ON ISOMORPHISMS OF INDUCTIVE LIMIT C*-ALGEBRAS

KLAUS THOMSEN

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Abstract. We prove that for a large class of inductive limit C*-algebras, including inductive limits of finite direct sums of interval and circle algebras, any *-isomorphism is induced from an approximate intertwining, in the sense of Elliott, between the inductive systems defining the algebras.

In [3] Elliott introduced a notion, called an approximate intertwining, between two sequences of C*-algebras, and used it successfully to extend, beyond the AF-algebras, the class of C*-algebras for which K-theory is a complete invariant. The purpose of this note is to show that for a considerable class of inductive limit C*-algebras, including the inductive limits of finite direct sums of interval algebras, C[0,1] \otimes M_n, and circle algebras, C(T) \otimes M_n, any *-isomorphism is induced by an approximate intertwining. So with this notion Elliott has grasped all isomorphisms of such C*-algebras. This result shows to what extent the inductive limit C*-algebra reflects the inductive system defining it and gives a useful tool for the study of the structure of such inductive limit C*-algebras.

Fix two sequences

(A) \[ A_1 \xrightarrow{\phi_1} A_2 \xrightarrow{\phi_2} A_3 \xrightarrow{\phi_3} A_4 \rightarrow \cdots \]

and

(B) \[ B_1 \xrightarrow{\psi_1} B_2 \xrightarrow{\psi_2} B_3 \xrightarrow{\psi_3} B_4 \rightarrow \cdots \]

of C*-algebras and *-homomorphisms. And fix subsets \( F_i \subseteq A_i \) and \( G_i \subseteq B_i \), which generate \( A_i \) and \( B_i \), respectively, as C*-algebras, \( i \in \mathbb{N} \). For \( i > j \) we set \( \phi_{i,j} = \phi_{i-1} \circ \phi_{i-2} \circ \cdots \circ \phi_j \) and \( \psi_{i,j} = \psi_{i-1} \circ \psi_{i-2} \circ \cdots \circ \psi_j \). We define \( \phi_{i,i} \) and \( \psi_{i,i} \) to be the identity on \( A_i \) and \( B_i \), respectively. Let \( A = \varinjlim A_i \) and \( B = \varinjlim B_i \) denote the corresponding inductive limit C*-algebras and \( \mu^A_i : A_i \rightarrow A \) and \( \mu^B_i : B_i \rightarrow B \) be the canonical *-homomorphisms. We emphasize that the connecting *-homomorphisms are not assumed to be injective.

The following is one of several possible elaborations on Remark 2.3 of [3].

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Lemma 1 (Elliott). Let \( \{n(i)\} \) and \( \{m(i)\} \) be strictly increasing sequences in \( \mathbb{N} \) and \( \{\delta_n\} \) a sequence in \([0, \infty[\) such that \( \sum_{n=1}^{\infty} \delta_n < \infty \). Let \( \alpha_i: A_{n(i)} \to B_{m(i)}, \) \( i \in \mathbb{N}, \) be \(*\)-homomorphisms such that

\[
\|\psi_{m(i+1),m(i)} \circ \alpha_i \circ \phi_{n(i),n(k)}(x) - \alpha_{i+1} \circ \phi_{n(i+1),n(k)}(x)\| < \delta_i,
\]

whenever \( k \leq i \) and \( x \in F_{n(k)}. \)

Then the sequence \( \{\mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)}(x)\}, \) \( i \geq k, \) converges in \( B \) for each \( x \in A_k \) and all \( k \in \mathbb{N}. \) Furthermore, there is a \(*\)-homomorphism \( \alpha: A \to B \) such that

\[
\alpha(\mu_k^A(x)) = \lim_{i \to \infty} \mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)}(x),
\]

\( x \in A_k, \) \( k = 1, 2, 3, \ldots. \)

