt{2\pi}$, then $M$ is either the equatorial sphere or the Clifford torus.

1. Notations and auxiliary results

Let $M$ be a compact connected minimally immersed surface in the unit sphere $S^3$. Following the notations of [1], denote by $h = (h_{ij})$ the second fundamental form of $M$, and by $S$ the square of the length of $h$, $S = \sum h_{ij}^2$. We need the following auxiliary results.

Lemma 1 [1]. $\frac{1}{2} \Delta S = S(2 - S) + \sum h_{ijk}^2$, where $h_{ijk}$ denote the covariant derivatives of $h_{ij}$.
Lemma 2 [3]. The set of all zeros of $S$ is either the whole space $M$ or at most a finite set of points.

Lemma 3. $|\nabla S|^2 = 2S \sum h_{ijk}^2$.

From Lemmas 1 and 3, we see that if $S$ is constant, then either $S \equiv 0$ or $S \equiv 2$.

Lemma 4. If $g \geq 1$, then

$$
\lim_{\epsilon \to 0} \sum_{i=1}^{k} \int_{\partial B\epsilon(p_i)} \frac{S_r}{S} = 16\pi(g - 1)
$$

where $p_1, p_2, \ldots, p_k$ constitute all the zeros of $S$ and $S_r$ denotes the derivative of $S$ on $\partial B\epsilon(p_i)$ in the radial direction from $p_i$.

Proof. At the points where $S$ is positive, by Lemma 3, we get

$$(1) \quad \Delta \log S = 2(2 - S).$$

Integrating (1) over $Me = M \setminus \bigcup_{i=1}^{k} B\epsilon(p_i)$, we get from the Gauss equation

$$(2) \quad 2K = 2 - S,$$

where $K$ is the Gauss curvature of $M$, the assertion by Stokes’s theorem and the theorem of Gauss-Bonnet. □

Lemma 5.

$$
\int_{M} \sqrt{\frac{S}{2}} + \left( \frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \geq (g+1)\pi^2.
$$

Proof. Regard $M$ as an immersed surface of $\mathbb{R}^4$. Then the total absolute curvature of $M$, in the sense of [2], is given by

$$
\int_{M} \int_{0}^{2\pi} \left( (\sin \theta)^2 - \frac{S}{2} (\cos \theta)^2 \right) \, d\theta \, dV
$$

$$
= \int_{M} 2\sqrt{2S} + \left( \pi - 4 \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2}.
$$

By a well-known inequality of Chern-Lashof [2], we have

$$
\int_{M} \sqrt{\frac{S}{2}} + \left( \frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}} \right) \frac{2-S}{2} \geq \frac{\pi^2}{2} (b_0 + b_1 + b_2),
$$

where $b_i$ denotes the $i$th Betti number relative to the real field, for $i = 0, 1, 2$. Since $M$ is of two-dimensional, $b_0 = 1, b_1 = 2g$, and $b_2 = 1$. □
2. Proof of theorem

We may assume that $S$ is positive except possibly at a finite set of points (Lemma 2). By using (1) and Lemmas 3 and 4, we get

$$\int_M \left(1 + \frac{1}{4} h_{ijk}^2 - \sqrt{\frac{S}{2}} - \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2}\right) \right) \Delta \log S + \frac{\nabla S^2}{8S}$$

$$\lim_{\varepsilon \to 0} \int_{M_{\varepsilon}} \nabla \left(\frac{1}{2} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}}\right)\right) \Delta \log S + \frac{\nabla S^2}{8S}$$

$$\lim_{\varepsilon \to 0} \int_{\partial M_{\varepsilon}} \nabla \left(\frac{1}{2} - \frac{1}{2} \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}}\right)\right) \frac{S_r}{S}$$

$$= - (4 - \pi)\pi(g-1) + \lim_{\varepsilon \to 0} \int_{M_{\varepsilon}} \left(\frac{1}{4} - \frac{1}{(2+S)(\sqrt{2+S})^2}\right) \frac{\nabla S^2}{2S}$$

$$= - (4 - \pi)\pi(g-1) + \int_M \left(\frac{1}{4} - \frac{1}{(2+S)(\sqrt{2+S})^2}\right) h_{ijk}^2$$

$$\geq - (4 - \pi)\pi(g-1),$$

where the equality sign holds if and only if $S$ is constant. According to Lemma 5, we get

$$A + \frac{1}{4} \int_M h_{ijk}^2 \geq -(4 - \pi)\pi(g-1) + \int_M \sqrt{\frac{S}{2}} + \left(\frac{\pi}{4} - \sin^{-1} \sqrt{\frac{S}{2+S}}\right) \frac{2-S}{2}$$

$$\geq 2g\pi^2 - 4\pi(g-1),$$

where $A$ denotes the area of $M$. By combining (2) with the inequality (3), it follows that

$$\int_M 2S + h_{ij}^2 = 4A + 16\pi(g-1) + \int_M h_{ijk}^2 \geq 8g\pi^2.$$

The desired inequality now follows from Lemma 1.

References


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