Proof. To prove that \( \{\mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)}(y)\} \) converges for all \( y \in A_{n(k)}, \) it clearly suffices to consider the case that \( y \in F_{n(k)}. \) But in this case, the sequence is Cauchy by our assumption. It follows that the sequence \( \{\mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)}(x)\} = \{\mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)} \circ \phi_{n(k),n(k)}(x)\} \) converges for all \( x \in A_k. \)

Therefore we can define \( \alpha'_k: A_k \to B \) by \( \alpha'_k(x) = \lim_{i \to \infty} \mu_{m(i)}^B \circ \alpha_i \circ \phi_{n(i),n(k)}(x). \)
Since \( \alpha'_{k+1} \circ \phi_k = \alpha'_k, \) we obtain a \(*\)-homomorphism \( \alpha: A \to B \) with the stated property. \( \square \)

The assumption of the lemma can be visualized schematically as follows.

\[
\begin{array}{c}
A_{n(1)} \rightarrow \cdots \rightarrow A_{n(2)} \rightarrow A_{n(3)} \rightarrow A_{n(4)} \rightarrow \cdots \\
\downarrow \alpha_1 \downarrow \delta_1 \downarrow \alpha_2 \downarrow \delta_2 \downarrow \alpha_3 \downarrow \delta_3 \downarrow \alpha_4 \downarrow \\
B_{m(1)} \rightarrow \cdots \rightarrow B_{m(2)} \rightarrow B_{m(3)} \rightarrow B_{m(4)} \rightarrow \cdots
\end{array}
\]

A \(*\)-homomorphism \( \alpha: A \to B \) obtained from \(*\)-homomorphisms \( \alpha_i \) as in Lemma 1 is called \emph{approximately filtered} and denoted by \( \alpha = \lim_{i \to \infty} \alpha_i. \)

Definition 2. By an \emph{approximate intertwining} between the sequences (A) and (B), we mean two increasing sequences \( \{n(i)\} \) and \( \{m(i)\} \) in \( \mathbb{N}, \) together with \(*\)-homomorphisms \( \alpha_i: A_{n(i)} \to B_{m(i)} \) and \( \beta_i: B_{m(i)} \to A_{n(i+1)} \) such that

\[
\|\alpha_i \circ \beta_{i-1} \circ \psi_{m(i-1),m(k)}(y) - \psi_{m(i+1),m(k)}(x)\| < 2^{-i},
\]

for \( x \in \bigcup_{k \leq i} \psi_{m(i),m(k)}(F_{m(k)}), \) \( x \in \bigcup_{k \leq i} \alpha_i \circ \phi_{n(i),n(k)}(F_{n(k)}), \)

\[
\|\beta_i \circ \alpha_{i+1} \circ \psi_{m(i+1),m(k)}(y) - \phi_{n(i+1),n(k)}(x)\| < 2^{-i},
\]

for \( y \in \bigcup_{k \leq i} \phi_{n(i),n(k)}(F_{n(k)}), \) \( y \in \bigcup_{k < i} \beta_{i-1} \circ \psi_{m(i-1),m(k)}(G_{m(k)}), \) \( i \in \mathbb{N}. \)
This definition should be compared with that in [3]. Schematically, an approximate intertwining can be visualized as follows.

This diagram is analogous to the diagram in [1, p. 206], which has played such an important role in the study of AF-algebras. The difference is that the triangles do not commute exactly, but only better and better as one approaches infinity in the diagram.

**Theorem 3** (Elliott). An approximate intertwining between the diagrams (A) and (B) induces a $\ast$-isomorphism between $A = \varinjlim A_i$ and $B = \varinjlim B_i$.

**Proof.** From the norm estimates in the definition of an approximate intertwining, it follows that

$$
\|\psi_{m(k),m(i)} \circ \alpha_k(x) - \alpha_k \circ \phi_{n(k),n(i)}(x)\| \leq 2^{-i+2}, \quad x \in \bigcup_{j \leq i} \phi_{n(i),n(j)}(F_{n(j)}),
$$

and

$$
\|\phi_{n(k),n(i+1)} \circ \beta_i(x) - \beta_{i-1} \circ \psi_{m(k-1),m(i)}(x)\| \leq 2^{-i+1}, \quad x \in \bigcup_{j \leq i} \psi_{m(i),m(j)}(G_{m(j)}).
$$

Therefore Lemma 1 can be applied to get approximately filtered $\ast$-homomorphisms $\alpha = \varinjlim \alpha_i : A \to B$ and $\beta = \varprojlim \beta_j : B \to A$. By using the original estimates from the definition of an approximate intertwining it is easily seen that $\alpha$ and $\beta$ are inverses of each other. □

**Lemma 4.** Assume that the generating sets $F_i \subseteq A_i$ and $G_i \subseteq B_i$ are all finite sets. Let $\alpha : A \to B$ be a $\ast$-isomorphism such that $\alpha$ and $\alpha^{-1}$ both are approximately filtered.

Then $\alpha$ is induced from an approximate intertwining of the sequences (A) and (B).

**Proof.** Assume that $\alpha$ and $\beta = \alpha^{-1}$ are derived from the data indicated by the following two diagrams:

$$
\begin{array}{ccccccc}
A_{n(1)} \rightarrow & A_{n(2)} \rightarrow & A_{n(3)} \rightarrow & A_{n(4)} \rightarrow & \cdots \\
\beta_1 \downarrow & \beta_2 \downarrow & \beta_3 \downarrow & \beta_4 \downarrow & \\
B_{m(1)} \rightarrow & B_{m(2)} \rightarrow & B_{m(3)} \rightarrow & B_{m(4)} \rightarrow & \cdots \\
\end{array}
$$
Let \( t_1, t_2, t_3, \ldots \) be the sequence in \([0, 1[\) determined recursively as follows:

\[
t_1 = 1/6 \quad \text{and} \quad 2t_{n+1} + t_n = 2^{-(n+1)}, \quad n \geq 1.
\]

We construct an approximate intertwining between \((A)\) and \((B)\), which can be described schematically as follows:

\[
\begin{array}{ccccccc}
A_{n(y_1)} & 
\xrightarrow{2^{-2}} &
A_{n(y_2)} & 
\xrightarrow{2^{-3}} &
A_{n(y_3)} & 
\xrightarrow{2^{-4}} &
A_{n(y_d)} \\
B_{k(x_1)} & 
\xleftarrow{2^{-2}} &
B_{k(x_2)} & 
\xleftarrow{2^{-3}} &
B_{k(x_3)} & 
\xleftarrow{2^{-4}} &
B_{k(x_d)}
\end{array}
\]

where the \(*\)-homomorphisms \( \alpha'_i : A_{n(y_i)} \to B_{k(x_i)} \) are related to the \( \alpha_i \)'s by

\[
\alpha'_i = \psi_{k(x_i)}, m(c_i) \circ \alpha_{c_i} \circ \phi_{n(c_i)}, n(y_i), \quad i \in \mathbb{N},
\]

for some strictly increasing sequence \( \{c_i\} \subseteq \mathbb{N} \) with \( n(y_i) < n(c_i) \) and \( m(c_i) < k(x_i) \). Once this is done it follows from Lemma 1 that \( \alpha = \lim \alpha'_i = \lim \alpha_i \), so that \( \alpha \) is the \(*\)-isomorphism induced by the constructed approximate intertwining.

The construction proceeds by induction. To construct \( \alpha'_i \), set \( y_1 = 1 \) and choose any \( c_1 \in \mathbb{N} \) such that \( n(1) < n(c_1) \) and

\[
\|\alpha(\mu^A_{n(1)}(x)) - \mu^B_{m(c_1)} \circ \alpha_{c_1} \circ \phi_{n(c_1)}, n(1)(x)\| < t_2,
\]

for \( x \in F_{n(1)} \). This can be done by the definition of \( \alpha = \lim \alpha_i \), since \( F_{n(1)} \) is a finite set. Then we choose \( x_1 \in \mathbb{N} \) such that \( m(c_1) < k(x_1) \) and set

\[
\alpha'_1 = \psi_{k(x_1)}, m(c_1) \circ \alpha_{c_1} \circ \phi_{n(c_1)}, n(1).
\]

Now assume that we have constructed the following piece of an approximate intertwining:

\[
\begin{array}{ccccccc}
A_{n(y_1)} & 
\xrightarrow{2^{-2}} &
A_{n(y_2)} & 
\xrightarrow{2^{-3}} &
A_{n(y_d)} \\
B_{k(x_1)} & 
\xleftarrow{2^{-2}} &
B_{k(x_2)} & 
\xleftarrow{2^{-3}} &
B_{k(x_d)}
\end{array}
\]

such that \( \alpha'_i, \ i \leq d, \) have the form indicated by (1) and such that

\[
\|\alpha(\mu^A_{n(y_d)}(z)) - \mu^B_{m(c_d)} \circ \alpha_{c_d} \circ \phi_{n(c_d)}, n(y_d)(z)\| < t_{2d},
\]
for \( z \in M_d \), where \( M_d \) is the union of
\[
\bigcup_{a \leq d} \phi_{n(y_d), n(y_d)}(F_{n(y_d)})
\]
and
\[
\bigcup_{a \leq d} \beta '_{d-1} \circ \psi_{k(x_{d-1}), k(x_d)}(G_{k(x_d)}).
\]
The induction step then proceeds as follows. Find \( b \) in \( \mathbb{N} \) such that \( l(b) > n(y_d) \), \( b > x_d \) and
\[
\|\beta(\mu_{k(x_d)}^B(y)) - \mu_{l(b)}^A \circ \beta_b \circ \psi_{k(b), k(x_d)}(y)\| < t_{2d+1},
\]
for \( y \in N_d \), where \( N_d \) is the union of \( \bigcup_{a \leq d} \psi_{k(x_d), k(x_d)}(G_{k(a)}) \) and \( \alpha'_d(M_d) \). This can be done by the definition of \( \beta = \lim \beta_i \), using that \( N_d \) is a finite set. Next find \( y_{d+1} \) in \( \mathbb{N} \) so large that
\[
\|\phi_{n(y_{d+1}), l(b)}(z)\| < t_{2d+1} + \|\mu_{l(b)}^A(z)\|,
\]
for all \( z \in (\beta_b \circ \psi_{k(b), k(x_d)} \circ \alpha'_d - \phi_{l(b), n(y_d)}(M_d)) \). This can be done because
\[
\|\mu_{l(b)}^A(z)\| = \lim_{i \to \infty} \|\phi_{n_i(l(b)), l(b)}(z)\| \text{ for each } z.
\]
Set \( \beta'_d = \phi_{n(y_{d+1}), l(b)} \circ \beta_b \circ \psi_{k(b), k(x_d)} \). For each \( x \in M_d \), we find the following estimates, using first (4), then (3) and (1), and finally (2):
\[
\|\beta'_d \circ \alpha'_d(x) - \phi_{n(y_{d+1}), n(y_d)}(x)\|
\leq t_{2d+1} + \|\mu_{l(b)}^A \circ \beta_b \circ \psi_{k(b), k(x_d)}(x)\| - \mu_{n(y_d)}^A(x)\|
\leq 2t_{2d+1} + \|\beta \circ \mu_{k(x_d)}^B \circ \alpha'_d(x) - \mu_{n(y_d)}^A(x)\|
= 2t_{2d+1} + \|\beta(\mu_{m(c_d)}^B \circ \alpha_{c_d} \circ \phi_{n(c_d), n(y_d)}(x)) - \mu_{n(y_d)}^A(x)\|
\leq 2t_{2d+1} + t_{2d} + \|\beta \circ \mu_{m(c_d)}^B \circ \alpha_{c_d} \circ \phi_{n(c_d), n(y_d)}(x)\| = 2t_{2d+1} + t_{2d} < 2^{-(d+1)}.
\]
Now choose \( c_{d+1} > c_d \) such that \( n(c_{d+1}) > n(y_{d+1}) \) and
\[
\|\mu_{m(c_{d+1})}^B \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})}(x) - \alpha_{c_d}(x)\| < t_{2d+2},
\]
for all \( x \in M_{d+1} \cup \beta'_d(N_d) \) and then \( x_{d+1} \) so that \( k(x_{d+1}) > m(c_{d+1}) \) and
\[
\|\psi_{k(x_{d+1}), m(c_{d+1})}(z)\| < t_{2d+2} + \|\mu_{m(c_{d+1})}^B(z)\|,
\]
for all \( z \in (\alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})} \circ \beta'_d - \psi_{m(c_{d+1}), k(x_d)}(N_d)) \).
Set \( \alpha'_{d+1} = \psi_{k(x_{d+1}), m(c_{d+1})} \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})} \). Then we find the following estimates for all \( x \in N_d \), by using first (6), then (5) and (3):
\[
\|\alpha'_{d+1} \circ \beta'_d(x) - \psi_{k(x_{d+1}), k(x_d)}(x)\|
\leq t_{2d+2} + \|\mu_{m(c_{d+1})}^B \circ \alpha_{c_{d+1}} \circ \phi_{n(c_{d+1}), n(y_{d+1})} \circ \beta'_d(x) - \mu_{k(x_d)}^B(x)\|
\leq 2t_{2d+2} + \|\alpha \circ \mu_{n(y_{d+1})}^A \circ \beta'_d(x) - \mu_{k(x_d)}^B(x)\|
\leq 2t_{2d+2} + t_{2d+1} < 2^{-(d+1)}.
\]
This completes the induction step and hence the proof. □

**Lemma 5.** Let \( \alpha : A \to B \) be a \(*\)-homomorphism and assume that the generating subsets \( F_i \subseteq A_i \) are finite. Assume, furthermore, that for each \( i \in \mathbb{N} \) and each \( \varepsilon > 0 \), there is a \( k \in \mathbb{N} \) and a \(*\)-homomorphism \( \alpha_i : A_i \to B_k \) such that

\[
\| \mu_k \circ \alpha_i(x) - \alpha \circ \mu_i^A(x) \| < \varepsilon, \quad x \in F_i.
\]

Then \( \alpha \) is approximately filtered.

**Proof.** For each \( i \in \mathbb{N} \), the set \( F_i' = \bigcup_{j \leq i} \phi_{i,j}(F_j) \) is finite. Our assumptions are therefore strong enough to give us a strictly increasing sequence \( \{k(i)\} \) in \( \mathbb{N} \) and \(*\)-homomorphisms \( \alpha_i : A_i \to B_{k(i)} \) such that

\[
(7) \quad \| \mu_{k(i)} \circ \alpha_i(x) - \alpha \circ \mu_i^A(x) \| < 2^{-i}, \quad x \in F_i'.
\]

We use this to construct by induction a strictly increasing sequence \( \{m(i)\} \) in \( \mathbb{N} \) and \(*\)-homomorphisms \( \alpha_i' : A_i \to B_{m(i)} \) such that \( m(i) > k(i) \), \( \alpha_i' = \psi_{m(i), k(i)} \circ \alpha_i \), and

\[
\| \psi_{m(i+1), m(i)} \circ \alpha_i'(x) - \alpha_i' \circ \phi_{i}(x) \| < 2^{-i} + 2^{-i-1}, \quad x \in F_i'.
\]

We leave the reader to start the induction and concentrate on the induction step. So assume that \( m(i) \) and \( \alpha_i' \), \( i \leq d \), have been found. (7) yields the estimate

\[
\| \mu_{k(d+1)} \circ \alpha_d'(x) - \alpha_d \circ \mu_d^A(x) \| < 2^{-d} + 2^{-d-1}, \quad x \in F_d'.
\]

for \( l = \max\{m(d), k(d+1)\} \). It follows that there is a \( m(d+1) > l \) such that

\[
\| \psi_{m(d+1), m(d)} \circ \alpha_d'(x) - \psi_{m(d+1), k(d+1)} \circ \alpha_{d+1} \circ \phi_{d}(x) \| < 2^{-d} + 2^{-d-1},
\]

\( x \in F_d' \). Set \( \alpha_{d+1}' = \psi_{m(d+1), k(d+1)} \circ \alpha_{d+1} \). This completes the induction step.

By Lemma 1 we obtain a filtered \(*\)-homomorphism \( \alpha' = \lim \alpha_i' \). Since

\[
\alpha' (\mu_k^A(x)) = \lim_{i \to \infty} \mu_{m(i)}^B \circ \alpha_i' \circ \phi_{i,k}(x)
\]

\[
= \lim_{i \to \infty} \mu_{m(i)}^B \circ \psi_{m(i), k(i)} \circ \alpha_i \circ \phi_{i,k}(x)
\]

\[
= \lim_{i \to \infty} \mu_{k(i)}^B \circ \alpha_i \circ \phi_{i,k}(x)
\]

and

\[
\| \mu_{k(i)} \circ \alpha_i \circ \phi_{i,k}(x) - \alpha \circ \mu_i^A \circ \phi_{i,k}(x) \| < 2^{-i},
\]

\( x \in F_k, \ i \geq k \), we conclude that \( \alpha' (\mu_k^A(x)) = \alpha \circ \mu_k^A(x), \ x \in F_k, \ k \in \mathbb{N} \). Hence \( \alpha = \alpha' \) and the proof is complete. □
Let $\mathcal{C}$ denote the class of $C^*$-algebras $A$ that meet the following conditions:

(i) $A$ is finitely generated, and

(ii) when $B = \lim_n B_n$ is the inductive limit of a sequence of $C^*$-algebras $B_n$, $\mu_n : B_n \to B$ the canonical $\ast$-homomorphisms, $\alpha : A \to B$ a $\ast$-homomorphism, $F \subseteq A$ a finite set and $\varepsilon > 0$, then there is a $k \in \mathbb{N}$ and a $\ast$-homomorphism $\beta : A \to B_k$ such that $\|\mu_k \circ \beta(x) - \alpha(x)\| < \varepsilon$, $x \in F$.

If all the $C^*$-algebras in the sequence (A) are in the class $\mathcal{C}$, the second assumption of Lemma 5 is automatically fulfilled. So we get the following result.

**Theorem 6.**  (i) Assume that the $C^*$-algebras occurring in the sequence (A) are in $\mathcal{C}$. Then every $\ast$-homomorphism $\alpha : A \to B$ is approximately filtered.

(ii) Assume that the $C^*$-algebras occurring in the sequences (A) and (B) are in $\mathcal{C}$. Then any $\ast$-isomorphism $\alpha : A \to B$ is induced from an approximate intertwining of the sequences (A) and (B).

**Proof.** Combine Lemma 4 and Lemma 5. $\square$

By modifying arguments from [2], it is not difficult to prove the following assertions:

(a) $\mathbb{C}$, $C_0(\mathbb{R})$, $C(\mathbb{T})$, $O_A \in \mathcal{C}$ (where $O_A$ are the Cuntz-Krieger algebras).

(b) When $A$, $B \in \mathcal{C}$ are unital, then $A \oplus B \in \mathcal{C}$.

(c) When $A \in \mathcal{C}$ is unital and $F$ is finite-dimensional, then $A \otimes F \in \mathcal{C}$.

(d) When $A$, $B \in \mathcal{C}$ are unital and $F \subseteq A$, $F \subseteq B$ is a common unital finite-dimensional $C^*$-subalgebra, then (the amalgamated free product) $A \ast_F B \in \mathcal{C}$.

Furthermore, it is easy to show that $\mathcal{C}$ contains $C[0, 1]$ and $C^*(F_n)$ for all $n$. In particular, $\mathcal{C}$ contains all finite direct sums of circle and interval algebras.

**References**


Matematisk Institut, Ny Munkegade, 8000 Aarhus C, Denmark